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Some New Biparametric Distance Measures on Single-Valued Neutrosophic Sets with Applications to Pattern Recognition and Medical Diagnosis

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Abstract: Single-valued neutrosophic sets (SVNSs) handling the uncertainties characterized by truth, indeterminacy, and falsity membership degrees, are a more flexible way to capture uncertainty. In this paper, some new types of distance measures, overcoming the shortcomings of the existing measures, for SVNSs with two parameters are proposed along with their proofs. The various desirable relations between the proposed measures have also been derived. A comparison between the proposed and the existing measures has been performed in terms of counter-intuitive cases for showing its validity. The proposed measures have been illustrated with case studies of pattern recognition as well as medical diagnoses, along with the effect of the different parameters on the ordering of the objects.

Keywords: decision-making; single-valued neutrosophic sets; distance measure; pattern recognition; uncertainties

1. Introduction

The classical measure theory has been widely used to represent uncertainties in data. However, these measures are valid only for precise data, and hence they may be unable to give accurate judgments for data uncertain and imprecise in nature. To handle this, fuzzy set (FS) theory, developed by Zadeh [1], has received much attention over the last decades because of its capability of handling uncertainties. After this, Atanassov [2] proposed the concept of an intuitionistic fuzzy set (IFS), which extends the theory of FSs with the addition of a degree of non-membership. As IFS theory has widely been used by researchers [3–16] in different disciplines for handling the uncertainties in data, hence its corresponding analysis is more meaningful than FSs' crisp analysis. Nevertheless, neither the FS nor IFS theory are able to deal with indeterminate and inconsistent information. For instance, we take a person giving their opinion about an object with 0.5 being the possibility that the statement is true, 0.7 being the possibility that the statement is false and 0.2 being the possibility that he or she is not sure. To resolve this, Smarandache [17] introduced a new component called the "indeterminacy-membership function" and added the "truth membership function" and "falsity membership function", all which are independent components lying in $]0^-, 1^+[$, and hence the corresponding set is known as a neutrosophic set (NS), which is the generalization of the IFS and FS. However, without specification, NSs are difficult to apply to real-life problems. Thus, a particular case of the NS called a single-valued NS (SVNS) has been proposed by Smarandache [17], Wang et al. [18].

After this pioneering work, researchers have been engaged in extensions and applications to different disciplines. However, the most important task for the decision-maker is to rank the objects so as to obtain the desired object(s). For this, researchers have made efforts to enrich the concept of information measures in neutrosophic environments. Broumi and Smarandache [19] introduced the Hausdorff distance, while Majumdar [20] presented the Hamming and Euclidean distance for comparing the SVNSs. Ye [21] presented the concept of correlation for single-valued neutrosophic numbers (SVNNs). Additionally, Ye [22] improved the concept of cosine similarity for SVNSs, which was firstly introduced by Kong et al. [23] in a neutrosophic environment. Nancy and Garg [24] presented an improved score function for ranking the SVNNs and applied them to solve the decision-making problem. Garg and Nancy [25] presented the entropy measure of order α and applied them to solve decision-making problems. Recently, Garg and Nancy [26] presented a technique for order preference by similarity to ideal solution (TOPSIS) method under an interval NS environment to solve decision-making problems. Aside from these, various authors have incorporated the idea of NS theory into the similarity measures [27,28], distance measures [29,30], the cosine similarity measure [19,22,31], and aggregation operators [22,31–40].

Thus, on the basis of the above observations, it has been observed that distance or similarity measures are of key importance in a number of theoretical and applied statistical inference and data processing problems. It has been deduced from studies that similarity, entropy and divergence measures could be induced by the normalized distance measure on the basis of their axiomatic definitions. On the other hand, SVNSs are one of the most successful theories to handle the uncertainties and certainties in the system, but little systematic research has explored these problems. The gap in the research motivates us to develop some families of the distance measures of the SVNS to solve the decision-making problem, for which preferences related to different alternatives are taken in the form of neutrosophic numbers. The main contributions of this work are summarized as follows: (i) to highlight the shortcomings of the various existing distance measures under the single-valued neutrosophic information through illustrative examples; (ii) to overcome the shortcomings of the existing measures, this paper defines some new series of biparametric distance measures between SVNSs, which depend on two parameters, namely, p and t, where p is the L_p norm and t identifies the level of uncertainty. The various desirable relations between these have been investigated in detail. Then, we utilized these measures to solve the problem of pattern recognition as well as medical diagnosis and compared their performance with that of some of the existing approaches.

The rest of this paper is organized as follows. Section 2 briefly describes the concepts of NSs, SVNSs and their corresponding existing distance measures. Section 3 presents a family of the normalized and weighted normalized distance measures between two SVNSs. Some of their desirable properties have also been investigated in detail, while generalized distance measures have been proposed in Section 4. The defined measures are illustrated, by an example in Section 5, using the field of pattern recognition and medical diagnosis for demonstrating the effectiveness and stability of the proposed measures. Finally, a concrete conclusion has been drawn in Section 6.

2. Preliminaries

An overview of NSs and SVNSs is addressed here on the universal set X.

2.1. Basic Definitions

Definition 1 ([17,41])**.** *A neutrosophic set (NS) A in X is defined by its truth membership function* $(T_A(x))$*, an indeterminacy-membership function* $(I_A(x))$ *and a falsity membership function* $(F_A(x))$*, where all are subsets of* $]0^-, 1^+[$*. There is no restriction on the sum of* $T_A(x)$*,* $I_A(x)$ *and* $F_A(x)$*; thus* $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ *for all* $x \in X$*. Here,* sup *represents the supremum of the set.*

Wang et al. [18], Smarandache [41] defined the SVNS, which is an instance of a NS.

Definition 2 ([18,41])**.** *A single-valued neutrosophic set (SVNS) A is defined as*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$$

where $T_A : X \to [0,1]$, $I_A : X \to [0,1]$ and $F_A : X \to [0,1]$ with $T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The values $T_A(x)$, $I_A(x)$ and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of x to A, respectively. The pairs of these are called single-valued neutrosophic numbers (SVNNs), which are denoted by $\alpha = \langle \mu_A, \rho_A, \nu_A \rangle$, and class of SVNSs is denoted by $\Phi(X)$.

Definition 3. *Let $A = \langle \mu_A, \rho_A, \nu_A \rangle$ and $B = \langle \mu_B, \rho_B, \nu_B \rangle$ be two single-valued neutrosophic sets (SVNSs). Then the following expressions are defined by [18]:*

(i) $A \subseteq B$ *if and only if (iff)* $\mu_A(x) \leq \mu_B(x), \rho_A(x) \geq \rho_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for all x in X;*
(ii) $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$;
(iii) $A^c = \{\langle \nu_A(x), 1 - \rho_A(x), \mu_A(x) \mid x \in X \rangle\}$;
(iv) $A \cap B = \langle \min(\mu_A(x), \mu_B(x)), \max(\rho_A(x), \rho_B(x)), \max(\nu_A(x), \nu_B(x)) \rangle$;
(v) $A \cup B = \langle \max(\mu_A(x), \mu_B(x)), \min(\rho_A(x), \rho_B(x)), \min(\nu_A(x), \nu_B(x)) \rangle$.

2.2. Existing Distance Measures

Definition 4. *A real function $d : \Phi(X) \times \Phi(X) \to [0,1]$ is called a distance measure [19], where d satisfies the following axioms for $A, B, C \in \Phi(X)$:*

(P1) $0 \leq d(A,B) \leq 1$;
(P2) $d(A,B) = 0$ *iff* $A = B$;
(P3) $d(A,B) = d(B,A)$;
(P4) *If* $A \subseteq B \subseteq C$, *then* $d(A,C) \geq d(A,B)$ *and* $d(A,C) \geq d(B,C)$.

