The Exact Solution of Gauss's Problem on the Number of Integer "Points" in a Circular and Spherical "Layers"

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In the article, the Gauss's problem on the number of integer points for a circle and a ball in the framework of an integer lattice is reformulated in an equivalent way and reduces to solving two combinatorial tasks for a circular and spherical "layer" in the framework of Quantum Discrete Space. These tasks are solved using trigonometric functions defined on a set of integers whose range of values is also integers, and other new mathematical tools. It comes not about evaluative solutions, but about exact solutions, which, if necessary, can be transferred to a circle and a ball. In doing so not only specific formulas for determine the exact number of solutions are presented, but also the formulas for enumerating the corresponding pairs and triples of integers. The importance of obtained solutions lies in the fact that they determine the analytical likenesses of not only the circumference and the sphere in the Quantum Discrete Space, but also point to the possibility of constructing of the likenesses of ellipse, cone, hyperboloid and other figures.

Keywords: Quantum Discrete Space; Gauss's Problem; Circular Passage; Spherical Passage

Introduction

Gauss's problem is put in the following way: "*Definition*. The point M with coordinates (x, y) is called an integer point if the numbers x and y are integers. We consider the circle $x^2+y^2\leqslant R$, and we denote by K(R) the number of integer points inside this circle. For large R the value of K(R) is approximately πR, the area of the circle. We denote by $\Delta(R)$ the difference between K(R) and πR, that is, $\Delta(R)=K(R)-\pi R$. Gauss's problem on the number of integer points inside a circle is to determine the correct order of magnitude of $\Delta(R)$ as $R\to\infty$." [1] By Gauss and then by other mathematicians, various asymptotic estimates have been proposed, for example, in the works of Voronoi G.F., Landau E.H., Huxley M.N., Hardy G. H. and Littlewood J.E. And the problem of the determining of integer points in a ball was being solved, for example, in the works of Vinogradov I.M., Chamizo F. and Iwaniec H., Chen J.R. In doing so some of obtained estimates are called unimprovable, and it is believed that there is no the exact solution of Gauss's problem.

Yes, indeed, *there is no the exact solution of Gauss's problem within the framework of the integer lattice of points, but the exact solution exists within the framework of* ***Quantum Discrete Space***. The best proof of this statement is the presenting of an exact solution within the framework of Quantum Discrete Space. It is talking namely about this solution in this article. In doing so Gauss's problem for a circle and a ball is reformulated in an equivalent way and reduces to solving the following two combinatorial tasks for a circular and spherical "layer", respectively.

Task 1: Specify the quantity and list the pairs of numbers (x_i,y_i), where $i=0\div n$, $n\in N$, such that the following conditions are fulfilled (the sign "÷" means – successively runs through the values):

$$\begin{cases}(R-1)^2<x_i^2+y_i^2<(R+1)^2\\ -R\leqslant x_i\leqslant R\\ -R\leqslant y_i\leqslant R\text{, where } R\geqslant 0.\end{cases}$$

Task 2: Specify the quantity and list the triples of numbers (x_i,y_i,z_i), where $i=0\div n$, $n\in N$, such that the following conditions are fulfilled:

$$\begin{cases}(R-1)^2<x_i^2+y_i^2+z_i^2<(R+1)^2\\ -R\leqslant x_i\leqslant R\\ -R\leqslant y_i\leqslant R\\ -R\leqslant z_i\leqslant R\text{, where } R\geqslant 0.\end{cases}$$

The solutions of these tasks for a circular and spherical "layer", if necessary, can be transferred to a circle and a ball. Before start to the solving of tasks, we shall previously introduce the basic concepts and definitions with respect to the Quantum Discrete Space and trigonometric functions defined on a set of integers whose range of values is also integers. Other necessary definitions will be introduced in the course of solving tasks 1 and 2.

1. The Basic Concepts of Quantum Discrete Space

<u>Axiom (the beginning of the Universe)</u>: At the beginning of origin of the physical component of the Universe, there was **the unique unit cube**.

<u>Definition 1 (the space)</u>: A unit cube without inner content is called **a unit empty cube**. An ordered set of unit empty cubes, each of which, at any moment of time, is combining its own facets with six others, forms **the space**.

<u>Definition 2 (unit layer of space)</u>: An ordered set of unit empty cubes, each of which, at any moment of time, is combining its own facets with four others, and the other two, **free** facets, are opposite, forms **a unit layer of space** that can be frontal, vertical or horizontal. The space is a totality of like-named unit layers of space.

<u>Definition 3 (unit channel of space)</u>: Two unlike-named unit layers of space intersect, forming **the unit channel of space**. The unit layer of space is the totality of the corresponding unit channels of space.

<u>Definition 4 (the quantum of space)</u>: Three unlike-named unit layers of space intersect, forming the common unit empty cube, which is **the quantum of space**. The unit channel of space is the totality of the corresponding quanta of space.

<u>Definition 5 (Cartesian coordinate system)</u>: Every quantum of space is unique, and therefore can be regarded as some beginning, relative to which the position of geometric or physical body in space is determined. Through the chosen quantum of space, which we shall assume to be zero, three unlike-named unit layers of space are passing, which pairwise have common unit channels of space. Let's specify the directions for these channels and sequentially numbered the quanta of space from which they consist, by positive numbers from zero in the side of chosen direction and by negative numbers in the opposite direction. In doing so the minus sign is formal, and indicates only on the direction, opposite to the chosen one. We call these three channels by **coordinate axes**, and their common quantum by **the origin of coordinates**. The frontal coordinate axis directed at us will be called **the abscissa axis** and denoted by **X**, and the corresponding numbers of coordinate quanta will be called by **abscises**. The horizontal coordinate axis directed to the right will be called **the ordinate axis** and denoted by **Y**, and the corresponding numbers of coordinate quanta will be called by **ordinates**. The vertical coordinate axis directed upwards will be called the **applicate axis** and denote by **Z**, and the corresponding numbers of coordinate quanta will be called by **applicates**. The chosen origin of coordinates and the three coordinate axes, which are passing through it, will be called **the Cartesian coordinate system**. The unit layer of space, in which two coordinate axes lie, is called **the main coordinate layer XY**, **XZ** or **YZ**, respectively. Any other frontal, vertical or horizontal unit layer of space contains in itself one and only one coordinate quantum of the coordinate axis of abscissa, ordinate or applicate respectively, and hence any quantum of space can be unambiguously indicated in the chosen Cartesian coordinate system.

<u>Definition 6 (coordinate packet)</u>: A unit layer of space, which is passing through a given non-zero quantum of the coordinate axis, will be called **an additional coordinate layer**. For example, the notation XY (z=-5) will mean that the additional coordinate layer is parallel to the main coordinate layer XY, and the corresponding two coordinate axes pass through the coordinate quantum of the applicate axis with the number n=-5. The totality of additional coordinate layers, which are passing through the quanta of the corresponding coordinate axis with numbers from *a* to *b*, where a;bЄZ, is called a **coordinate packet**, and is designated, for example, as follows: XY (z=a÷b).

