Interesting expansion based on matching definite integrals of derivatives: simple, elegant, but unexplored

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Abstract

A novel method of function expansion is presented. It is based on matching the definite integrals of the derivatives of the function to be approximated by appropriate polynomials. The method is fully integral-based, it is easy to construct and it presumably slightly outperforms Taylor series in the convergence rate.

1 Introduction

Different ideas can be forwarded when discussing function approximations. For example, the case of Fourier series is often interpreted as a decomposition of a vector (function) into the basis of the vector space (sines and cosines). Yet, a more general view can be adopted: if a function $g(x)$ is to be approximated by an approximation $A_g(x)$ then one usually tries to formulate an infinite number of conditions/restrictions respected by the function g and build A_g such as to respect them too. As examples:

- Taylor series, Neumann series (of Bessel functions) and Padé approximation are based on matching derivatives
$$\frac{d^n}{dx^n}A_g(x)|_{x=x_0} = \frac{d^n}{dx^n}g(x)|_{x=x_0}, \quad n = 0, 1, 2, \ldots$$
- Fourier and Fourier-like (Legendre polynomials) series match integrals of the function to be approximated with a predefined function set
$$\int_0^{2\pi} \sin(nx) A_g(x)\, dx = \int_0^{2\pi} \sin(nx) g(x)\, dx,$$
$$\int_0^{2\pi} \cos(nx) A_g(x)\, dx = \int_0^{2\pi} \cos(nx) g(x)\, dx,$$
$$n = 0, 1, 2, \ldots$$

If, in addition to an infinite set of conditions, further requirements are adopted (integrability, smoothness, analyticity) one may be lucky to get an approximation which works for a large set of functions. The approximation very often converges for some trivial cases (e.g. $g(x) = 0$) but, also usually, fails on a large set of functions[1].

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I am willing to publish any of my ideas presented here in a journal, if someone (an editor) judges them interesting enough. Journals in the "*Current Contents*" database are strongly preferred.

[1]Non-analytic smooth functions in case of derivative matching or non-periodic functions when Fourier series are concerned.

In case of the Taylor and Fourier series a huge number of results exists and specific criteria on convergence are established. This is not the case of the approximation I present in this text. This text should be seen as “brainstorming”: the principle is presented without actually providing nice criteria for convergence or proving the convergence for specific cases. Well, for some trivial scenarios, like $g(x) = 0$, the convergence is guaranteed.

2 Approximation method

A function g can be characterized by a set of numbers

$$c_n[g] = \int_a^b \left[g^{(n)}(x)\right] dx,$$

where

$$g^{(n)}(x) = \frac{d^n}{dx^n} g(x)$$

and $a < b$ are arbitrary real numbers on which the expansion depends. A rather natural choice is $a = 0,\ b = 1$. By construction, the coefficients c_n can be computed easily

$$c_n = \left[g^{(n-1)}(x)\right]_a^b = g^{(n-1)}(b) - g^{(n-1)}(a).$$

The approximation $A_g(x)$ is built requiring the same characterization

$$c_n[A_g] = c_n[g] \text{ for each } n.$$

Clearly, for some choice of a and b, two different functions can have the same characterization, e.g. function $g_1(x) = \sin(x)$ and $g_2(x) = 0$ when $a = 0$ and $b = 2\pi$, leading so to a failure of the approximation. Yet, for a different interval, the approximation can be valid. One may notice a similarity with Taylor polynomials: the coefficients c_n are as easy to determine as the Taylor series coefficients, only the derivatives need to be computed at two points instead of one.

A more complicated task one needs to fulfill to guarantee the simplicity of the approximation is to find a set of functions $b_n(x)$ having the delta-property with respect to the presented characterization

$$c_n[b_m(x)] = \delta_{n,m}.$$

Once these functions are found the approximation can be very easily written as

$$A_g(x) = \sum_{n=0}^{\infty} c_n b_n(x),$$

where

$$c_n = c_n[g] = g^{(n-1)}(b) - g^{(n-1)}(a).$$

Indeed, one has

$$\begin{aligned} c_n[A_g] &= c_n\left[\sum_{m=0}^{\infty} c_m[g]\, b_m(x)\right] \\ &= \sum_{m=0}^{\infty} c_m[g]\, c_n[b_m(x)] \qquad \text{(because } c_n[f] \text{ is linear in } f\text{)} \\ &= \sum_{m=0}^{\infty} c_m[g]\, \delta_{n,m} \\ &= c_n[g] \end{aligned}$$

I would like to bring the attention to the nature of the method: it is a fully "integral" method and in this sense it can be compared to the Fourier series. And yet, it is based on differentiation. The "integral" approach is however applied to derivatives which might lead to a specific behavior: different regions of the (a, b) interval might have different weights, in fact in some situations a very small region can have a large effect on the approximation[2]. I believe this is an interesting mix of properties which makes the method unique and unlike others.