On the basis of this, several researchers have addressed the various types of distance and similarity measures between two SVNSs $A = \langle x_i, \mu_A(x_i), \rho_A(x_i), \nu_A(x_i) | x_i \in X \rangle$ and $B = \langle x_i, \mu_B(x_i), \rho_B(x_i), \nu_B(x_i) | x_i \in X \rangle$, $i = 1, 2, ..., n$, which are given as follows:

(i) The extended Hausdorff distance [19]:

$$D_H(A,B) = \frac{1}{n}\sum_{i=1}^{n} \max\Big\{|\mu_A(x_i) - \mu_B(x_i)|, |\rho_A(x_i) - \rho_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|\Big\} \tag{1}$$

(ii) The normalized Hamming distance [20]:

$$D_{NH}(A,B) = \frac{1}{3n}\sum_{i=1}^{n} \Big\{|\mu_A(x_i) - \mu_B(x_i)| + |\rho_A(x_i) - \rho_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|\Big\} \tag{2}$$

(iii) The normalized Euclidean distance [20]:

$$D_{NE}(A,B) = \left(\frac{1}{3n}\sum_{i=1}^{n} \Big\{(\mu_A(x_i) - \mu_B(x_i))^2 + (\rho_A(x_i) - \rho_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2\Big\}\right)^{1/2} \tag{3}$$

(iv) The cosine similarities [22]:

$$S_{CS1}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi\big(|\mu_A(x_i) - \mu_B(x_i)| \vee |\rho_A(x_i) - \rho_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)|\big)}{2}\right] \tag{4}$$

and

$$S_{CS2}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi\big(|\mu_A(x_i) - \mu_B(x_i)| + |\rho_A(x_i) - \rho_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|\big)}{6}\right] \tag{5}$$

and their corresponding distances denoted by $D_{CS1} = 1 - S_{CS1}$ and $D_{CS2} = 1 - S_{CS2}$.

(v) The tangent similarities [42]:

$$S_{T1}(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n}\tan\left[\frac{\pi\left(|\mu_A(x_i) - \mu_B(x_i)| \vee |\rho_A(x_i) - \rho_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)|\right)}{4}\right] \quad (6)$$

and

$$S_{T2}(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n}\tan\left[\frac{\pi\left(|\mu_A(x_i) - \mu_B(x_i)| + |\rho_A(x_i) - \rho_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|\right)}{12}\right] \quad (7)$$

and their corresponding distances denoted by $D_{T1} = 1 - S_{T1}$ and $D_{T2} = 1 - S_{T2}$.

2.3. Shortcomings of the Existing Measures

The above measures have been widely used; however, simultaneously they have some drawbacks, which are illustrated with the numerical example that follows.

Example 1. *Consider two known patterns A and B, which are represented by SVNSs in a universe X given by* $A = \langle x, 0.5, 0.0, 0.0 \mid x \in X\rangle$, $B = \langle x, 0.0, 0.5, 0.0 \mid x \in X\rangle$. *Consider an unknown pattern* $C \in SVNSs(X)$, *which is recognized where* $C = \langle x, 0.0, 0.0, 0.5 \mid x \in X\rangle$; *then the target of this problem is to classify the pattern C in one of the classes A or B. If we apply the existing measures [19,20,22,42] defined in Equations (1)–(7) above, then we obtain the following:*

Pair	D_H	D_{NH}	D_{NE}	D_{CS1}	D_{CS2}	D_{T1}	D_{T1}
(A,C)	0.5	0.3333	0.4048	0.2929	0.1340	0.4142	0.2679
(B,C)	0.5	0.3333	0.4048	0.2929	0.1340	0.4142	0.2679

Thus, from this, we conclude that these existing measures are unable to classify the pattern C with A and B. Hence these measures are inconsistent and unable to perform ranking.

Example 2. *Consider two SVNSs defined on the universal set X given by* $A = \langle x, 0.3, 0.2, 0.3 \mid x \in X\rangle$ *and* $B = \langle x, 0.4, 0.2, 0.4 \mid x \in X\rangle$. *If we replace the degree of falsity membership of A (0.3) with 0.4, and that of B (0.4) with 0.3, then we obtain new SVNSs as* $C = \langle x, 0.3, 0.2, 0.4 \mid x \in X\rangle$ *and* $D = \langle x, 0.4, 0.2, 0.3 \mid x \in X\rangle$. *Now, by using the distance measures defined in Equations (1)–(7), we obtain their corresponding values as follows:*

Pair	D_H	D_{NH}	D_{NE}	D_{CS1}	D_{CS2}	D_{T1}	D_{T1}
(A,B)	0.1	0.066	0.077	0.013	0.006	0.078	0.052
(C,D)	0.1	0.066	0.077	0.013	0.006	0.078	0.052

Thus, it has been concluded that by changing the falsity degree of SVNSs and keeping the other degrees unchanged, the values of their corresponding measures remain the same. Thus, there is no effect of the degree of falsity membership on the distance measures. Similarly, we can observe the same for the degree of the truth membership functions.

This seems to be worthless to calculate distance using the measures mentioned above. Thus, there is a need to build up a new distance measure that overcomes the shortcomings of the existing measures.

3. Some New Distance Measures between SVNSs

In this section, we present the Hamming and the Euclidean distances between SVNSs, which can be used in real scientific and engineering applications.

Letting $\Phi(X)$ be the class of SVNSs over the universal set X, then we define the distances for SVNSs, $A = \langle \mu_A(x_i), \rho_A(x_i), \nu_A(x_i) \mid x_i \in X\rangle$ and $B = \langle \mu_B(x_i), \rho_B(x_i), \nu_B(x_i) \mid x_i \in X\rangle$, by considering the uncertainty parameter t, as follows:

(i) Hamming distance:

$$d_1(A,B)=\frac{1}{3(2+t)}\sum_{i=1}^{n}\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| \\ +|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ +|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix} \quad (8)$$

(ii) Normalized Hamming distance:

$$d_2(A,B)=\frac{1}{3n(2+t)}\sum_{i=1}^{n}\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| \\ +|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ +|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix} \quad (9)$$

(iii) Euclidean distance:

$$d_3(A,B)=\left(\frac{1}{3(2+t)^2}\sum_{i=1}^{n}\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2 \\ +|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \\ +|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix}\right)^{1/2} \quad (10)$$

(iv) Normalized Euclidean distance:

$$d_4(A,B)=\left(\frac{1}{3n(2+t)^2}\sum_{i=1}^{n}\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2 \\ +|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \\ +|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix}\right)^{1/2} \quad (11)$$

where $t \geq 3$ is a parameter.

Then, on the basis of the distance properties as defined in Definition 4, we can obtain the following properties:

Proposition 1. *The above-defined distance $d_2(A,B)$, between two SVNSs A and B, satisfies the following properties (P1)–(P4):*

(P1) $0 \leq d_2(A,B) \leq 1, \forall A,B \in \Phi(X)$;
(P2) $d_2(A,B) = 0$ *iff* $A = B$;
(P3) $d_2(A,B) = d_2(B,A)$;
(P4) *If* $A \subseteq B \subseteq C$, *then* $d_2(A,C) \geq d_2(A,B)$ *and* $d_2(A,C) \geq d_2(B,C)$.

Proof. For two SVNSs A and B, we have

(P1) $0 \leq \mu_A(x_i), \mu_B(x_i) \leq 1$, $0 \leq \rho_A(x_i), \rho_B(x_i) \leq 1$ and $0 \leq \nu_A(x_i), \nu_B(x_i) \leq 1$. Thus, $|\ \mu_A(x_i) - \mu_B(x_i)\ | \leq 1$, $|\ \rho_A(x_i) - \rho_B(x_i)\ | \leq 1$, $|\ \nu_A(x_i) - \nu_B(x_i)\ | \leq 1$ and $|\ t(\mu_A(x_i) - \mu_B(x_i))\ | \leq t$. Therefore,

$$\begin{aligned} &|\ (t\mu_A(x_i)-\nu_A(x_i)-\rho_A(x_i))-(t\mu_B(x_i)-\nu_B(x_i)-\rho_B(x_i))\ | \leq (2+t) \\ &|\ (t\rho_A(x_i)+\nu_A(x_i)-\mu_A(x_i))-(t\rho_B(x_i)+\nu_B(x_i)-\mu_B(x_i))\ | \leq (2+t) \\ &|\ (t\nu_A(x_i)+\rho_A(x_i)-\mu_A(x_i))-(t\nu_B(x_i)+\rho_B(x_i)-\mu_B(x_i))\ | \leq (2+t) \end{aligned}$$

Hence, by the definition of d_2, we obtain $0 \leq d_2(A,B) \leq 1$.