<u>Definition 7 (the main trigonometric rhombus)</u>: The totality of the quanta of space with coordinates {(R-i,i,-R+i),…,(-i,R-i,i),…,(-R+i,-i,R-i),…,(i,-R+i,-i),…}, where RЄN, i=0÷R-1, is called **the main trigonometric rhombus of radius R**, and is designated as follows way: $R_{\Diamond}$. The main trigonometric rhombus consists of 4R quanta of space, which can be figuratively represented if in the cube with edge 2R+1, whose central quantum coincides with the origin of coordinates of the chosen Cartesian coordinate system, consistently to connect four quanta with the following coordinates: (R,0,-R), (0,R,0), (-R,0,R), (0,-R,0).

<u>Definition 8 (trigonometric square)</u>: The totality of the quanta of space lying in one coordinate layer with a given coordinate, the corresponding two other coordinates of which are equal to – {(R-i,i),…,(-i,R-i),…,(-R+i,-i),…,(i,-R+i),…}, where R⩾0, i=0÷R-1, i⩾0, is called **the trigonometric square of radius R**, briefly **T-square(R)** (Fig.1), and is designated as follows way: $\mathbf{R}_{\square}$. T-square (R) consists of the 4R quanta of space. Any quantum of space can be considered as the T-square of zero radius – $0_{\square}$.

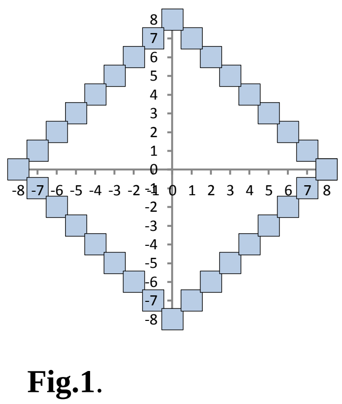

Fig.1. **T-square(R=8)**.

Fig. 2.
T-Octahedron(R=7).
The model has been composed up of wooden cubes with the edge of 2 cm.

Definition 9 (trigonometric octahedron): Let there be a coordinate packet XY(z=-R÷R). Further, let a T-square(R-|z|) is constructed in the each of 2R+1 coordinate layer of the given coordinate packet according to the following scheme: $XY(z=-R\div R)\rightarrow(R-|z|)_{\square}$. The totality of T-squares obtained in this way is called **the trigonometric octahedron of radius R**, briefly **T-octahedron(R)** (Fig. 2), which also can be specified by the following two equisignificant schemes: $XZ(y=-R\div R)\rightarrow(R-|y|)_{\square}$, $YZ(x=-R\div R)\rightarrow(R-|x|)_{\square}$. The coordinates of quanta of the T-octahedron(R) constructed on the coordinate packet XY(z=-R÷R) can be specified as follows way: {(R-|z|-i,i,z),…,(-i,R-|z|-i,z),…,(-R+|z|+i,-i,z),…,(i,-R+|z|+i,z), …}, where i=0÷R-|z|-1 for every z, i⩾0.

Definition 10 (cosine, sine, versine): The quantum's abscissa of the main trigonometric rhombus of radius R is called **the cosine** of number n of the given quantum of radius R, and is designated as follows way: $\mathbf{cos_R n}$. The quantum's ordinate of the main trigonometric rhombus of radius R is called **the sine** of number n of the given quantum of radius R, and is designated as follows way: $\mathbf{sin_R n}$. The quantum's applicate of the main trigonometric rhombus of radius R is called **the versine** (from Latin *vertebralis* – vertical) of number n of the given quantum of radius R, and is designated as follows way: $\mathbf{ver_R n}$.

Definition 11 (bypass of the quanta): We put in correspondence to the abstract zero the quantum of trigonometric rhombus with the coordinates (R,0,-R), and to each number n=1÷4R-1, where R∈N, the quantum, according to the location indicated in Definition 7. The ordered totality of quanta obtained in this way is called **the positive trigonometric rhombus of radius R**, and is designated as follows way: $\mathbf{+R_{\lozenge}}$. In the $+R_{\lozenge}$, without changing the initial quantum, we will perform the reverse streamlining, that is, we change the arrangement of quanta as follows way: 1→4R-1, 2→4R-2, …, 4R-1→1. The ordered totality of quanta obtained in this way is called **the negative trigonometric rhombus of radius R**, and is designated as follows way: $\mathbf{-R_{\lozenge}}$. Bypass of the quanta in $+R_{\lozenge}$ is called by **bypass counterclockwise**, and bypass of the quanta in $-R_{\lozenge}$ is called by **bypass clockwise**. The quantum (R,0,-R) is the initial quantum of bypass in both cases.

Definition 12 (periodic sequence): Let's will bypass $+R_{\lozenge}$ k times (k∈N), numbering the initial quantum by abstract zero, and each subsequent quantum by the corresponding number from the natural series. Thus, each quantum of $+R_{\lozenge}$ will be periodically numbered k times with a period equal to 4R. The obtained sequence of numbers, which can be specified recurrently – $a_0=0$, $a_n=a_{n-1}+1$ for n=1÷4R-1, $a_n=4Rk+a_{n-4Rk}$ for n=4Rk÷4R(k+1)-1, is called **the positive periodic sequence** (PPS), and is designated by $\mathbf{a_n(+R_{\lozenge})}$. Let's will bypass $-R_{\lozenge}$ k times (k∈N), numbering the initial quantum by abstract zero, and each subsequent quantum by the corresponding number from the series Z-. Each quantum of $-R_{\lozenge}$ will be periodically numbered k times with a period equal to (-4R). The obtained sequence of numbers, which can be specified recursively – $a_0=0$, $a_n=a_{n+1}-1$ for n=-1÷-4R+1, $a_n=-4Rk+a_{n+4Rk}$ for n =-4Rk÷-4R(k+1)+1, is called **the negative periodic sequence** (NPS), and is designated by $\mathbf{a_n(-R_{\lozenge})}$. The PPS and the NPS of the main trigonometric rhombus of radius R and of the trigonometric square of the same radius R are coinciding, that is, $\mathbf{a_n(+R_{\lozenge})=a_n(+R_{\square}), a_n(-R_{\lozenge})=a_n(-R_{\square})}$. The union of the PPS and the NPS forms a set of integers Z

Definition 13 (trigonometric functions): To each integer x∈Z={PPS, NPS}, we can associate a certain number y, which is the cosine of corresponding quantum of the main trigonometric rhombus of radius R. Then y is by **the trigonometric function cosine of radius R** defined on the set Z, which is designated as follows way: $\mathbf{y=cos_R x}$. The range of values of the cosine function can be specified graphically on the coordinate layer. To each integer x∈Z={PPS, NPS}, we can associate a certain number y, which is the sine of corresponding quantum of the main trigonometric rhombus of radius R. Then y is by **the trigonometric function sine of radius R** defined on the set Z, which is designated as follows way: $\mathbf{y=sin_R x}$. The range of values of the sine function can be specified graphically on the coordinate layer. To each integer x∈Z={PPS, NPS}, we can associate a certain number y, which is the versine of corresponding quantum of the main trigonometric rhombus of radius R. Then y is by **the trigonometric function versine of radius R** defined on the set Z, which is designated as follows way: $\mathbf{y=ver_R x}$. The range of values of the versine function can be specified graphically on the coordinate layer. To each integer x∈Z={PPS, NPS}, we can associate a certain number $y=-sin_R x$. Then y is by **the trigonometric function opsine of radius R** (from the Latin *oppositus* – opposite) defined on the set Z, which is designated as follows way: $\mathbf{y=ops_R x}$. The range of values of the opsine function can be specified graphically on the coordinate layer (Fig. 3).