Now the focus should be given to the construction of the functions b_n. Without loss of generality I assume $a = 0$ and $b = 1$. Then one is free to chose a function $b_{-1}(x)$ which satisfies

$$\int_0^1 \left[b_{-1}^{(n)}(x)\right] dx = 0 \text{ for any } n \geq 0.$$

As already stated this choice is ambiguous, an infinite number of function fulfills this requirement. In what follows I adopt the most natural choice

$$b_{-1}(x) = 0.$$

Then, by integration with carefully chosen integration constants, one can progressively build a sequence of functions having the property

$$\int_0^1 b_0(x)\, dx = 1,$$
$$\int_0^1 b_n(x)\, dx = 0 \text{ for } n > 0,$$

with

$$\frac{d}{dx} b_n(x) = b_{n-1}(x).$$

Here is an important point to stress: the choice of the integration constant is unique and therefore (in the case $b_{-1} \equiv 0$) the polynomials $b_n(x)$ are unique with no other polynomial existing which would satisfy a given characterization. The actual construction leads to

$$\begin{aligned}
b_0(x) &= 1, \\
b_1(x) &= x - \frac{1}{2}, \\
b_2(x) &= \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12}, \\
b_3(x) &= \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12}, \\
b_4(x) &= \frac{x^4}{24} - \frac{x^3}{12} + \frac{x^2}{24} - \frac{1}{720}, \\
b_5(x) &= \frac{x^5}{120} - \frac{x^4}{48} + \frac{x^3}{72} - \frac{x}{720}, \\
&\dots
\end{aligned}$$

The only reference to such polynomials I was able to find on internet is a french exercise book [1] and I do not know whether these polynomials have been given a name. I the cited reference it is shown that they can be related to the generalized harmonic numbers $H_{n,-m}$, $0 < m$. The cited reference also provides the way these polynomials can be written using a recursive rule

$$b_n(x) = \sum_{k=0}^{n} \frac{a_k}{(n-k)!} x^{(n-k)}$$

[2] Consider a situation where some derivative is close to a delta function - and it can well be situated somewhere inside the (a, b) interval.

with

$$\begin{aligned} a_0 &= 1, \\ a_n &= -\sum_{j=0}^{n-1} \frac{a_j}{(n+1-j)!}. \end{aligned} \tag{1}$$

The functions b_n can be easily scaled to any interval (a,b):

$$b_n^{a,b}(x) = (b-a)^{n-1} b_n\left(\frac{x-a}{b-a}\right). \tag{2}$$

Six lowest b_n functions are shown in Fig. 1. Quickly they become very similar to (scaled) trigonometric functions. The authors of [1] characterize these polynomials as the only polynomials b_n satisfying

$$\begin{aligned} b_0(x) &= 1, \\ b'_{n+1}(x) &= b_n(x), \\ b_n(0) &= b_n(1) \text{ for all } n > 2. \end{aligned}$$

The last property clearly indicates that the only b -polynomial which can be responsible for $A_g(0) \neq A_g(1)$ is the polynomial b_1.

3 Numerical results, observations, comments, conjectures

Approximation of polynomials

As stressed before, the polynomials b_n are unique[3] and therefore the following statement has to hold: any polynomial P_n, if approximated by the presented method, leads to an exact expression

$$A_{P_n}(x) = P_n(x) = \sum_{m=0}^{n} c_m b_m(x).$$

This expansion always exists and is unique. It follows from the "vector space" point of view and can be seen as linear algebra problem of combining the basal vectors b_m to construct the vector P_n.