(P2) Firstly, we assume that $A = B$, which implies that $\mu_A(x_i) = \mu_B(x_i)$, $\rho_A(x_i) = \rho_B(x_i)$, and $\nu_A(x_i) = \nu_B(x_i)$ for $i = 1,2,..,n$. Thus, by the definition of d_2, we obtain $d_2(A,B) = 0$. Conversely, assuming that $d_2(A,B) = 0$ for two SVNSs A and B, this implies that

$$\left.\begin{array}{rl} & |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| \\ + & |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ + & |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{array}\right\} = 0$$

or

$$\begin{aligned} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| &= 0 \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| &= 0 \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| &= 0 \end{aligned}$$

After solving, we obtain $\mu_A(x_i)-\mu_B(x_i)=0$, $\rho_A(x_i)-\rho_B(x_i)=0$ and $\nu_A(x_i)-\nu_B(x_i)=0$, which implies $\mu_A(x_i)=\mu_B(x_i)$, $\rho_A(x_i)=\rho_B(x_i)$ and $\nu_A(x_i)=\nu_B(x_i)$. Therefore, $A=B$. Hence $d_2(A,B)=0$ iff $A=B$.

(P3) This is straightforward from the definition of d_2.

(P4) If $A\subseteq B\subseteq C$, then $\mu_A(x_i)\le\mu_B(x_i)\le\mu_C(x_i)$, $\rho_A(x_i)\ge\rho_B(x_i)\ge\rho_C(x_i)$ and $\nu_A(x_i)\ge\nu_B(x_i)\ge\nu_C(x_i)$, which implies that $\mu_A(x_i)-\mu_B(x_i)\ge\mu_A(x_i)-\mu_C(x_i)$, $\nu_A(x_i)-\nu_B(x_i)\le\nu_A(x_i)-\nu_C(x_i)$, and $\rho_A(x_i)-\rho_B(x_i)\le\rho_A(x_i)-\rho_C(x_i)$.

Therefore,

$$\begin{aligned} &|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| \\ &\qquad\le|-t(\mu_A(x_i)-\mu_c(x_i))+(\rho_A(x_i)-\rho_C(x_i))+(\nu_A(x_i)-\nu_C(x_i))| \\ &|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ &\qquad\le|-t(\rho_A(x_i)-\rho_C(x_i))-(\nu_A(x_i)-\nu_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))| \\ &|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ &\qquad\le|-t(\nu_A(x_i)-\nu_C(x_i))-(\rho_A(x_i)-\rho_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))| \end{aligned}$$

By adding, we obtain $d_2(A,B)\le d_2(A,C)$. Similarly, we obtain $d_2(B,C)\le d_2(A,C)$.

□

Proposition 2. *Distance d_4 as defined in Equation (11) is also a valid measure.*

Proof. For two SVNSs A and B, we have

(P1) $0\le\mu_A(x_i),\mu_B(x_i)\le 1$, $0\le\rho_A(x_i),\rho_B(x_i)\le 1$ and $0\le\nu_A(x_i),\nu_B(x_i)\le 1$. Thus, $|\mu_A(x_i)-\mu_B(x_i)|\le 1$, $|\rho_A(x_i)-\rho_B(x_i)|\le 1$, $|\nu_A(x_i)-\nu_B(x_i)|\le 1$ and $|t(\mu_A(x_i)-\mu_B(x_i))|\le t$. Therefore,

$$\begin{aligned} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2 &\le(2+t)^2 \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 &\le(2+t)^2 \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 &\le(2+t)^2 \end{aligned}$$

Hence, by the definition of d_4, we obtain $0\le d_4(A,B)\le 1$.

(P2) Assuming that $A=B$ implies that $\mu_A(x_i)=\mu_B(x_i)$, $\rho_A(x_i)=\rho_B(x_i)$ and $\nu_A(x_i)=\nu_B(x_i)$ for $i=1,2,\ldots,n$, and hence using Equation (11), we obtain $d_4(A,B)=0$. Conversely, assuming that $d_4(A,B)=0$ implies

$$\begin{aligned} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2 &= 0 \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 &= 0 \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 &= 0 \end{aligned}$$

After solving these, we obtain $\mu_A(x_i) - \mu_B(x_i) = 0$, $\rho_A(x_i) - \rho_B(x_i) = 0$ and $\nu_A(x_i) - \nu_B(x_i) = 0$; that is, $\mu_A(x_i) = \mu_B(x_i)$, $\rho_A(x_i) = \rho_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$ for $t \geq 3$. Hence $A = B$. Therefore, $d_4(A, B) = 0$ iff $A = B$.

(P3) This is straightforward from the definition of d_4.

(P4) If $A \subseteq B \subseteq C$, then $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$, $\rho_A(x_i) \geq \rho_B(x_i) \geq \rho_C(x_i)$, and $\nu_A(x_i) \geq \nu_B(x_i) \geq \nu_C(x_i)$. Therefore

$$\begin{aligned}
&|-t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))|^2 \\
&\qquad \leq |-t(\mu_A(x_i) - \mu_c(x_i)) + (\rho_A(x_i) - \rho_C(x_i)) + (\nu_A(x_i) - \nu_C(x_i))|^2 \\
&|-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^2 \\
&\qquad \leq |-t(\rho_A(x_i) - \rho_C(x_i)) - (\nu_A(x_i) - \nu_C(x_i)) + (\mu_A(x_i) - \mu_C(x_i))|^2 \\
&|-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^2 \\
&\qquad \leq |-t(\nu_A(x_i) - \nu_C(x_i)) - (\rho_A(x_i) - \rho_C(x_i)) + (\mu_A(x_i) - \mu_C(x_i))|^2
\end{aligned}$$

Hence by the definition of d_4, we obtain $d_4(A, B) \leq d_4(A, C)$. Similarly, we obtain $d_4(B, C) \leq d_4(A, C)$.

□

Now, on the basis of these proposed distance measures, we conclude that this successfully overcomes the shortcomings of the existing measures as described above.

Example 3. *If we apply the proposed distance measures d_2 and d_4 on the data considered in Example 1 to classify the pattern C, then corresponding to the parameter $t = 3$, we obtain $d_2(A, C) = 0.3333$, $d_2(B, C) = 0.1333$, $d_4(A, C) = 0.3464$ and $d_4(B, C) = 0.1633$. Thus, the pattern C is classified with the pattern B and hence is able to identify the best pattern.*

Example 4. *If we utilize the proposed distances d_2 and d_4 for the above-considered Example 2, then their corresponding values are $d_2(A, B) = 0.0267$, $d_2(C, D) = 0.0667$, $d_4(A, B) = 0.0327$ and $d_4(C, D) = 0.6930$. Therefore, there is a significant effect of the change in the falsity membership on the measure values and hence consequently on the ranking values.*

Proposition 3. *Measures d_1 and d_3 satisfy the following properties:*

(i) $0 \leq d_1 \leq n$;

(ii) $0 \leq d_3 \leq n^{1/2}$.

Proof. We can easily obtain that $d_1(A, B) = nd_2(A, B)$, and thus by Proposition 1, we obtain $0 \leq d_1(A, B) \leq n$. Similarly, we can obtain $0 \leq d_3(A, B) \leq n^{1/2}$. □

However, in many practical situations, the different sets may have taken different weights, and thus weight $\omega_i (i = 1, 2, \ldots, n)$ of the element $x_i \in X$ should be taken into account. In the following, we develop a weighted Hamming distance and the normalized weighted Euclidean distance between SVNSs.