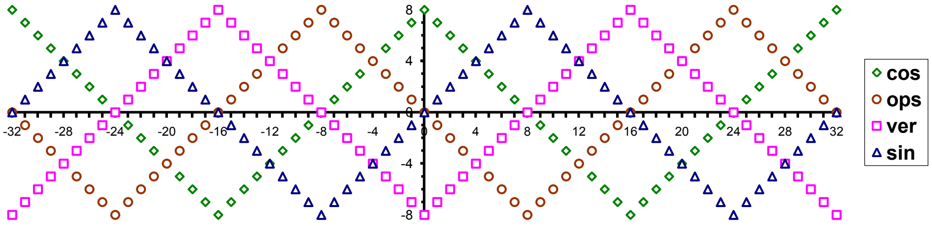

Fig. 3. The Graphs of Trigonometric Functions for R=8.

The main trigonometric formulas: $|\cos_R n|+|\sin_R n|=R$, $|\sin_R n|+|\mathrm{ver}_R n|=R$, $\cos_R n+\mathrm{ver}_R n=0$, $\sin_R n+\mathrm{ops}_R n=0$, where $n\in Z$, $R\geqslant 0$.

Parity and periodicity formulas: $\cos_R(-n)=\cos_R n$, $\sin_R(-n)=\mathrm{ops}_R n$, $\mathrm{ver}_R(-n)=\mathrm{ver}_R n$, $\mathrm{ops}_R(-n)=\sin_R n$; $\cos_R(n+4Rk)=\cos_R n$, $\sin_R(n+4Rk)=\sin_R n$, $\mathrm{ver}_R(n+4Rk)=\mathrm{ver}_R n$, $\mathrm{ops}_R(n+4Rk)=\mathrm{ops}_R n$, where $n,k\in Z$; $R\geqslant 0$.

The basic formulas on quarters: 1 qu. ($0\leqslant n\leqslant R$): $\cos_R n=R-n$, $\sin_R n=n$, $\mathrm{ver}_R n=n-R$, $\mathrm{ops}_R n=-n$; 2 qu. ($R\leqslant n\leqslant 2R$): $\cos_R n=R-n$, $\sin_R n=2R-n$, $\mathrm{ver}_R n=n-R$, $\mathrm{ops}_R n=n-2R$; 3 qu. ($2R\leqslant n\leqslant 3R$): $\cos_R n=n-3R$, $\sin_R n=2R-n$, $\mathrm{ver}_R n=3R-n$, $\mathrm{ops}_R n=n-2R$; 4 qu. ($3R\leqslant n\leqslant 4R$): $\cos_R n=n-3R$, $\sin_R n=n-4R$, $\mathrm{ver}_R n=3R-n$, $\mathrm{ops}_R n=4R-n$.

2. The Solution of Task 1

We shall solve the task 1 by using the main coordinate layer XY. In this case (x_i,y_i) can be considered as the coordinates of the corresponding quantum of space. We will seek solutions located in the first quarter, so how the other solutions will easily follow from the obtained solutions. Thus, we can say that $x_i=\cos_a t$, $y_i=\sin_a t$, where $0\leqslant t\leqslant a$, $a=R+j-1$, $1\leqslant j\leqslant R$ is some variable. Then the main condition of the task can be written as follows way: $(R-1)^2<\cos^2_a t+\sin^2_a t<(R+1)^2$ or $(R-1)^2<(a-t)^2+t^2<(R+1)^2$.

After the disclosure of brackets and the adduction of like summands, we obtain the following expression:

$$-b<2t(t-a)<4R-b, \text{ where } a=R+j-1,\ b=j^2+2Rj-2j. \qquad (1)$$

To solve the resulting double inequality, it is necessary to introduce a new mathematical operation, the essence of which is as follows.

<u>Definition 14 (gissing operation)</u>: Let's divide all natural numbers into groups according to the following criterion: the equal quantity of numbers that are the square of a natural number are located between any element of group and zero, not counting this element, if it itself is the square of a natural number. The operation of determining the group of natural number N with respect to the indicated criterion is called **the operation of gissing** (from the Latin *gis* – group), and is designated as follows way: **n=gisN**. The group's number of a natural number is equals to the base of the square of natural number closest to a given number, which is greater than a given number, or equal to it: $N\leqslant n^2$; and between N and n^2 there is no number that is the square of a natural number.

For clarity, the numbers from 1 to 25 are grouped by the indicated criterion and reduced into the Table 1:

Table 1. The Gissing of Numbers from 1 to 25.

Number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	k
The quantity of squares	0	0	1	1	1	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	n-1
Group's number	0	1	2			3					4							5									n
The quantity of numbers in a group	0	1	3			5					7							9									2n-1
The quantity of numbers up to the n-th group inclusively	0	1	4			9					16							25									n^2

For example: gis21=5; gis111=11; gis289=17; gis859=30; gis3975=64…

Inequality (1) is equipollent to the following system of inequalities:

$$\begin{cases} 2t^2-2at+b>0 & (2) \\ 2t^2-2at-(4R-b)<0. & (3) \end{cases}$$

Here and below, square brackets mean the union of solutions, and the curly brackets mean the intersection of solutions. Inequality (2) is equipollent to the following system of inequalities:

$$\left[\begin{array}{l} 0\leq t<½\left(a-\sqrt{a^2-2b}\right) \\ ½\left(a+\sqrt{a^2-2b}\right)<t\leq a \end{array}\right. \quad \text{or} \quad \left[\begin{array}{l} 0\leqslant t<½(a-\mathrm{gis}(a^2-2b)+1) \\ ½(a+\mathrm{gis}(a^2-2b)-1)<t\leqslant a. \end{array}\right.$$

The operation of gissing of the expression under the radical leads to a decrease in the right-hand side of the first inequality of system and an increase in the left-hand side of the second inequality of system, except for the case when the expression under the radical is the square of some natural number. Therefore, we are increasing the expression in brackets on the right-hand side of the first inequality of system by one, and reducing on the left-hand side of the second inequality of system by 1.