When approximation fails

As already stated, the approximation fails, in a convergent manner, when one has a non-zero function g which is characterized on the interval (a,b) by zeros $c_n[g] = 0$. From this, one can generalize to other functions: if a function f can be split to two non-zero functions $f = g_1 + g_2$ such that $A_{g_1}(x) = g_1(x)$ and $c_n[g_2] = 0$, then the approximation $A_f(x)$ fails to approximate f. This is a trivial statement which wants only to say that a non-working but convergent (to a different value) expansions do not happen only for function characterized by zeros but also for functions which have components (additive part) characterized by zeros. As an example: expansion for $g_1(x) = \exp(x)$ seems to work for $a = -1$ and $b = 1$ but the approximation fails for $g_1(x) = \exp(x) + \exp\left(-\frac{1}{1-x^2}\right)$ on the same interval[4].

It is unclear whether a different scenario exists, where a convergent but non-approximating expansion occurs. In other words: Can one from $A_g(x) \neq g(x)$ deduce $[A_g(x) - g(x)]^{(n)}(x)|_{x=a,b} = 0$ for any n?

[3] And linearly independent because of different order.

[4] A "standard" bump function is used here, where the value at $|x| = 1$ is assumed to be zero.

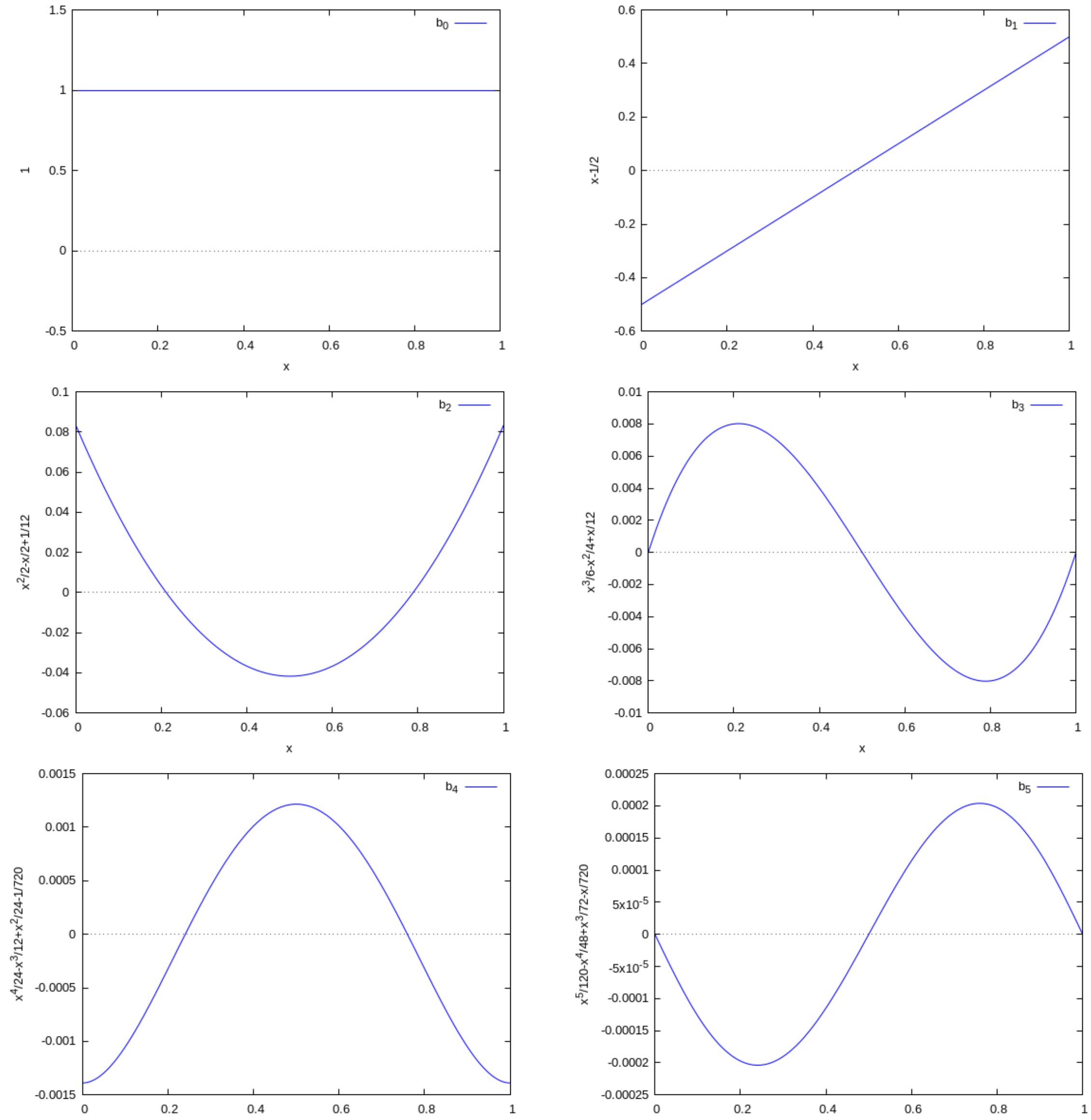

Figure 1: Six lowest b_n functions.