(i) The normalized weighted Hamming distance:

$$\begin{aligned}
&d_5(A, B) \\
= \quad &\frac{1}{3n(2+t)} \sum_{i=1}^{n} \omega_i \begin{pmatrix} |-t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))| \\ +|-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| \\ +|-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| \end{pmatrix} \qquad (12)
\end{aligned}$$

(ii) The normalized weighted Euclidean distance:

$$d_6(A,B) = \left\{ \frac{1}{3n(2+t)^2} \sum_{i=1}^{n} \omega_i \begin{pmatrix} \left| -t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i)) \right|^2 \\ + \left| -t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i)) \right|^2 \\ + \left| -t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i)) \right|^2 \end{pmatrix} \right\}^{1/2} \tag{13}$$

where $t \geq 3$ is a parameter.

It is straightforward to check that the normalized weighted distance $d_k(A,B)(k = 5,6)$ between SVNSs A and B also satisfies the above properties (P1)–(P4).

Proposition 4. *Distance measures d_2 and d_5 satisfy the relation $d_5 \leq d_2$.*

Proof. Because $\omega_i \geq 0, \sum_{i=1}^{n} \omega_i = 1$, then for any two SVNSs A and B, we have $d_5(A,B) = \frac{1}{3n(2+t)} \sum_{i=1}^{n} \omega_i \Big\{ \Big(| - t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))| + | - t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| + | - t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| \Big) \Big\} \leq \frac{1}{3n(2+t)} \sum_{i=1}^{n} \Big(| - t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))| + | - t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| + | - t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| \Big)$; that is, $d_5(A,B) \leq d_2(A,B)$. □

Proposition 5. *Let A and B be two SVNSs in X; then d_5 and d_6 are the distance measures.*

Proof. Because $\omega_i \in [0,1]$ and $\sum\limits_{i=1}^{n} \omega_i = 1$ then we can easily obtain $0 \leq d_5(A,B) \leq d_2(A,B)$. Thus, $d_5(A,B)$ satisfies (P1). The proofs of (P2)–(P4) are similar to those of Proposition 1. Similar is true for d_6. □

Proposition 6. *The distance measures d_4 and d_6 satisfy the relation $d_6 \leq d_4$.*

Proof. The proof follows from Proposition 4. □

Proposition 7. *The distance measures d_2 and d_4 satisfy the inequality $d_4 \leq \sqrt{d_2}$.*

Proof. For two SVNSs A and B, we have

$$\begin{aligned} &| - t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))|^2 \leq (2+t)^2 \\ &| - t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^2 \leq (2+t)^2 \\ &| - t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^2 \leq (2+t)^2 \end{aligned}$$

which implies that

$$\begin{aligned} &\left| \frac{-t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))}{2+t} \right|^2 \leq 1 \\ &\left| \frac{-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))}{2+t} \right|^2 \leq 1 \\ &\left| \frac{-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))}{2+t} \right|^2 \leq 1 \end{aligned}$$

For any $a \in [0,1]$, we have $a^2 \leq a$. Therefore,

$$\left|\frac{-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))}{2+t}\right|^2 \leq \left|\frac{-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))}{2+t}\right|$$

$$\left|\frac{-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|^2 \leq \left|\frac{-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|$$

and

$$\left|\frac{-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|^2 \leq \left|\frac{-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|$$

By adding these inequalities and by the definition of d_4, we have

$$\begin{aligned} d_4(A,B) &= \\ &\left[\frac{1}{3n(2+t)^2}\sum_{i=1}^{n}\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2 \\ +|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \\ +|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix}\right]^{1/2} \\ &\leq \left[\frac{1}{3n(2+t)}\sum_{i=1}^{n}\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| \\ +|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ +|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix}\right]^{1/2} \\ &\leq (d_2(A,B))^{1/2} \end{aligned}$$

As A and B are arbitrary SVNSs, thus we obtain $d_4 \leq \sqrt{d_2}$. □

Proposition 8. *Measures d_6 and d_5 satisfy the inequality $d_6 \leq \sqrt{d_5}$.*

Proof. The proof follows from Proposition 7. □

The Hausdroff distance between two non-empty closed and bounded sets is a measure of the resemblance between them. For example, we consider $A = [x_1, x_2]$ and $B = [y_1, y_2]$ in the Euclidean domain R; the Hausdroff distance in the additive set environment is given by the following [8]:

$$H(A,B) = \max\{\, |\, x_1 - y_1 \,|, |\, x_2 - y_2 \,|\, \}$$

Now, for any two SVNSs A and B over $X = \{x_1, x_2, \ldots, x_n\}$, we propose the following utmost distance measures:

- Utmost normalized Hamming distance:

$$d_1^H(A,B) = \frac{1}{3n(2+t)}\sum_{i=1}^{n}\max_i\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|, \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|, \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix} \quad (14)$$

- Utmost normalized weighted Hamming distance:

$$d_2^H(A,B) = \frac{1}{3n(2+t)}\sum_{i=1}^{n}\omega_i \max_i \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|, \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|, \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix} \quad (15)$$

- Utmost normalized Euclidean distance:

$$d_3^H(A,B) = \left\{ \frac{1}{3n(2+t)^2}\sum_{i=1}^{n} \max_i \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2, \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2, \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix} \right\}^{1/2} \quad (16)$$

- Utmost normalized weighted Euclidean distance:

$$d_4^H(A,B) = \left\{ \frac{1}{3n(2+t)^2}\sum_{i=1}^{n}\omega_i \max_i \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2, \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2, \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix} \right\}^{1/2} \quad (17)$$

Proposition 9. *The distance $d_1^H(A,B)$ defined in Equation (14) for two SVNSs A and B is a valid distance measure.*

Proof. The above measure satisfies the following properties:

(P1) As A and B are SVNSs, so $|\ \mu_A(x_i)-\mu_B(x_i)\ |\leq 1$, $|\ \rho_A(x_i)-\rho_B(x_i)\ |\leq 1$ and $|\ \nu_A(x_i)-\nu_B(x_i)\ |\leq 1$. Thus,

$$\begin{aligned} &|\ (t\mu_A(x_i)-\nu_A(x_i)-\rho_A(x_i))-(t\mu_B(x_i)-\nu_B(x_i)-\rho_B(x_i))\ |\leq (2+t) \\ &|\ (t\rho_A(x_i)+\nu_A(x_i)-\mu_A(x_i))-(t\rho_B(x_i)+\nu_B(x_i)-\mu_B(x_i))\ |\leq (2+t) \\ &|\ (t\nu_A(x_i)+\rho_A(x_i)-\mu_A(x_i))-(t\nu_B(x_i)+\rho_B(x_i)-\mu_B(x_i))\ |\leq (2+t) \end{aligned}$$

Hence, by the definition of d_1^H, we obtain $0\leq d_1^H(A,B)\leq 1$.

(P2) Similar to the proof of Proposition 1.

(P3) This is clear from Equation (14).

(P4) Let $A\subseteq B\subseteq C$, which implies $\mu_A(x_i)\leq\mu_B(x_i)\leq\mu_C(x_i)$, $\rho_A(x_i)\geq\rho_B(x_i)\geq\rho_C(x_i)$ and $\nu_A(x_i)\geq\nu_B(x_i)\geq\nu_C(x_i)$. Therefore, $|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|\ \leq\ |-t(\mu_A(x_i)-\mu_c(x_i))+(\rho_A(x_i)-\rho_C(x_i))+(\nu_A(x_i)-\nu_C(x_i))|$, $|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|\leq|-t(\rho_A(x_i)-\rho_C(x_i))-(\nu_A(x_i)-\nu_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|$ and $|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|\ \leq\ |-t(\nu_A(x_i)-\nu_C(x_i))-(\rho_A(x_i)-\rho_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|$, which implies that $\max_i\big(|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|, |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|, |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|\big)$ $\leq\max_i\big(|-t(\mu_A(x_i)-\mu_c(x_i))+(\rho_A(x_i)-\rho_C(x_i))+(\nu_A(x_i)-\nu_C(x_i))|, |-t(\rho_A(x_i)-\rho_C(x_i))-(\nu_A(x_i)-\nu_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|$ and $|-t(\nu_A(x_i)-\nu_C(x_i))-(\rho_A(x_i)-\rho_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|\big)$. Hence $d_1^H(A,B)\leq d_1^H(A,C)$. Similarly, we obtain $d_1^H(B,C)\leq d_1^H(A,C)$.