Inequality (3) is equipollent to the following double inequality:

$$½(a-\sqrt{a^2+8R-2b})<t<½(a+\sqrt{a^2+8R-2b}) \quad \text{or}$$
$$½(a-gis(a^2+8R-2b))<t<½(a+gis(a^2+8R-2b)).$$

Here the left-hand side of inequality decreases, and the right-hand side increases, and therefore the inequality is not violated. Thus, inequality (1) is equipollent to the following system of inequalities:

$$\begin{cases}\left[\begin{array}{l}0\leqslant t<½(a-gis(a^2-2b)+1)\\ ½(a+gis(a^2-2b)-1)<t\leqslant a\end{array}\right.\\ ½(a-gis(a^2+8R-2b))<t<½(a+gis(a^2+8R-2b)).\end{cases} \tag{4}$$

Definition 15 (d- and g-functions): Let $F(x_i)$ and $f(x_i)$, where $i=1\div n$, $n\in N$ – some mathematical expressions, which after substitution of numerical values of x_i are some nonnegative numbers. Then:

$d = \begin{cases}0, \text{ if } F(x_i) \text{ is an even number,}\\ 1, \text{ if } F(x_i) \text{ is an odd number,}\end{cases}$ is called **the d-function**;

$g = \begin{cases}0, \text{ if } f(x_i)\neq c^2,\\ 1, \text{ if } f(x_i)=c^2,\end{cases}$ where $c\geqslant 0$, is called **the g-function**.

In the task under consideration, $F(x_1,x_2)$ is one of the following four expressions: $a\pm gis(a^2-2b)$, $a\pm gis(a^2+8R-2b)$, and $f(x_1,x_2)=a^2-2b$.

Further we will use the following designation: if a> b, c and f are integers, and $a=b\times c+f$, then $c=[a:b]$ is the integer part off the dividing of *a* by *b*. With a view to replace the strict inequalities of system (4) by non-strict ones, we will write the system (4) in another equipollent format:

$$\begin{cases}\left[\begin{array}{l}0\leqslant t\leqslant [a-gis(a^2-2b)-d:2]-g\\ [a+gis(a^2-2b)+d:2]+g\leqslant t\leqslant a\end{array}\right.\\ [a-gis(a^2+8R-2b)-d:2]+1\leqslant t\leqslant [a+gis(a^2+8R-2b)+d:2]-1\\ 0\leqslant t\leqslant a, \text{ if } a<gis(a^2+8R-2b).\end{cases} \tag{5}$$

The equipollent of systems (4) and (5) is clearly shown below, in Fig. 5, which will be more understandable after determining the corresponding solutions of system 5. The required pairs of numbers have the form: $(\cos_a t, \sin_a t)$,, where $a=R+j-1$, t are the solutions of system (5).

For clarity and to find out the essence of the variable j, we will consider a concrete example for R=17.

1. $j=1 \Rightarrow b=33$, $a=17$. Inequality (1) takes the form: $-33<2t(t-17)<35$; and the system (5):

$$\begin{cases}\left[\begin{array}{l}0\leqslant t\leqslant [17-gis223-d:2]-0=[17-15-0:2]=1\\ [17+15+0:2]=16\leqslant t\leqslant 17\end{array}\right.\\ 7<gis359=19 \Rightarrow 0\leqslant t\leqslant 17,\end{cases} \qquad \begin{cases}\left[\begin{array}{l}0\leqslant t\leqslant 1\\ 16\leqslant t\leqslant 17\end{array}\right.\\ 0\leqslant t\leqslant 17 .\end{cases}$$

We have 4 solutions: t=0;1;16;17, and hence 4 pairs:
$(\cos_{17}0, \sin_{17}0)=(17,0)$; $(\cos_{17}1, \sin_{17}1)=(16,1)$;
$(\cos_{17}16, \sin_{17}16)=(1,16)$; $(\cos_{17}17, \sin_{17}17)=(0,17)$.

2. $j=2 \Rightarrow b=68$, $a=18$. Inequality (1) takes the form: $-68<2t(t-18)<0$; and the system (5):

$$\begin{cases}\left[\begin{array}{l}0\leqslant t\leqslant [18-gis188-d:2]-0=[18-14-0:2]=2\\ [18+14+0:2]=16\leqslant t\leqslant 18\end{array}\right.\\ [18-gis324-d:2]+1=[18-18-0:2]+1=1\leqslant t\leqslant [18+18+0:2]-1=17,\end{cases} \qquad \begin{cases}\left[\begin{array}{l}0\leqslant t\leqslant 2\\ 16\leqslant t\leqslant 18\end{array}\right.\\ 1\leqslant t\leqslant 17.\end{cases}$$

We have 4 solutions: t=1;2;16;17, and hence 4 pairs:
$(\cos_{18}1, \sin_{18}1)=(17,1)$; $(\cos_{18}2, \sin_{18}2)=(16,2)$;
$(\cos_{18}16, \sin_{18}16)=(2,16)$; $(\cos_{18}17, \sin_{18}17)=(1,17)$.

3. $j=3 \Rightarrow b=105$, $a=19$. Inequality (1) takes the form: $-105<2t(t-19)<-37$; and the system (5):

$$\begin{cases}\left[\begin{array}{l}0\leqslant t\leqslant [19-gis151-d:2]-0=[19-13-0:2]=3\\ [19+13+0:2]=16\leqslant t\leqslant 19\end{array}\right.\\ [19-gis287-d:2]+1=[19-17-0:2]+1=2\leqslant t\leqslant [19+17+0:2]-1=17,\end{cases} \qquad \begin{cases}\left[\begin{array}{l}0\leqslant t\leqslant 3\\ 16\leqslant t\leqslant 19\end{array}\right.\\ 2\leqslant t\leqslant 17.\end{cases}$$

We have 4 solutions: t=2;3;16;17, and hence 4 pairs:
$(\cos_{19}2,\sin_{19}2)=(17,2)$; $(\cos_{19}3,\sin_{19}3)=(16,3)$;
$(\cos_{19}16,\sin_{19}16)=(3,16)$; $(\cos_{19}17,\sin_{19}17)=(2,17)$.

4. j=4 $\Rightarrow$ b=144, a=20. Inequality (1) takes the form: $-144<2t(t-20)<-76$; and the system (5):

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant [20-gis112-d:2]-0=[20-11-1:2]=4 \\ [20+11+1:2]=16\leqslant t\leqslant 20 \end{array}\right. \\ [20-gis248-d:2]+1=[20-16-0:2]+1=3\leqslant t\leqslant [20+16+0:2]-1=17, \end{cases} \qquad \begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant 4 \\ 16\leqslant t\leqslant 20 \end{array}\right. \\ 3\leqslant t\leqslant 17. \end{cases}$$

We have 4 solutions: t=3;4;16;17, and hence 4 pairs:
$(\cos_{20}3,\sin_{20}3)=(17,3)$; $(\cos_{20}4,\sin_{20}4)=(16,4)$;
$(\cos_{20}16,\sin_{20}16)=(4,16)$; $(\cos_{20}17,\sin_{20}17)=(3,17)$.