Determining coefficient c_0

A very nice feature of the proposed expansion lies in the fact that one does not need to integrate (unlike in Fourier series, for example). The exception is the coefficient c_0:

$$c_0 = g^{(-1)}(b) - g^{(-1)}(a),$$

where the integral of g appears. Clearly, the "c_0" issue is only an issue of an absolute offset (because b_0 is constant). To avoid integration a different strategy for its determination can be adopted: One first determines c_n for $n > 0$, builds the series and then adjusts c_0 such as to match the function value at some point

$$c_0 \text{ such as to provide } A_g(x_0) = g(x_0)$$

where some natural choice of x_0 can be made, such as

$$x_0 = a,\ b,\ \frac{a+b}{2}\dots$$

Relation to Taylor series

An obvious idea comes to the mind: the approximation is based on the evaluation the derivatives at two points, a and b, and therefore, maybe, it can be expressed as some function of the Taylor series at these points, $T_a(x)$ and $T_b(x)$. Let me now give some arguments that a simple link between the proposed expansion and Taylor series is improbable.

First, the coefficients inside the b_n polynomials, see (1), are computed in a rather complicated recursive way. It seems unlikely that some simple combination of Taylor polynomials could lead to the same result as the summation of the b_n functions.

In addition, the numerical results seem to also dismiss this hypothesis. The numerical observation is as follows: the proposed approximation, when applied to the exponential function, works for $a = 0$ and $b = 1$ but fails for $a = -5$ and $b = 5$. On the other hand, the Taylor series $T_a(x)$ and $T_b(x)$ converge expanded at any point. If the method presented in this text depends on the Taylor series expanded at the endpoints of the integral, then this dependence is clearly a complicated one: it does or does not provide a converging result depending on a and b, while, at least in the case of the exponential function, there is nothing special about Taylor expansions at any of these points whatever the distance between them is.

An interesting scenario is the limit $a \to b$. In this limit the integrals (i.e. the c_n coefficients) go to zero but the functions b_n are scaled also, see (2). The multiplicative factor makes them smaller, the argument scaling greater. Because many changing parameters are in game, it is difficult to see what actually happens. It might well be that the Taylor series is obtained.

Clearly, the question about how is the presented expansion related with Taylor series is a good one. One cannot conclude that a simple relation is impossible; however, in the defense of the originality of the new method I try to provide (some rough) arguments that it is improbable.

Convergence

Let me first focus on the case $a = 0$ and $b = 1$. For this setting the numerical observation is as follows: When the absolute value of all derivatives is bounded (by a common bound, uniformly bounded) then the approximation converges . This is, obviously, a very constraining criterion which implies that the method does not do better then the Taylor series: bounded derivatives imply analyticity. Yet, the previous statement is an implication. Numerical observation is, that even growing derivatives may lead to a convergent result, but the growth cannot be very fast. Let the toy function be $\exp(\alpha x)$. Its derivatives are bounded by the value at $x = 1$ and, if $\alpha > 1$, the bound is growing in geometrical progression (with growing derivative order). It seems the convergence is lost somewhere around $\alpha \approx 6.2$. One should notice that the b_n functions are greater then the $\frac{1}{n!}x^n$ functions (for high values of n), the latter being the basic building blocks of the

Taylor series. In both cases one may expect the coefficients to be of the order of the derivatives[5], thus the b_n functions have greater chances to diverge when comparable coefficients are used. This are rather pessimistic statements about the new expansion and naturally lead to the question: are there any cases when the new expansion outperforms the Taylor series[6]? I did not find such an example for now and hopes are rather small: a non-analytic smooth function should be targeted (because here the Taylor expansion fails), yet, such functions have to have unbounded derivatives... can their growth be slow enough?

