A Simple Proof that $\zeta(n \geq 2)$ is Irrational

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Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. The limit of the partials is $z_n$ and the limit of the exclusions leaves only irrational numbers. Thus $z_n$ is proven to be irrational.

1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [2]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. It generalizes only to the $\zeta(3)$ case. Here we give a simpler proof that uses just basic number theory (early chapters of Apostol and Hardy, [1, 3]) and treats all cases at once.

We use the following notation: for positive integers $n > 1$,

$$ z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \quad \text{and} \quad s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}. $$

2 Decimals using denominators

Our aim in this section is to show that the reduced fractions that give the partial sums of $z_n$ require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of $z_n$ can’t be expressed as a finite decimal using for a base the denominators of any of the
Lemma 1. If \( s^k_n = r/s \) with \( r/s \) a reduced fraction, then \( 2^n \) divides \( s \).

Proof. The set \( \{2, 3, \ldots, k\} \) will have a greatest power of 2 in it, \( a \); the set \( \{2^n, 3^n, \ldots, k^n\} \) will have a greatest power of 2, \( na \). Also \( k! \) will have a powers of 2 divisor with exponent \( b \); and \((k!)^n\) will have a greatest power of 2 exponent of \( nb \). Consider

\[
\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}.
\]

The term \((k!)^n/2^{na}\) will pull out the most 2 powers of any term, leaving a term with an exponent of \( nb - na \) for 2. As all other terms but this term will have more than an exponent of \( 2^{nb-na} \) in their prime factorization, we have the numerator of (1) has the form

\[2^{nb-na}(2A + B),\]

where \( 2 \nmid B \) and \( A \) is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term \((k!)^n/2^{na}\). The denominator, meanwhile, has the factored form

\[2^{nb}C,\]

where \( 2 \nmid C \). This leaves \( 2^{na} \) as a factor in the denominator with no powers of 2 in the numerator, as needed. \( \square \)

Lemma 2. If \( s^k_n = r/s \) with \( r/s \) a reduced fraction and \( p \) is a prime such that \( k > p > k/2 \), then \( p^n \) divides \( s \).

Proof. First note that \((k, p) = 1\). If \( p|k \) then there would have to exist \( r \) such that \( rp = k \), but by \( k > p > k/2 \), \( 2p > k \) making the existence of a natural number \( r > 1 \) impossible.

The reasoning is much the same as in Lemma 1. Consider

\[
\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}.
\]

As \((k, p) = 1\), only the term \((k!)^n/p^n\) will not have \( p \) in it. The sum of all such terms will not be divisible by \( p \), otherwise \( p \) would divide \((k!)^n/p^n\). As \( p < k \), \( p^n \) divides \((k!)^n\), the denominator of \( r/s \), as needed. \( \square \)
Theorem 1. If \( s_k^n = \frac{r}{s} \), with \( r/s \) reduced, then \( s > k^n \).

Proof. Bertrand’s postulate states that for any \( k \geq 2 \), there exists a prime \( p \) such that \( k < p < 2k \) [3]. For even \( k \), we are assured that there exists a prime \( p \) such that \( k > p > k/2 \). If \( k \) is odd, \( k - 1 \) is even and we are assured of the existence of prime \( p \) such that \( k - 1 > p > (k - 1)/2 \). As \( k - 1 \) is even, \( p \neq k - 1 \) and \( p > (k - 1)/2 \) assures us that \( 2p > k \), as \( 2p = k \) implies \( k \) is even, a contradiction.

For both odd and even \( k \), using Bertrand’s postulate, we have assurance of the existence of a \( p \) that satisfies Lemma 2. Using Lemmas 1 and 2, we have \( 2^n p^n \) divides the denominator of \( r/s \) and as \( 2^n p^n > k^n \), the proof is completed. \( \square \)

In light of this result we give the following definitions and corollary.

Definition 1.\[ D_{j^n} = \{0, 1/j^n, \ldots, (j^n - 1)/j^n\} = \{0, 1, \ldots, (j^n - 1)\} \] base \( j^n \)

Definition 2.\[ \bigcup_{j=2}^{k} D_{j^n} = \Xi_{k}^n \]

Corollary 1.\[ s_k^n \notin \Xi_{k}^n \]

Proof. Reduced fractions are unique. Suppose, to obtain a contradiction, that there exists \( a/b \in \Xi_{k}^n \) such that \( a/b = r/s \) then \( b < s \) by Theorem 1. If \( a/b \) is not reduced, reduce it: \( a/b = a_1/b_1 \). A reduced fraction must have a smaller denominator than the unreduced form so \( b_1 \leq b < s \) and this contradicts the uniqueness of the denominator of a reduced fraction. \( \square \)

3 A Suggestive Table

The result of applying Corollary 1 to all partial sums of \( z_2 \) is given in Table 1.\(^1\) The table shows that adding the numbers above each \( D_{k^n} \), for all \( k \geq 2 \) gives results not in \( D_{k^n} \) or any previous rows’ such sets. So, for example, \( 1/4 + 1/9 \) is not in \( D_4 \), \( 1/4 + 1/9 \) is not in \( D_4 \) or \( D_9 \), \( 1/4 + 1/9 + 1/16 \) is not in \( D_4 \), \( D_9 \), or \( D_{16} \), etc.. That’s what Corollary 1 says.

\(^1\)Table 1 might remind readers of Cantor’s diagonal method. We don’t pursue this idea in this article.
Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of $z_2$ are excluded from sets below and to the upper left of the partial.