□

Proposition 10. *For $A,B\in\Phi(X)$, d_2^H, d_3^H and d_4^H are the distance measures.*

Proof. The proof follows from the above proposition. □

Proposition 11. *The measures d_2^H and d_1^H satisfy the following inequality: $d_2^H \leq d_1^H$.*

Proof. Because $w_i \in [0,1]$, therefore

$$\begin{aligned} d_2^H(A,B) &= \frac{1}{3n(2+t)}\sum_{i=1}^{n} w_i\Bigg(\max_i\Big(|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|,\\ &\quad |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|,|-t(\nu_A(x_i)\\ &\quad -\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|\Big)\Bigg)\\ &\leq \frac{1}{3n(2+t)}\sum_{i=1}^{n}\max_i\Big(|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|,\\ &\quad |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|,|-t(\nu_A(x_i)\\ &\quad -\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|\Big)\\ &= d_1^H(A,B) \end{aligned}$$

Hence, $d_2^H \leq d_1^H$. □

Proposition 12. *The measures d_3^H and d_4^H satisfy the inequality $d_4^H(A,B) \leq d_3^H(A,B)$.*

Proof. The proof follows from Proposition 11. □

Proposition 13. *The measures d_3^H and d_1^H satisfy the inequality $d_3^H \leq \sqrt{d_1^H}$.*

Proof. Because for any $a \in [0,1]$, $a^2 \leq a \leq a^{1/2}$, the remaining proof follows from Proposition 7. □

Proposition 14. *The measures d_4^H and d_2^H satisfy the inequality $d_4^H \leq \sqrt{d_2^H}$.*

Proof. The proof follows from Proposition 13. □

Proposition 15. *The measures d_1^H and d_2 satisfy the following inequality:*

$$d_1^H \leq d_2$$

Proof. For positive numbers a_i, $i = 1,2,...,n$, we have $\max_i\{a_i\} \leq \sum_{i=1}^{n} a_i$. Thus, for any two SVNSs A and B, we have $d_1^H(A,B) = \frac{1}{3n(2+t)}\sum_{i=1}^{n}\max_i\Big(|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|,|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|,|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|\Big) \leq \frac{1}{3n(2+t)}\sum_{i=1}^{n}|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|+|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|+|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| = d_2(A,B)$. Hence $d_1^H \leq d_2$. □

Proposition 16. *The measures d_3^H and d_4 satisfy the following inequality:*

$$d_3^H \leq d_4$$

Proof. The proof follows from Proposition 15. □

Proposition 17. *The measures d_2, d_5 and d_1^H satisfy the following inequalities:*

(i) $d_2 \geq \frac{d_5+d_1^H}{2}$;
(ii) $d_2 \geq \sqrt{d_5 \cdot d_1^H}$.

Proof. Because $d_2 \geq d_5$ and $d_2 \geq d_1^H$, by adding these inequalities, we obtain $d_2 \geq \frac{d_5 + d_1^H}{2}$. On the other hand, by multiplying these, we obtain $d_2 \geq \sqrt{d_5 \cdot d_1^H}$. □

4. Generalized Distance Measure

The above-defined Hamming and Euclidean distance measures are generalized for the two SVNSs A and B on the universal set X as follows:

$$\begin{aligned} d^p(A,B) \quad = \quad & \Bigg\{ \frac{1}{3n(2+t)^p} \sum_{i=1}^{n} \Big(|-t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))|^p \\ & + |-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^p \\ & + |-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^p \Big) \Bigg\}^{1/p} \end{aligned} \tag{18}$$

where $p \geq 1$ is an L_p norm and $t \geq 3$ represents the uncertainty index parameters.

In particular, if $p = 1$ and $p = 2$, then the above measure, given in Equation (18), reduces to measures d_2 and d_4 defined in Equations (9) and (11), respectively.

Proposition 18. *The above-defined distance $d^p(A,B)$, between SVNSs A and B, satisfies the following properties (P1)–(P4):*

(P1) $0 \leq d^p(A,B) \leq 1, \forall A, B \in \Phi(X)$;
(P2) $d^p(A,B) = 0$, *iff* $A = B$;
(P3) $d^p(A,B) = d^p(B,A)$;
(P4) *If* $A \subseteq B \subseteq C$, *then* $d^p(A,C) \geq d^p(A,B)$ *and* $d^p(A,C) \geq d^p(B,C)$.

Proof. For $p \geq 1$ and $t \geq 3$, we have the following:

(P1) For SVNSs, $\mid \mu_A(x_i) - \mu_B(x_i) \mid \leq 1$, $\mid \rho_A(x_i) - \rho_B(x_i) \mid \leq 1$ and $\mid \nu_A(x_i) - \nu_B(x_i) \mid \leq 1$. Thus, we obtain

$$\begin{aligned} -(2+t) \leq t(\mu_A(x_i) - \mu_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) \leq (2+t) \\ -(2+t) \leq -t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i)) \leq (2+t) \\ -(2+t) \leq -t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i)) - \mu_B(x_i) \leq (2+t) \end{aligned}$$

which implies that

$$\begin{aligned} 0 \leq \left|t(\mu_A(x_i) - \mu_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i))\right|^p \leq (2+t)^p \\ 0 \leq \left|-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))\right|^p \leq (2+t)^p \\ 0 \leq \left|-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i)) - \mu_B(x_i)\right|^p \leq (2+t)^p \end{aligned}$$

Thus, by adding these inequalities, we obtain $0 \leq d^p(A,B) \leq 1$.

(P2) Assuming that $A = B \Leftrightarrow \mu_A(x) = \mu_B(x_i)$, $\rho_A(x_i) = \rho_B(x_i)$, and $\nu_A(x) = \nu_B(x_i)$, thus, $d^p(A,B) = 0$.

Conversely, assuming that $d^p(A,B) = 0$ implies that

$$
\begin{aligned}
|-t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))| &= 0 \\
|-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| &= 0 \\
|-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| &= 0
\end{aligned}
$$

and hence, after solving, we obtain $\mu_A(x_i) = \mu_B(x_i)$, $\rho_A(x_i) = \rho_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$. Thus, $A = B$.

(P3) This is straightforward.

(P4) Let $A \subseteq B \subseteq C$; then $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$, $\rho_A(x_i) \geq \rho_B(x_i) \geq \rho_C(x_i)$ and $\nu_A(x_i) \geq \nu_B(x_i) \geq \nu_C(x_i)$. Thus, $\mu_A(x_i) - \mu_B(x_i) \geq \mu_A(x_i) - \mu_C(x_i)$, $\rho_A(x_i) - \rho_B(x_i) \leq \rho_A(x_i) - \rho_C(x_i)$ and $\nu_A(x_i) - \nu_B(x_i) \leq \nu_A(x_i) - \nu_C(x_i)$. Hence, we obtain

$$
\begin{aligned}
&|-t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))|^p \\
&\qquad \leq |-t(\mu_A(x_i) - \mu_c(x_i)) + (\rho_A(x_i) - \rho_C(x_i)) + (\nu_A(x_i) - \nu_C(x_i))|^p \\
&|-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^p \\
&\qquad \leq |-t(\rho_A(x_i) - \rho_C(x_i)) - (\nu_A(x_i) - \nu_C(x_i)) + (\mu_A(x_i) - \mu_C(x_i))|^p \\
\text{and}\quad &|-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^p \\
&\qquad \leq |-t(\nu_A(x_i) - \nu_C(x_i)) - (\rho_A(x_i) - \rho_C(x_i)) + (\mu_A(x_i) - \mu_C(x_i))|^p
\end{aligned}
$$

Thus, we obtain $d^p(A,B) \leq d^p(A,C)$. Similarly, $d^p(B,C) \leq d^p(A,C)$.