5. j=5 $\Rightarrow$ b=185, a=21. Inequality (1) takes the form: $-185<2t(t-21)<-117$; and the system (5):

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant [21-gis71-d:2]-0=[21-9-0:2]=6 \\ [21+9+0:2]=15\leqslant t\leqslant 21 \end{array}\right. \\ [21-gis207-d:2]+1=[21-15-0:2]+1=4\leqslant t\leqslant [21+15+0:2]-1=17, \end{cases} \qquad \begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant 6 \\ 15\leqslant t\leqslant 21 \end{array}\right. \\ 4\leqslant t\leqslant 17. \end{cases}$$

We have 6 solutions: t=4;5;6;15;16;17, and hence 6 pairs:
$(\cos_{21}4,\sin_{21}4)=(17,4)$; $(\cos_{21}5,\sin_{21}5)=(16,5)$; $(\cos_{21}6,\sin_{21}6)=(15,6)$;
$(\cos_{21}15,\sin_{21}15)=(6,15)$; $(\cos_{21}16,\sin_{21}16)=(5,16)$; $(\cos_{21}17,\sin_{21}17)=(4,17)$.

6. j=6 $\Rightarrow$ b=228, a=22. Inequality (1) takes the form: $-228<2t(t-22)<-160$; and the system (5):

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant [22-gis28-d:2]-0=[22-6-0:2]=8 \\ [22+6+0:2]=14\leqslant t\leqslant 22. \end{array}\right. \\ [22-gis164-d:2]+1=[22-13-1:2]+1=5\leqslant t\leqslant [22+13+1:2]-1=17, \end{cases} \qquad \begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant 8 \\ 14\leqslant t\leqslant 22 \end{array}\right. \\ 5\leqslant t\leqslant 17. \end{cases}$$

We have 8 solutions: t=5;6;7;8;14;15;16;17, and hence 8 pairs:
$(\cos_{22}5,\sin_{22}5)=(17,5)$; $(\cos_{22}6,\sin_{22}6)=(16,6)$; $(\cos_{22}7,\sin_{22}7)=(15,7)$;
$(\cos_{22}8,\sin_{22}8)=(14,8)$; $(\cos_{22}14,\sin_{22}14)=(8,14)$; $(\cos_{22}15,\sin_{22}15)=(7,15)$;
$(\cos_{22}16,\sin_{22}16)=(6,16)$; $(\cos_{22}17,\sin_{22}17)=(5,17)$.

7. j=7 $\Rightarrow$ b=273, a=23. Inequality (1) takes the form: $-273<2t(t-23)<-205$; and the system (5):

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant [23-gis(-17)-d:2]-g,\ gis(-17)\text{ not determined,} \\ [23+gis(-17)+d:2]+g\leqslant t\leqslant 23 \text{ – not determined,} \end{array}\right. \\ [23-gis119-d:2]+1=[23-11-0:2]+1=7\leqslant t\leqslant [23+11+0:2]-1=16, \end{cases} \qquad 7\leqslant t\leqslant 16.$$

We have 10 solutions: t=7;8;9;10;11;12;13;14;15;16, and hence 10 pairs:
$(\cos_{23}7,\sin_{23}7)=(16,7)$; $(\cos_{23}8,\sin_{23}8)=(15,8)$; $(\cos_{23}9,\sin_{23}9)=(14,9)$; $(\cos_{23}10,\sin_{23}10)=(13,10)$;
$(\cos_{23}11,\sin_{23}11)=(12,11)$; $(\cos_{23}12,\sin_{23}12)=(11,12)$; $(\cos_{23}13,\sin_{23}13)=(10,13)$; $(\cos_{23}14,\sin_{23}14)=(9,14)$;
$(\cos_{23}15,\sin_{23}15)=(8,15)$; $(\cos_{23}16,\sin_{23}16)=(7,16)$.

8. j=8 $\Rightarrow$ b=320, a=24. Inequality (1) takes the form: $-320<2t(t-24)<-252$; and the system (5):

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant [24-gis(-64)-d:2]+g,\ gis(-64)\text{ not determined,} \\ [24+gis(-64)+d:2]+g\leqslant t\leqslant 24 \text{ – not determined,} \end{array}\right. \\ [24-gis72-d:2]+1=[24-9-1:2]+1=8\leqslant t\leqslant [24+9+1:2]-1=16, \end{cases} \qquad 8\leqslant t\leqslant 16.$$

We have 9 solutions: t=8;9;10;11;12;13;14;15;16, and hence 9 pairs:
$(\cos_{24}8,\sin_{24}8)=(16,8)$; $(\cos_{24}9,\sin_{24}9)=(15,9)$; $(\cos_{24}10,\sin_{24}10)=(14,10)$;
$(\cos_{24}11,\sin_{24}11)=(13,11)$; $(\cos_{24}12,\sin_{24}12)=(12,12)$; $(\cos_{24}13,\sin_{24}13)=(11,13)$;
$(\cos_{24}14,\sin_{24}14)=(10,14)$; $(\cos_{24}15,\sin_{24}15)=(9,15)$; $(\cos_{24}16,\sin_{24}16)=(8,16)$.

9. j=9 $\Rightarrow$ b=369, a=25. Inequality (1) takes the form: $-369<2t(t-25)<-301$; and the system (5):

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant [25-gis(-113)-d:2]+g,\ gis(-113)\text{ not determined,} \\ [25+gis(-113)+d:2]+g\leqslant t\leqslant 25 \text{ – not determined,} \end{array}\right. \\ [25-gis23-d:2]+1=[25-5-0:2]+1=11\leqslant t\leqslant [25+5+0:2]-1=14, \end{cases} \qquad 11\leqslant t\leqslant 14.$$

We have 4 solutions: t=11;12;13;14, and hence 4 pairs:
$(\cos_{25}11,\sin_{25}11)=(14,11)$; $(\cos_{25}12,\sin_{25}12)=(13,12)$;
$(\cos_{25}13,\sin_{25}13)=(12,13)$; $(\cos_{25}14,\sin_{25}14)=(11,14)$.

10. j=10 $\Rightarrow$ b=420, **a**=26. Inequality (1) takes the form: $-420<2t(t-26)<-352$; and the system (5):

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t\leqslant [26-gis(-164)-d:2]-g,\ gis(-164)\text{ not determined,} \\ [26+gis(-164)+d:2]+g\leqslant t\leqslant 26 \text{ – not determined,} \end{array}\right. \\ [26-gis(-28)-d:2]+1\leqslant t\leqslant [26+gis(-28)+d:2]-1,\ \ gis(-28)\text{ not determined.} \end{cases}$$

The solution of system – $\varnothing$.

All solutions for the first quarter are found. Total – 53 pairs. Then the quantity of pairs for the entire coordinate layer at R=17, after deduction of overlaps, is equals: $N_{R=17}=53x4-4=208$.