<table>
<thead>
<tr>
<th>$+1/4$</th>
<th>$+1/9$</th>
<th>$+1/16$</th>
<th>$+1/25$</th>
<th>$+1/(k-1)^2$</th>
<th>$+1/k^2$</th>
<th>$+1$</th>
<th>$\vdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+1/4$</td>
<td>$+1/9$</td>
<td>$+1/16$</td>
<td>$+1/25$</td>
<td>$+1/(k-1)^2$</td>
<td>$+1/k^2$</td>
<td>$+1$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\notin D_4$</td>
<td>$\notin D_9$</td>
<td>$\notin D_{16}$</td>
<td>$\notin D_{25}$</td>
<td>$\notin D_{36}$</td>
<td>$\notin D_{k^2}$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

Theorem 2. $z_n$ is irrational.

4 Set theoretical proof

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ and $\mathbb{Q}(0,1) \cap \mathbb{H}(0,1) = \emptyset$ in the following.

**Theorem 2.** $z_n$ is irrational.
Proof. Idea: Corollary 1 implies \( s_k^n \in \mathbb{R}(0, 1) \setminus \Xi_k^n \). As \( \lim_{k \to \infty} s_k^n = z_n \), using Lemma 3, we have

\[
z_n \in \mathbb{R}(0, 1) \setminus \mathbb{Q}(0, 1) = \mathbb{H}(0, 1).
\]

That is \( z_n \) is irrational.

Details: \( \mathbb{R}(0, 1) \setminus \Xi_k^n \) consists of a union of open intervals with rational endpoints given by elements of \( \Xi_k^n \). So for example, considering \( z_2 \), we have

\[
I_1 = \mathbb{R}(0, 1) \setminus D_4.
\]

This gives gives \( I_1 = (0, 1/4) \cup (1/4, 2/4) \cup (2/4, 3/4) \cup (3/4, 1) \). This is \( (0, 1) \) with rational points of the form \( x/4, x = 1, 2, \) and 3 removed. Now let

\[
I_2 = \mathbb{R}(0, 1) \setminus D_4 \cup D_5.
\]

When the fractions are sorted in ascending order they are

\[
\begin{array}{ccccccccccc}
1 & 2 & 1 & 3 & 4 & 1 & 5 & 6 & 3 & 7 & 8 \\
9 & 9 & 4 & 9 & 9 & 2 & 9 & 4 & 9 & 9 & 9
\end{array}
\]

so

\[
I_2 = (0, 1/9) \cup (1/9, 2/9) \cup (2/9, 1/4) \cup (1/4, 3/9) \cup (3/9, 4/9) \text{ and so on.}
\]

Clearly the diameter of each of these open intervals is going to 0 and all rational numbers in \( (0, 1) \) occur as endpoints in these intervals.

As \( s_k \in \mathbb{R}(0, 1) \setminus \Xi_k^n \), this implies that \( \frac{p}{q} < s_k < \frac{r}{s} \) where \( (\frac{p}{q}, \frac{r}{s}) \) is an open interval and \( \frac{p}{q} \) and \( \frac{r}{s} \) are in \( \Xi_k^n \). For any such unequal rationals \( |\frac{p}{q} - \frac{r}{s}| > \epsilon \), for every \( \epsilon \). So \( z_n \) is forced into an interval of diameter 0 with all rational values in \( (0, 1) \) either less than it or greater. It must be irrational.

\[\square\]

5 Conclusion

The proof is very much like that of Sondow’s geometric proof of the irrationality of \( e \) [6]. One can understand his proof by considering the series given by \( \mathfrak{T} \) base 4:

\[
\sum_{j=1}^{\infty} \frac{1}{4^j} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots.
\]
Divide the unit interval into \( \frac{1}{4} \). Skip the first interval as we know the convergence point must be in \((.1,.2)\), base 4. Then divide \([.1,.2]\) into four intervals and skip the first one to give \([.11,.12]\). Repeat this process. The diameter of the intervals goes to 0. Leaving the convergence point between all finite base 4 decimals between 0 and 1: it is \( \frac{1}{5} \), a number not expressible as a finite decimal base 4. Sondow does the same thing dividing up the unit interval with \( n! \) rather than fourths. Skipping the first two terms in the series for \( e \), the unit interval is divided into \( 3! = 6 \), the first interval is skipped, and the next sixth is divided into 4 giving \( \frac{1}{24} \) intervals. The first of these is skipped and the process continues. Once again the intervals shrink to 0 with all possible rational convergence points to the left and right of the point. \( e - 2 \) must be irrational, so \( e \) is. Sondow is using results from analysis: in particular, [5, page 82, problem 21].

The situation for \( z_n \) is more complicated than \( T \), base 4, and the irrationality of \( e \). For these numbers the intervals are perpetually offset, moving to the right. None the less the diameters of the intervals are shrinking and the excluded finite decimals in all bases are increasing to \( \mathbb{Q}(0,1) \), so all rationals are excluded. These numbers must be irrational. It’s a nest within a nest: referring to (4), \( I_{n+1} \subset I_n \) and within each of these intervals is the one that contains \( s_k \). As each of the finite union of intervals has a diameter that is going to 0, the union’s diameter suddenly vanishes at \( \infty \). That’s the mystery of (3). But one can infer in the case of \( e \) and \( z_n \) their irrationality without using nested intervals: their irrational because they define Dedekind cuts for irrational numbers [5, chapter 1 appendix].

References