□

If the weight vector $\omega_i, (i = 1, 2, \ldots, n)$ of each element is considered such that $\omega_i \in [0,1]$ and $\sum_i \omega_i = 1$, then a generalized parametric distance measure between SVNSs A and B takes the following form:

$$
\begin{aligned}
d_w^p(A,B) = &\left(\frac{1}{3n(2+t)^p} \sum_{i=1}^{n} \omega_i \Big\{ \Big(|-t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))|^p \right. \\
&+ |-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^p \\
&\left. + |-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^p \Big) \Big\} \right)^{1/p} \qquad (19)
\end{aligned}
$$

In particular, if $p = 1$ and $p = 2$, Equation (19) reduces to Equations (12) and (13), respectively.

Proposition 19. *Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $x_i, (i = 1, 2, \ldots, n)$ with $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$; then the generalized parametric distance measure between the SVNSs A and B defined by Equation (19) satisfies the following:*

(P1) $0 \leq d_w^p(A,B) \leq 1, \forall A, B \in \Phi(X)$;

(P2) $d_w^p(A,B) = 0$ *iff* $A = B$;

(P3) $d_w^p(A,B) = d_w^p(B,A)$;

(P4) $A \subseteq B \subseteq C$ *then* $d_w^p(A,C) \geq d_w^p(A,B)$ *and* $d_w^p(A,C) \geq d_w^p(B,C)$.

Proof. The proof follows from Proposition 18. □

5. Illustrative Examples

In order to illustrate the performance and validity of the above-proposed distance measures, two examples from the fields of pattern recognition and medical diagnosis have been taken into account.

5.1. Example 1: Application of Distance Measure in Pattern Recognition

Consider three known patterns A_1, A_2 and A_3, which are represented by the following SVNSs in a given universe $X = \{x_1, x_2, x_3, x_4\}$:

$$A_1 = \{\langle x_1, 0.7, 0.0, 0.1\rangle, \langle x_2, 0.6, 0.1, 0.2\rangle, \langle x_3, 0.8, 0.7, 0.6\rangle, \langle x_4, 0.5, 0.2, 0.3\rangle\}$$
$$A_2 = \{\langle x_1, 0.4, 0.2, 0.3\rangle, \langle x_2, 0.7, 0.1, 0.0\rangle, \langle x_3, 0.1, 0.1, 0.6\rangle, \langle x_4, 0.5, 0.3, 0.6\rangle\}$$
$$A_3 = \{\langle x_1, 0.5, 0.2, 0.2\rangle, \langle x_2, 0.4, 0.1, 0.2\rangle, \langle x_3, 0.1, 0.1, 0.4\rangle, \langle x_4, 0.4, 0.1, 0.2\rangle\}$$

Consider an unknown pattern $B \in SVNS(X)$, which will be recognized where

$$B = \{\langle x_1, 0.4, 0.1, 0.4\rangle, \langle x_2, 0.6, 0.1, 0.1\rangle, \langle x_3, 0.1, 0.0, 0.4\rangle, \langle x_4, 0.4, 0.4, 0.7\rangle\}$$

Then the target of this problem is to classify the pattern B in one of the classes A_1, A_2 or A_3. For this, proposed distance measures, $d_1, d_2, d_3, d_4, d_1^H$ and d_3^H, have been computed from B to $A_k (k = 1, 2, 3)$ corresponding to $t = 3$, and the results are given as follows:

$$\begin{aligned}
d_1(A_1, B) &= 0.5600; \quad d_1(A_2, B) = 0.2932; \quad d_1(A_3, B) = 0.4668\\
d_2(A_1, B) &= 0.1400; \quad d_2(A_2, B) = 0.0733; \quad d_2(A_3, B) = 0.1167\\
d_3(A_1, B) &= 0.3499; \quad d_3(A_2, B) = 0.1641; \quad d_3(A_3, B) = 0.3120\\
d_4(A_1, B) &= 0.1749; \quad d_4(A_2, B) = 0.0821; \quad d_4(A_3, B) = 0.1560\\
d_1^H(A_1, B) &= 0.0633; \quad d_1^H(A_2, B) = 0.0300; \quad d_1^H(A_3, B) = 0.0567\\
d_3^H(A_1, B) &= 0.1252; \quad d_3^H(A_2, B) = 0.0560; \quad d_3^H(A_3, B) = 0.1180
\end{aligned}$$

Thus, from these distance measures, we conclude that the pattern B belongs to the pattern A_2. On the other hand, if we assume that the weights of x_1, x_2, x_3 and x_4 are 0.3, 0.4, 0.2 and 0.1, respectively, then we utilize the distance measures d_5, d_6, d_2^H and d_4^H for obtaining the most suitable pattern as follows:

$$\begin{aligned}
d_5(A_1, B) &= 0.0338; \quad d_5(A_2, B) = 0.0162; \quad d_5(A_3, B) = 0.0233\\
d_6(A_1, B) &= 0.0861; \quad d_6(A_2, B) = 0.0369; \quad d_6(A_3, B) = 0.0604\\
d_2^H(A_1, B) &= 0.0148; \quad d_2^H(A_2, B) = 0.0068; \quad d_2^H(A_3, B) = 0.0117\\
d_4^H(A_1, B) &= 0.0603; \quad d_4^H(A_2, B) = 0.0258; \quad d_4^H(A_3, B) = 0.0464
\end{aligned}$$

Thus, the ranking order of the three patterns is A_2, A_3 and A_1, and hence A_2 is the most desirable pattern to be classified with B. Furthermore, it can be easily verified that these results validate the above-proposed propositions on the distance measures.

Comparison of Example 1 Results with Existing Measures

The above-mentioned measures have been compared with some existing measures under a NS environment for showing the validity of the approach whose results are summarized in Table 1. From these results, it has been shown that the final ordering of the pattern coincides with the proposed measures, and hence it shows the conservative nature of the measures.

Table 1. Ordering value of Example 1.

Methods	Measure Value of B from A_1	A_2	A_3	Ranking Order
D_H (defined in Equation (1)) [19]	0.3250	0.1250	0.2500	$A_2 \succ A_3 \succ A_1$
Correlation coefficient [19]	0.7883	0.9675	0.8615	$A_2 \succ A_3 \succ A_1$
D_{NE} (defined in Equation (3)) [20]	0.5251	0.7674	0.6098	$A_1 \succ A_3 \succ A_2$
S_{CS1} (defined in Equation (4)) [22]	0.8209	0.9785	0.8992	$A_2 \succ A_3 \succ A_1$
S_{CS2} (defined in Equation (5)) [22]	0.8949	0.9911	0.9695	$A_2 \succ A_3 \succ A_1$
S_{T1} (defined in Equation (6)) [42]	0.7275	0.9014	0.7976	$A_2 \succ A_3 \succ A_1$
S_{T2} (defined in Equation (7)) [42]	0.9143	0.9673	0.9343	$A_2 \succ A_3 \succ A_1$

5.2. Example 2: Application of Distance Measure in Medical Diagnosis

Consider a set of diseases $Q = \{Q_1(\text{Viral fever}), Q_2(\text{Malaria}), Q_3(\text{Typhoid}), Q_4(\text{Stomach Problem}), Q_5\ (\text{Chest problem})\}$ and a set of symptoms $S = \{s_1\ (\text{Temperature}), s_2\ (\text{HeadAche}), s_3\ (\text{Stomach Pain}), s_4\ (\text{Cough}), s_5\ (\text{Chest pain})\}$. Suppose a patient, with respect to all the symptoms, can be represented by the following SVNS:

$$P(\text{Patient}) = \{(s_1, 0.8, 0.2, 0.1), (s_2, 0.6, 0.3, 0.1), (s_3, 0.2, 0.1, 0.8), (s_4, 0.6, 0.5, 0.1), (s_5, 0.1, 0.4, 0.6)\}$$

and each diseases $Q_k (k = 1, 2, 3, 4, 5)$ is as follows:

$$Q_1(\text{Viral fever}) = \{(s_1, 0.4, 0.6, 0.0), (s_2, 0.3, 0.2, 0.5), (s_3, 0.1, 0.3, 0.7), (s_4, 0.4, 0.3, 0.3), (s_5, 0.1, 0.2, 0.7)\}$$
$$Q_2(\text{Malaria}) = \{(s_1, 0.7, 0.3, 0.0), (s_2, 0.2, 0.2, 0.6), (s_3, 0.0, 0.1, 0.9), (s_4, 0.7, 0.3, 0.0), (s_5, 0.1, 0.1, 0.8)\}$$
$$Q_3(\text{Typhoid}) = \{(s_1, 0.3, 0.4, 0.3), (s_2, 0.6, 0.3, 0.1), (s_3, 0.2, 0.1, 0.7), (s_4, 0.2, 0.2, 0.6), (s_5, 0.1, 0.0, 0.9)\}$$
$$Q_4(\text{Stomach problem}) = \{(s_1, 0.1, 0.2, 0.7), (s_2, 0.2, 0.4, 0.4), (s_3, 0.8, 0.2, 0.0), (s_4, 0.2, 0.1, 0.7), (s_5, 0.2, 0.1, 0.7)\}$$
$$Q_5(\text{Chest problem}) = \{(s_1, 0.1, 0.1, 0.8), (s_2, 0.0, 0.2, 0.8), (s_3, 0.2, 0.0, 0.8), (s_4, 0.2, 0.0, 0.8), (s_5, 0.8, 0.1, 0.1)\}$$