In the case $a \neq 0$, $b \neq 1$ further factors come into the play: the b_n functions are scaled and the factor $(b-a)^{n-1}$ becomes important, see (2). It makes function smaller if $b-a<1$, it makes the function greater if $b-a>1$. In the latter case the convergence become "worse", the function which has a converging approximation on $(a,b)=(0,1)$ might not have it on a larger interval. Let me chose the test function to be $\exp(x)$ and the interval in the form $(a,b)=(0,b)$. It turns out[7] that the convergence is lost somewhere around $b \approx 6.2$, which is a similar number[8] to the one from the previous paragraph. It seems the convergence scales with scaling the argument. One cannot approximate a larger portion of function by scaling it to a smaller interval. What is gained by the interval becoming smaller, that is lost the increase of the function derivatives: the convergence remains the same.

Approximation power

I use to test the approximation power on four elementary functions and make comparison to Taylor polynomials. The functions are x^2, $\exp(x)$, $\sin(x)$ and $\ln(x+1)$. Here, however, the function x^2 is by both methods reconstructed exactly, therefore it is dropped. The method presented in this text also needs a choice of the interval limits a and b. To be symmetric with respect to zero, I adopt the choice $a=-1$ and $b=1$ (and use scaled b_n functions). This choice is made for $\exp(x)$ and $\sin(x)$, the function $\ln(x+1)$ is not defined at -1. A different choice needs to be made: it turns out that in addition to the divergence at -1, the function $\ln(x+1)$ needs a small interval for the expansion to converge, e.g. the expansion on the interval $(-0.5,0.5)$ seems to diverge. Therefore I shrink the interval enough to get a convergent regime: the function $\ln(x+1)$ is expanded on the interval $(a,b)=(-0.1,0.1)$.

With this settings and using 11 terms for each series I get descriptions shown on Fig. 2. At last some good news: it seems that in all three cases the new expansion outperforms the Taylor series, the approximation stays longer with the original function that its concurrent. The expansion proposed in this text seems to depart more quickly from the logarithm function on the positive side, the zoom however reveals that, on the most of the interval where the approximation is good (up to $x \approx 0.92$) it actually stays closer to the original function then the Taylor series. One must honestly admit that the improvement with respect to the Taylor series is very modest.

4 Summary, conclusion

In this text I presented a method of approximation based on matching the definite integrals of derivatives. It is (presumably) an original method which has several appealing features: it is fully integral-based, it is simple to construct and, it seems, it performs somewhat better then the Taylor series when the rate of the convergence is concerned. This text is based only on numerical evidence, a future development of a rigorous treatment is desirable.

[5] Exactly derivatives in case of the Taylor series.
[6] Converge in a scenario where the Taylor series does not.
[7] Pure numerical observation.
[8] Is not that close to 2π?