From the example considered, it becomes to clear the essence of the variable j. It represents the **orbit's** number, on which the corresponding solutions of inequality (1) are distributed. And the orbits themselves are the T-squares of radius (R+j-1).

The possible values of j can be determined from the following conditions:

$$\begin{cases} 0\leqslant a^2-2b\leqslant a^2 \\ 0\leqslant a^2+8R-2b\leqslant a^2. \end{cases}$$

Substituting into this system the values a=R+j-1 and $b=j^2+2Rj-2j$, we will obtain:

$$\left[\begin{array}{l} \begin{cases} j^2+2(R-1)j-(R^2-2R+1)\leqslant 0 \\ j^2+2(R-1)j\geqslant 0 \end{cases} \\ \begin{cases} j^2+2(R-1)j-(R^2+6R+1)\leqslant 0 \\ j^2+2(R-1)j-4R\geqslant 0. \end{cases} \end{array}\right.$$

The solution of the obtained system has the form:

$$\left[\begin{array}{l} j_1=1\leqslant j\leqslant gis(2(R-1)^2)-R=j_2 \\ j_3=gis((R+1)^2)-R+1=2\leqslant j\leqslant gis(2(R+1)^2)-R=j_4. \end{array}\right. \quad (6)$$

The values of j_1 and j_2 indicate on the limits of action of the first and second conditions of system (5), and the values of j_3 and j_4 indicate on the limits of action of the third condition of system (5). For $j_1\leqslant j<j_3$ the third condition of system (5) does not act, but the fourth condition act. The maximum value of j_4 indicates on the number of T-squares with radius from R to $R+j_4-1$, on which the solutions of inequality (1) are dispersed. The expression $j_4=gis(2(R+1)^2)-R$ is the numerical sequence whose R-th term is the quantity of T-squares of the dispersal of the solutions of inequality (1), the radius of the first of which is equals to R-1.The first 25 terms of this sequence are equal: 1,2,3,3,3,4,4,5,5,6,6,6,7,7,8,8,9,9,9,10,10,11,11,11,12.

Thus, the general solution of Task 1 for the coordinate layer, in the part of enumeration of pairs, is looks as follows way:

$$\left[\begin{array}{l} \mathbf{(cos_a t,\ sin_a t)} \\ \mathbf{(ver_a t,\ sin_a t),\ t\neq R\ for\ j=1} \\ \mathbf{(ver_a t,\ ops_a t),\ t\neq 0\ for\ j=1} \\ \mathbf{(cos_a t,\ ops_a t),\ t\neq 0;R\ for\ j=1,\ where\ 0\leqslant t\leqslant a,\ a=R+j-1,\ R\geqslant 0,} \end{array}\right. \quad (7)$$

j runs through the values from j_1 to j_4, which are determined from the conditions –

$$\left[\begin{array}{l} \mathbf{j_1=1\leqslant j\leqslant gis(2(R-1)^2)-R=j_2} \\ \mathbf{j_3=2\leqslant j\leqslant gis(2(R+1)^2)-R=j_4,} \end{array}\right.$$

$b=j^2+2Rj-2j$, t – the solutions of inequality -b<2t(t-a)<4R-b for the first quarter, which are determined from the conditions –

$$\begin{cases} \left[\begin{array}{l} \mathbf{0\leqslant t\leqslant [(a-gis(a^2-2b)-d:2]-g} \\ \mathbf{[a+gis(a^2-2b)+d:2]+g\leqslant t\leqslant a} \end{array}\right. \\ \mathbf{[a-gis(a^2+8R-2b)-d:2]+1\leqslant t\leqslant [(a+gis(a^2+8R-2b)+d:2]-1} \\ \mathbf{0\leqslant t\leqslant a,\ if\ a<gis(a^2+8R-2b).} \end{cases}$$

The solution of task in part of the enumeration of pairs can be visualized with the help of function $(y_{ij})=2t_i(t_i-a_j)$, the graph of which is **a cascade of parabolas**, the quantity of which is equals to j_4. But first I will give the necessary definitions of the function, by the argument and value of which are numerical series.

Definition 16 (1N-, 2N-, 3N-series): The totality of numbered from 1 to i, i⩾1, quanta of the coordinate axis Y is called **the linear natural series i**, briefly the **1N-series(i)**. Let the numbered from -1 to -j, where j⩾1, quanta of coordinate axis Z are renumbered by the corresponding positive numbers from 1 to j. Then the totality of these quanta and quanta of the 1N-series(i) is called **the planar natural series ixj**, briefly **the 2N-series(ixj)**. Let the numbered from -1 to -k, where k⩾1, quanta of coordinate axis X are renumbered by the corresponding positive numbers from 1 to k. Then the totality of these quanta and quanta of the 2N-series(ixj) is called **the spatial natural series ixjxk**, briefly **the 3N-series(ixjxk)**. The quantum with coordinates (0,0,0) is called the zero quantum of indicated series. The arrangement of these series in the space may be other.

Definition 17 (matrix): We put in correspondence to each quantum located under the 1N-series(i) a certain number n_i. The totality of numbers obtained in this way is called **the linear numeric matrix n_i**, briefly **the Matrix(n_i)**. We put in correspondence to each quantum located at the intersection of the vertical unit channel passing through the quantum i and the horizontal unit channel passing through the quantum j of the 2N-series(ixj) a certain number n_{ij}. The totality of numbers obtained in this way is called **the planar numeric matrix n_{ij}**, briefly **the Matrix(n_{ij})**. In doing so the horizontal channels are called **the strings**, and the vertical channels are called **the columns** of Matrix(n_{ij}). We put in correspondence to each quantum located at the intersection of the vertical unit layer passing through the quantum i, the horizontal unit layer passing through the quantum j and the frontal unit layer passing through the quantum k of the 3N-series(ixjxk) a certain number n_{ijk}. The totality of numbers obtained in this way is called **the spatial numeric matrix n_{ijk}**, briefly **the Matrix(n_{ijk})**. In doing so the vertical layers are called **vertical slices**, the horizontal layers are called **horizontal slices**, and the frontal layers are called the **frontal slices** of the Matrix(n_{ijk}). The numbers n_i, n_{ij}, n_{ijk} are called **the values** of the corresponding matrices.

Definition 18 (1F0-, 2F0-, 3F0-matrix): The Matrix(a), together with the empty series of quanta located under the series of numbers **a_i**, is called **the empty linear functional matrix (a_i)**, briefly the **1F0-matrix(a_i)**. We will transform the Matrix(n_{ij}) as follows way: the horizontal natural series begins from the quantum (0,2,0), under which is located the series of numbers a_i; the vertical natural series begins from the quantum (0,0,-2), to the right of which is located the series of numbers b_j; all other quanta are empty. The Matrix(n_{ij}) transformed in this way is called **the empty planar functional matrix (n_{ij})**, briefly **the 2F0-matrix(n_{ij})**. We will transform the Matrix(n_{ijk}) as follows way: the horizontal natural series begins from the quantum (0,1,1), under which is located the series of numbers a_i; the vertical natural series begins from the quantum (0,-1,-1), to the right of which is located the series of numbers b_j; the frontal natural series begins from the quantum (-1,0,1), under which is located the series of numbers c_k; all other quanta are empty. The Matrix(n_{ijk}) transformed in this way is called **the empty spatial functional matrix (n_{ijk})**, briefly **the 3F0-matrix(n_{ijk})**.