Now, the target is to diagnose the disease of patient P among Q_1, Q_2, Q_3, Q_4 and Q_5. For this, proposed distance measures, $d_1, d_2, d_3, d_4, d_1^H$ and d_3^H, have been computed from P to $Q_k (k = 1, 2, \ldots, 5)$ and are given as follows:

$$d_1(Q_1, P) = 0.6400; \quad d_1(Q_2, P) = 0.9067; \quad d_1(Q_3, P) = 0.6333; \quad d_1(Q_4, P) = 1.4600; \quad d_1(Q_5, P) = 1.6200$$
$$d_2(Q_1, P) = 0.1280; \quad d_2(Q_2, P) = 0.1813; \quad d_2(Q_3, P) = 0.1267; \quad d_2(Q_4, P) = 0.2920; \quad d_2(Q_5, P) = 0.3240$$
$$d_3(Q_1, P) = 0.3626; \quad d_3(Q_2, P) = 0.4977; \quad d_3(Q_3, P) = 0.4113; \quad d_3(Q_4, P) = 0.7566; \quad d_3(Q_5, P) = 0.8533$$
$$d_4(Q_1, P) = 0.1622; \quad d_4(Q_2, P) = 0.2226; \quad d_4(Q_3, P) = 0.1840; \quad d_4(Q_4, P) = 0.3383; \quad d_4(Q_5, P) = 0.3816$$
$$d_1^H(Q_1, P) = 0.0613; \quad d_1^H(Q_2, P) = 0.0880; \quad d_1^H(Q_3, P) = 0.0627; \quad d_1^H(Q_4, P) = 0.1320; \quad d_1^H(Q_5, P) = 0.1400$$
$$d_3^H(Q_1, P) = 0.1175; \quad d_3^H(Q_2, P) = 0.1760; \quad d_3^H(Q_3, P) = 0.1373; \quad d_3^H(Q_4, P) = 0.2439; \quad d_3^H(Q_5, P) = 0.2661$$

Thus, from these distance measures, we conclude that the patient P suffers from the disease Q_3.

On the other hand, if we assign weights 0.3, 0.2, 0.2, 0.1 and 0.2 corresponding to $Q_k (k = 1, 2, \ldots, 5)$, respectively, then we utilize the distance measures d_5, d_6, d_2^H and d_4^H for obtaining the most suitable pattern as

$$d_5(Q_1, P) = 0.0284; \quad d_5(Q_2, P) = 0.0403; \quad d_5(Q_3, P) = 0.0273; \quad d_5(Q_4, P) = 0.0625; \quad d_5(Q_5, P) = 0.0684$$
$$d_6(Q_1, P) = 0.0795; \quad d_6(Q_2, P) = 0.1101; \quad d_6(Q_3, P) = 0.0862; \quad d_6(Q_4, P) = 0.1599; \quad d_6(Q_5, P) = 0.1781$$
$$d_2^H(Q_1, P) = 0.0135; \quad d_2^H(Q_2, P) = 0.0200; \quad d_2^H(Q_3, P) = 0.0129; \quad d_2^H(Q_4, P) = 0.0276; \quad d_2^H(Q_5, P) = 0.0289$$
$$d_4^H(Q_1, P) = 0.0572; \quad d_4^H(Q_2, P) = 0.0885; \quad d_4^H(Q_3, P) = 0.0636; \quad d_4^H(Q_4, P) = 0.1139; \quad d_4^H(Q_5, P) = 0.1226$$

Thus, on the basis of the ranking order, we conclude that the patient P suffers from the disease Q_3.

Comparison of Example 2 Results with Existing Approaches

In order to verify the feasibility of the proposed decision-making approach based on the distance measure, we conducted a comparison analysis based on the same illustrative example. For this, various measures as presented in Equations (1)–(7) were taken, and their corresponding results are summarized in Table 2, which shows that the patient P suffers from the disease Q_1.

Table 2. Comparison of diagnosis result using existing measures.

Approach	Ranking Order
D_H (defined in Equation (1)) [19]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
Correlation [19]	$Q_1 \succ Q_2 \succ Q_3 \succ Q_4 \succ Q_5$
Distance measure [27]	
$p = 1$	$Q_3 \succ Q_1 \succ Q_2 \succ Q_4 \succ Q_5$
$p = 2$	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
$p = 3$	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
$p = 5$	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
D_{NH} (defined in Equation (2)) [20]	$Q_3 \succ Q_1 \succ Q_2 \succ Q_4 \succ Q_5$
D_{NH} (defined in Equation (3)) [20]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
S_{CS1} (defined in Equation (4)) [22]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
S_{CS1} (defined in Equation (5)) [22]	$Q_1 \succ Q_2 \succ Q_3 \succ Q_4 \succ Q_5$
S_{T1} (defined in Equation (6)) [42]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
S_{T1} (defined in Equation (7)) [42]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$

5.3. Effect of the Parameters p and t on the Ordering

However, in order to analyze the effect of the parameters t and p on the measure values, an experiment was performed by taking different values of p ($p = 1, 1.5, 2, 3, 5, 10$) corresponding to a different value of the uncertainty parameter t ($t = 3, 5, 7$). On the basis of these different pairs of parameters, distance measures were computed, and their results are summarized in Tables 3 and 4, respectively, for Examples 1 and 2 corresponding to different criterion weights.

From these, the following have been computed:

(i) For a fixed value of p, it has been observed that the measure values corresponding to each alternative increase with the increase in the value of t. On the other hand, by varying the value of t from 3 to 7, corresponding to a fixed value of p, this implies that values of the distance measures of each diagnosis from the patient P increase.
(ii) It has also been observed from this table that when the weight vector has been assigned to each criterion weight, then the measure values are less than that of an equal weighting case.
(iii) Finally, it is seen from the table that the measured values corresponding to each alternative $Q_k (k = 1, 2, 3, 4, 5)$ are conservative in nature.

For each pair, the measure values lie between 0 and 1, and hence, on the basis of this, we conclude that the patient P suffers from the Q_1 disease. The ranking order for the decision-maker is shown in the table as (13245), which indicates that the order of the different attributes is of the form $Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$. Hence Q_1 is the most desirable, while Q_5 is the least desirable for different values of t and p.

Table 3. Results of classification of given sample using proposed distance measure.