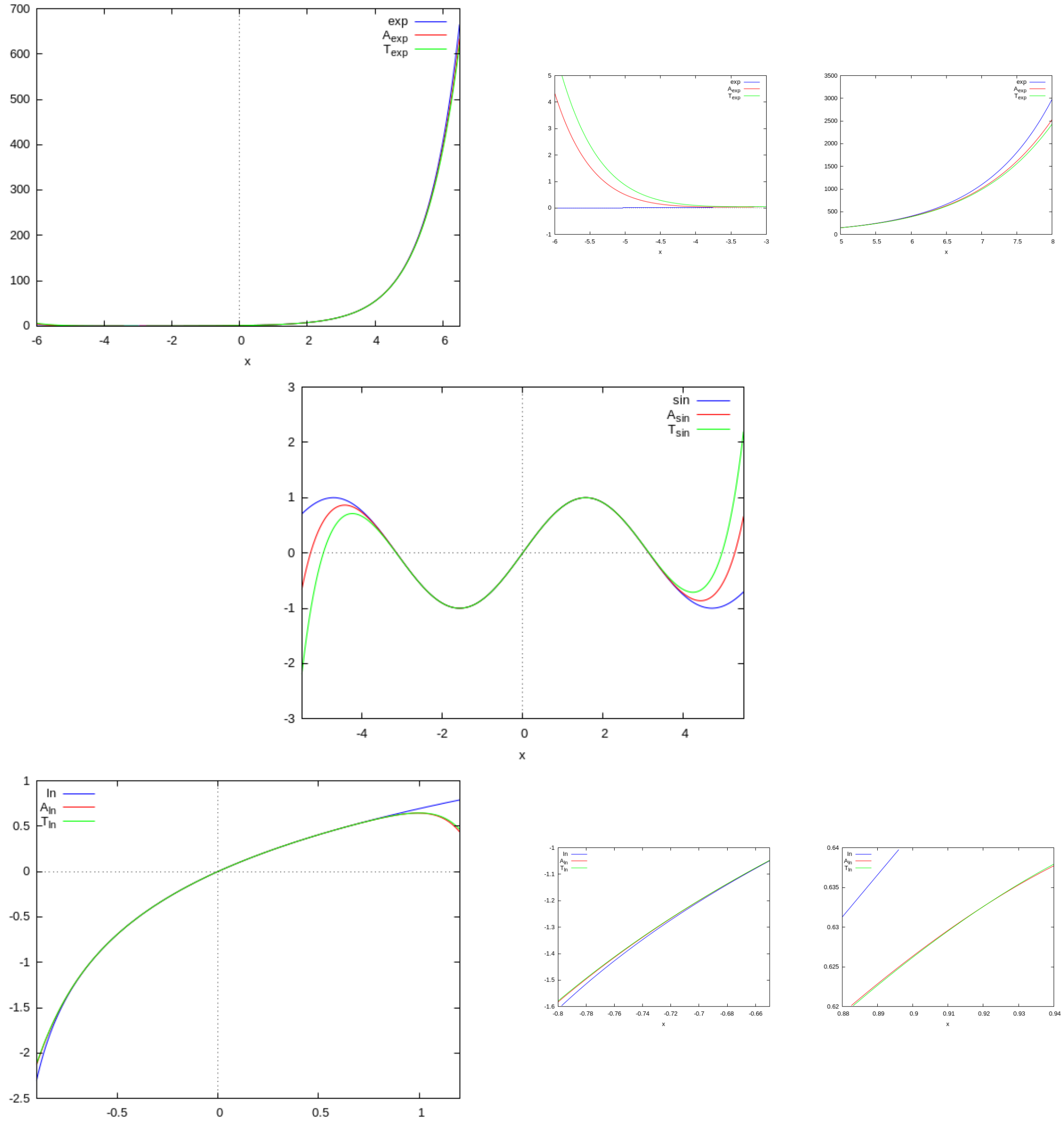

Figure 2: Approximations of three function $\exp(x)$, $\sin(x)$ and $\ln(x+1)$ by Taylor polynomials (T) and the expansion presented in this text (A) with 11 terms in both cases. For $\exp(x)$ and $\ln(x+1)$ a zoom into the tail-region area on both sides is provided.

References

[1] D. Guinin, B. Joppin,
"Mathématiques PCSI - Exercices", Editions Bréal, 2003, ISBN 978 2 7495 0176 5, pages 173, 174.
"Mathématiques MPSI - Exercices", Editions Bréal, 2003, ISBN 978 2 7495 0175 8, pages 90, 91.

A Program

Here I provide the WxMaxima code for getting the presented function expansion, also Taylor polynomials are displayed.

```
kill(all);

/**** START OF THE IMPUT ****/

    /* Enter function to expand */
fun(x) := exp(x)$

    /* Enter number of terms */
N: 5$

    /* Enter interval limits */
a : -1$
b : 1$

    /* Enter graphic window  interval */
gLL:-3$
gUL:2$

/**** END OF THE IMPUT ****/

dL : b-a$
N : N-1$

a[n] := if n=0 then 1 else -sum(a[k]/(n+1-k)!,k,0,n-1)$
P(n,x) := sum(a[k]*x^(n-k)/(n-k)!,k,0,n)$
sP(n,x) := dL^(n-1)*P(n, (x-a)/dL )$

cf[n] := at(diff(fun(x),x,n-1),x=b)-at(diff(fun(x),x,n-1),x=a)$

approx(x) := ''( sum(cf[n]*sP(n,x),n,1,N ) )$

mdp : (a+b)/2$
shift : fun(mdp)-approx(mdp)$

approx(x) := ''(approx(x) + shift)$
wxplot2d([fun(x),approx(x),taylor(fun(x), x, 0, N)],[x,gLL,gUL],[legend,"f","A_{f}","T_{f}"]);
```