Question 441 : eight series for pi

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March 5 , 2018

abstract
This note presents eight series for pi.

1. INTRODUCTION. The pi constant is defined by

$$\pi = 4\int_0^1 \sqrt{1-x^2}\,dx = 3.1415926535... \tag{1}$$

2. SOME FORMULAS FOR PI

$$\frac{\pi\left(\sqrt{2}-1\right)}{2} = -\sum_{n=1}^{\infty}\left(\left(\frac{2-\sqrt{2}+i\sqrt{2}}{2}\right)^n + \left(\frac{2-\sqrt{2}-i\sqrt{2}}{2}\right)^n\right)\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}} \tag{2}$$

$$\frac{\pi}{2} = \frac{1}{i}\sum_{n=1}^{\infty}\left(\left(\frac{2-\sqrt{2}+i\sqrt{2}}{2}\right)^n - \left(\frac{2-\sqrt{2}-i\sqrt{2}}{2}\right)^n\right)\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}} \tag{3}$$

$$\frac{\pi\left(2-\sqrt{3}\right)}{3} = -\sum_{n=1}^{\infty}\left(\left(\frac{2-\sqrt{3}+i}{2}\right)^n + \left(\frac{2-\sqrt{3}-i}{2}\right)^n\right)\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}} \tag{4}$$

$$\frac{\pi}{3} = \frac{1}{i}\sum_{n=1}^{\infty}\left(\left(\frac{2-\sqrt{3}+i}{2}\right)^n - \left(\frac{2-\sqrt{3}-i}{2}\right)^n\right)\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}} \tag{5}$$

$$\frac{\pi}{\sqrt{2}} = -\sum_{n=1}^{\infty}\left\{(1+i)\left(\frac{2-\sqrt{2}+i\sqrt{2}}{2}\right)^n + (1-i)\left(\frac{2-\sqrt{2}-i\sqrt{2}}{2}\right)^n\right\}\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}} \tag{6}$$

$$\frac{\pi\left(2-\sqrt{2}\right)}{2}=\sum_{n=1}^{\infty}\left\{(1-i)\left(\frac{2-\sqrt{2}+i\sqrt{2}}{2}\right)^{n}+(1+i)\left(\frac{2-\sqrt{2}-i\sqrt{2}}{2}\right)^{n}\right\}\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}} \qquad (7)$$

$$\frac{\pi\left(3-\sqrt{3}\right)}{3}=-\sum_{n=1}^{\infty}\left\{(1+i)\left(\frac{2-\sqrt{3}+i}{2}\right)^{n}+(1-i)\left(\frac{2-\sqrt{3}-i}{2}\right)^{n}\right\}\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}} \qquad (8)$$

$$\frac{\pi\left(\sqrt{3}-1\right)}{3}=\sum_{n=1}^{\infty}\left\{(1-i)\left(\frac{2-\sqrt{3}+i}{2}\right)^{n}+(1+i)\left(\frac{2-\sqrt{3}-i}{2}\right)^{n}\right\}\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}} \qquad (9)$$

Remark: $i=\sqrt{-1}$.

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