		When Equal Importance Is given to Each Criteria				When Weight Vector $(0.3, 0.4, 0.2, 0.1)^T$ Is Taken			
p	t	$d^p(A_1, B)$	$d^p(A_2, B)$	$d^p(A_3, B)$	Ranking	$d_w^p(A_1, B)$	$d_w^p(A_2, B)$	$d_w^p(A_3, B)$	Ranking
1	3	0.1400	0.0733	0.1167	$A_2 \succ A_3 \succ A_1$	0.0338	0.0162	0.0233	$A_2 \succ A_3 \succ A_1$
	5	0.1667	0.0762	0.1214	$A_2 \succ A_3 \succ A_1$	0.0387	0.0170	0.0248	$A_2 \succ A_3 \succ A_1$
	7	0.1815	0.0778	0.1241	$A_2 \succ A_3 \succ A_1$	0.0414	0.0175	0.0256	$A_2 \succ A_3 \succ A_1$
1.5	3	0.1598	0.0783	0.1374	$A_2 \succ A_3 \succ A_1$	0.0620	0.0277	0.0426	$A_2 \succ A_3 \succ A_1$
	5	0.1924	0.0817	0.1437	$A_2 \succ A_3 \succ A_1$	0.0723	0.0293	0.0452	$A_2 \succ A_3 \succ A_1$
	7	0.2116	0.0838	0.1480	$A_2 \succ A_3 \succ A_1$	0.0784	0.0304	0.0469	$A_2 \succ A_3 \succ A_1$
2	3	0.1749	0.0821	0.1560	$A_2 \succ A_3 \succ A_1$	0.0861	0.0369	0.0604	$A_2 \succ A_3 \succ A_1$
	5	0.2137	0.0859	0.1646	$A_2 \succ A_3 \succ A_1$	0.1021	0.0392	0.0644	$A_2 \succ A_3 \succ A_1$
	7	0.2374	0.0885	0.1705	$A_2 \succ A_3 \succ A_1$	0.1120	0.0408	0.0671	$A_2 \succ A_3 \succ A_1$
3	3	0.1970	0.0880	0.1875	$A_2 \succ A_3 \succ A_1$	0.1229	0.0507	0.0927	$A_2 \succ A_3 \succ A_1$
	5	0.2469	0.0929	0.02012	$A_2 \succ A_3 \succ A_1$	0.1497	0.0543	0.1000	$A_2 \succ A_3 \succ A_1$
	7	0.2785	0.0962	0.2098	$A_2 \succ A_3 \succ A_1$	0.1672	0.0566	0.1046	$A_2 \succ A_3 \succ A_1$
5	3	0.2240	0.0967	0.2314	$A_2 \succ A_1 \succ A_3$	0.1680	0.0689	0.1469	$A_2 \succ A_3 \succ A_1$
	5	0.2902	0.1041	0.2526	$A_2 \succ A_3 \succ A_1$	0.2128	0.0749	0.1605	$A_2 \succ A_3 \succ A_1$
	7	0.3326	0.1087	0.2650	$A_2 \succ A_3 \succ A_1$	0.2426	0.0786	0.1685	$A_2 \succ A_3 \succ A_1$
10	3	0.2564	0.1107	0.2830	$A_2 \succ A_1 \succ A_3$	0.2203	0.0939	0.2248	$A_2 \succ A_1 \succ A_3$
	5	0.3421	0.1231	0.3131	$A_2 \succ A_3 \succ A_1$	0.2915	0.1047	0.2487	$A_2 \succ A_3 \succ A_1$
	7	0.3942	0.1304	0.3301	$A_2 \succ A_3 \succ A_1$	0.3356	0.1109	0.2622	$A_2 \succ A_3 \succ A_1$

Table 4. Diagnosis result on basis of proposed distance measure.

		When Equal Importance Is Given to Each Criteria					When Weight Vector $(0.3, 0.2, 0.2, 0.1, 0.2)^T$ is Taken				
p	t	$d^p(Q_1, P)$	$d^p(Q_2, P)$	$d^p(Q_3, P)$	$d^p(Q_4, P)$	$d^p(Q_5, P)$	$d_w^p(Q_1, P)$	$d_w^p(Q_2, P)$	$d_w^p(Q_3, P)$	$d_w^p(Q_4, P)$	$d_w^p(Q_5, P)$
1	3	0.1280	0.1813	0.1267	0.2920	0.3240	0.0284	0.0403	0.0273	0.0625	0.0684
	5	0.1410	0.1867	0.1457	0.3076	0.3400	0.0304	0.0413	0.0300	0.0643	0.0700
	7	0.1481	0.1896	0.1563	0.3178	0.3489	0.0315	0.0419	0.0315	0.0656	0.070
1.5	3	0.1465	0.2023	0.1600	0.3175	0.3574	0.0553	0.0768	0.0579	0.1154	0.1282
	5	0.1612	0.2131	0.1794	0.3364	0.3778	0.0598	0.0808	0.0628	0.1202	0.1334
	7	0.1711	0.2205	0.1916	0.3492	0.3913	0.0630	0.0836	0.0658	0.1237	0.1369
2	3	0.1622	0.2226	0.1840	0.3383	0.3816	0.0795	0.1101	0.0862	0.1599	0.1781
	5	0.1787	0.2391	0.2038	0.3609	0.4052	0.0867	0.1183	0.0928	0.1686	0.1872
	7	0.1895	0.2501	0.2168	0.3760	0.4211	0.0914	0.1238	0.0972	0.1744	0.1933
3	3	0.1870	0.2601	0.2163	0.3715	0.4142	0.1182	0.1662	0.1312	0.2276	0.2509
	5	0.2061	0.2876	0.2376	0.4004	0.4421	0.1297	0.1842	0.1409	0.2436	0.2666
	7	0.2175	0.3047	0.2516	0.4185	0.4601	0.1365	0.1954	0.1475	0.2535	0.2765
5	3	0.2185	0.3187	0.2531	0.4170	0.4504	0.1675	0.2471	0.1892	0.3127	0.3354
	5	0.2405	0.3625	0.2782	0.4531	0.4826	0.1841	0.2817	0.2045	0.3384	0.3588
	7	0.2529	0.3877	0.2940	0.4740	0.5023	0.1934	0.3016	0.2145	0.3532	0.3729
10	3	0.2519	0.3980	0.2969	0.4731	0.4896	0.2215	0.3524	0.2599	0.4095	0.4235
	5	0.2771	0.4586	0.3271	0.5170	0.5252	0.2434	0.4063	0.2840	0.4464	0.4547
	7	0.2912	0.4624	0.3451	0.5420	0.5466	0.2556	0.4363	0.2981	0.4675	0.4730

5.4. Advantages of the Proposed Method

According to the above comparison analysis, the proposed method for addressing decision-making problems has the following advantages:

(i) The distance measure under the IFS environment can only handle situations in which the degree of membership and non-membership is provided to the decision-maker. This kind of measure is unable to deal with indeterminacy, which commonly occurs in real-life applications. Because SVNSs are a successful tool in handling indeterminacy, the proposed distance measure in the neutrosophic domain can effectively be used in many real applications in decision-making.

(ii) The proposed distance measure depends upon two parameters p and t, which help in adjusting the hesitation margin in computing data. The effect of hesitation will be diminished or almost neglected if the value of t is taken very large, and for smaller values of t, the effect of hesitation will rise. Thus, according to requirements, the decision-maker can adjust the parameter to handle incomplete as well as indeterminate information. Therefore, this proposed approach is more suitable for engineering, industrial and scientific applications.

(iii) As has been observed from existing studies, various existing measures under NS environments have been proposed by researchers, but there are some situations that cannot be distinguished by these existing measures; hence their corresponding algorithm may give an irrelevant result. The proposed measure has the ability to overcome these flaws; thus it is a more suitable measure to tackle problems.

6. Conclusions

SVNSs are applied to problems with imprecise, uncertain, incomplete and inconsistent information existing in the real world. Although several measures already exist to deal with such kinds of information systems, they have several flaws, as described in the manuscript. Here in this article, we overcome these flaws by proposing an alternative way to define new generalized distance measures between the two SVNNs. Further, a family of normalized and weighted normalized Hamming and Euclidean distance measures have been proposed for the SVNSs. Some desirable properties and their relations have been studied in detail. Finally, a decision-making method has been proposed on the basis of these distance measures. To demonstrate the efficiency of the proposed coefficients, numerical examples of pattern recognition as well as medical diagnosis have been taken. A comparative study, as well as the effect of the parameters on the ranking of the alternative, will support the theory and hence demonstrate that the proposed measures are an alternative way to solve the decision-making problems. In the future, we will extend the proposed approach to the soft set environment [43–45], the multiplicative environment [46–48], and other uncertain and fuzzy environments [7,49–53].

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