Definition 19 (function): Let a certain transformation exist, which to each number of the series a_i put in correspondence another number from the series y_i. Then the series y_i is called **the linear function off the series a_i with respect to the given transformation**, briefly **y_i=f(a_i)**. The linear function can be specified with the help of the 1F0-matrix(a_i), the empty quanta of which are filled with the corresponding numbers of the series y_i. The obtained matrix is called **the linear functional matrix y_i**, briefly **the 1F-matrix(y_i)**. Let a certain transformation exist, which to each admissible pair of numbers from the series a_i and b_j put in correspondence another number from the values of Matrix(y_{ij}). Then the Matrix(y_{ij}) is called **the planar function off the series a_i and b_j with respect to the given transformation**, briefly **(y_{ij})=f(a_i,b_j)**. The planar function can be specified with the help of the 2F0-matrix(n_{ij}), the empty quanta of which are filled with the corresponding values of the Matrix(y_{ij}). The obtained matrix is called **the planar functional matrix (y_{ij})**, briefly **the 2F-matrix(y_{ij})**. Let a certain transformation exist, which to each admissible triple of numbers from the series a_i, b_j and c_k put in correspondence another number from the values of Matrix(y_{ijk}). Then the Matrix(y_{ijk}) is called **the spatial function off the series a_i, b_j and c_k with respect to the given transformation**, briefly **(y_{ijk})=f(a_i,b_j,c_k)**. The spatial function can be specified with the help of the 3F0-matrix(n_{ijk}), the empty quanta of which are filled with the corresponding values of the Matrix(y_{ijk}). The obtained matrix is called **the spatial functional matrix (y_{ijk})**, briefly **the 3F-matrix(y_{ijk})**.

Definition 20 (function's graph): The reflection of the values of functional matrix relative to the corresponding series through the coordinates of a certain totality of quanta of the selected Cartesian coordinate system is called **the graphical mapping of the functional matrix**, and the figure formed by the specified totality of quanta is called **the function's graph**.

So, the graph of function (y_{ij})=2t_i(t_i-a_j) for R=17, i=1÷a_j+1, j=1÷9, a_j=16+j in the form of the corresponding cascade of parabolas is presented in Fig. 4, and the corresponding 2F-matrix(y_{ij}) is given in Table 2.

Table 2. 2F-matrix ((y_{ij})=2t_i(t_i-a_j)) for R=17.

0	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
j	t_i \ a_j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	17	0	-32	-60	-84	-104	-120	-132	-140	-144	-144	-140	-132	-120	-104	-84	-60	-32	0								
2	18	0	-34	-64	-90	-112	-130	-144	-154	-160	-162	-160	-154	-144	-130	-112	-90	-64	-34	0							
3	19	0	-36	-68	-96	-120	-140	-156	-168	-176	-180	-180	-176	-168	-156	-140	-120	-96	-68	-36	0						
4	20	0	-38	-72	-102	-128	-150	-168	-182	-192	-198	-200	-198	-192	-182	-168	-150	-128	-102	-72	-38	0					
5	21	0	-40	-76	-108	-136	-160	-180	-196	-208	-216	-220	-220	-216	-208	-196	-180	-160	-136	-108	-76	-40	0				
6	22	0	-42	-80	-114	-144	-170	-192	-210	-224	-234	-240	-242	-240	-234	-224	-210	-192	-170	-144	114	-80	-42	0			
7	23	0	-44	-84	-120	-152	-180	-204	-224	-240	-252	-260	-264	-264	-260	-252	-240	-224	-204	-180	-152	-120	-84	-44	0		
8	24	0	-46	-88	-126	-160	-190	-216	-238	-256	-270	-280	-286	-288	-286	-280	-270	-256	-238	-216	-190	-160	-126	-88	-46	0	
9	25	0	-48	-92	-132	-168	-200	-228	-252	-272	-288	-300	-308	-312	-312	-308	-300	-288	-272	-252	-228	-200	-168	-132	-92	-48	0

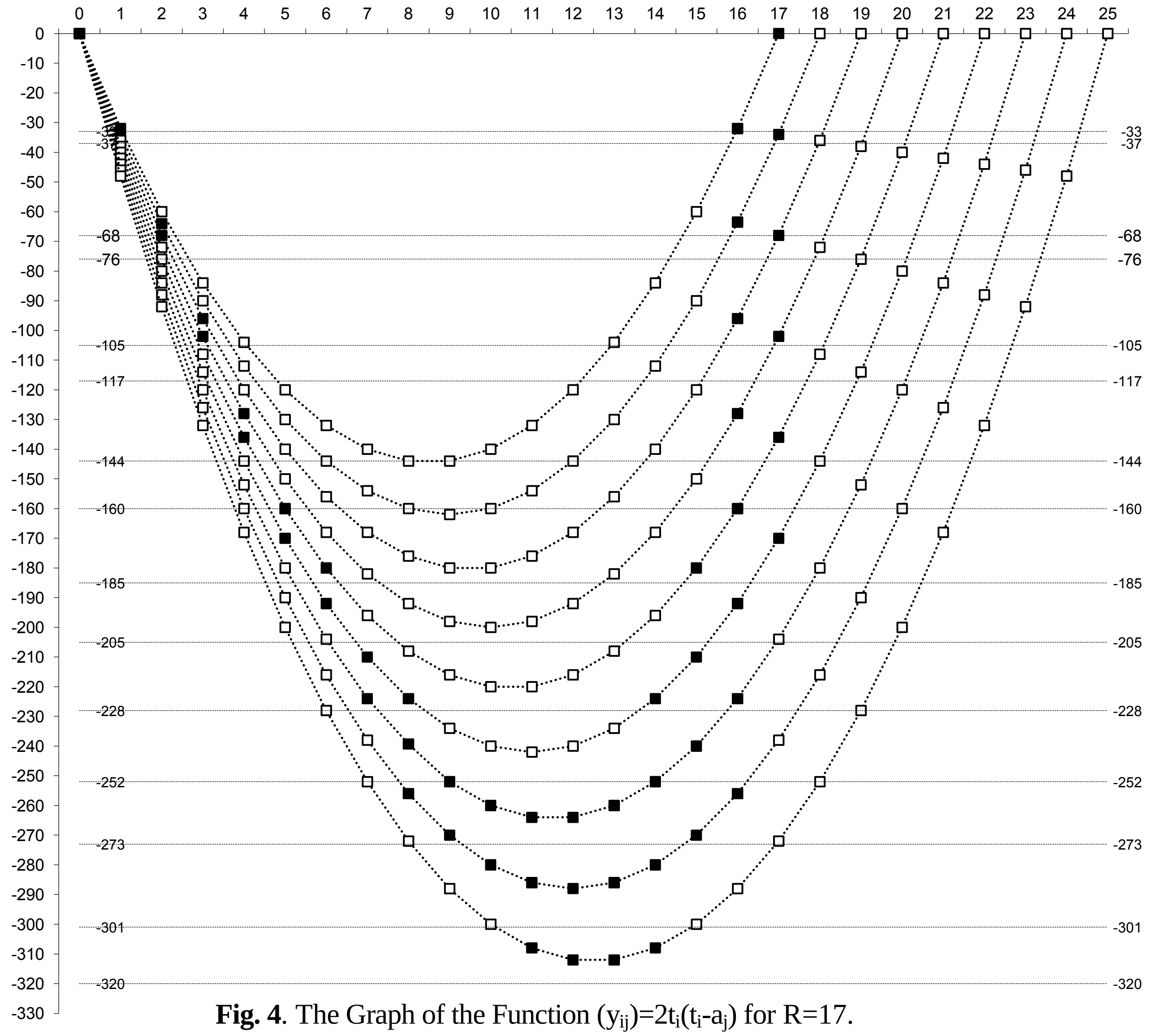

Fig. 4. The Graph of the Function $(y_{ij})=2t_i(t_i-a_j)$ for R=17.

All lines on the graph are for illustration only. All quanta of graph are the quanta of T-squares with the radiuses from R=17 to R=25, having been refracted by the function $(y_{ij})=2t_i(t_i-a_j)$, and the dedicated quanta are the solutions of inequality (1) for R=17. It is seen from the presented graph that the solutions of inequality (1) are between the corresponding channels y=-b and y=4R-b of the corresponding parabola of function's graph. The channel y=-b is located relative to the corresponding parabola in such a way that the nearest from top two parabola's quanta have values t equal to $[a-gis(a^2-2b)-d:2]-g$ and $[a+gis(a^2-2b)+d:2]+g$. The channel y=4R-b is located relative to the corresponding parabola in such a way that the nearest from below two parabola's quanta have values t equal to $[a-gis(a^2+8R-2b)-d:2]+1$ and $[a+gis(a^2+8R-2b)+d:2]-1$.Then the quantity of quanta located on a given orbit $j_1 \leqslant j \leqslant j_2$ can be determined from the following expression:

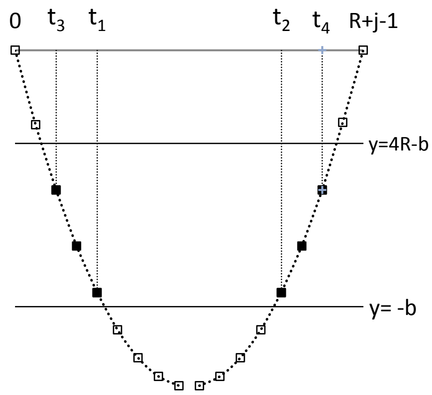

Fig.5.
Explanations to the Formula 8.

$$\mathbf{n_j=[a-gis(a^2-2b)-d:2]-[a-gis(a^2+8R-2b)-d:2]+[a+gis(a^2+8R-2b)+d:2]-[a+gis(a^2-2b)+d:2]-2g}. \quad (8)$$

Formula (8), as well as the equivalence of systems (4) and (5), becomes clearer from Fig. 5.

The ones shown in Fig. 5 the numbers t_1, t_2, t_3 and t_4 have the following values:
$t_1=[a-gis(a^2-2b)-d:2]-g$; $t_2=[a+gis(a^2-2b)+d:2]+g$; $t_3=[a-gis(a^2+8R-2b)-d:2]+1$; $t_4=[a+gis(a^2+8R-2b)+d:2]-1$.

For the $j_1=1$, the fourth condition of system (5) is acting. Then: $t_3=0$, $t_4=R$, a=R, b=2R-1 $\Rightarrow$ $n(j_1)=t_1-0+1+R-t_2+1=[R-gis((R-1)^2-(2R-1))-d:2]+R-[a+gis((R-1)^2-(2R-1))+d:2]-2g+2$. Further we are reasoning so: $(R-1)^2-(R-2)^2=2R-3<2R-1 \Rightarrow gis((R-1)^2-(2R-1))=R-2 \Rightarrow n(j_1)=1+R-(R-1)-0+2=4$.

If $j_2+1 \leqslant j \leqslant j_4$, then the distribution of quanta along the orbits can be determined from the following expression: $\mathbf{n_j=[a+gis(a^2+8R-2b)+d:2]-[a-gis(a^2+8R-2b)-d:2]-1}$. (9)

Let us verify the validity of formulas (8) and (9) for R=17. For $2 \leqslant j \leqslant j_2=6$ we have:
j=2. [18-gis188-d:2]-[18-gis324-d:2]+[18+gis324+d:2]-[18+gis188+d:2]-2x0=2-0+18-16-0=4;
j=3. [19-gis151-d:2]-[19-gis287-d:2]+[19+gis287+d:2]-[19+gis151+d:2]-2x0=3-1+18-16-0=4;
j=4. [20-gis112-d:2]-[20-gis248-d:2]+[20+gis248+d:2]-[20+gis112+d:2]-2x0=4-2+18-16-0=4;
j=5. [21-gis71-d:2]-[21-gis207-d:2]+[21+gis207+d:2]-[21+gis71+d:2]-2x0=6-3+18-15-0=6;
j=6. [22-gis28-d:2]-[22-gis164-d:2]+[22+gis164+d:2]-[22+gis28+d:2]-2x0=8-4+18-14-0=8.
For $j_2+1=7 \leqslant j \leqslant j_4=gis(2(R+1)^2)-R=gis648-17=26-17=9$ we have:
j=7. [23+gis119+d:2]-[23-gis119-d:2]-1=17-6-1=10;
j=8. [24+gis72+d:2]-[24-gis72-d:2]-1=17-7-1=9;
j=9. [25+gis23+d:2]-[25-gis23-d:2]-1=15-10-1=4.

The obtained result is in full agreement with the previously obtained distribution without applying formulas (8) and (9) – (4+4+4+4+6+8+10+9+4=53).

The total quantity of the pairs of numbers satisfying the conditions of Task 1 can be determined from the following formula:

$$N_R = 4\left(4+\sum_{j=2}^{j_2} n_j + \sum_{j=j_2+1}^{j_4} n_j\right) - 4 = \sum_{j=2}^{j_2}(t_1 - t_3 + t_4 - t_2 + 2) + \sum_{j=j_2+1}^{j_4}(t_4 - t_3 + 1) + 12, \quad (10)$$

where t_1=[a-gis(a^2-2b)-d:2]-g; t_2=[a+gis(a^2-2b)+d:2]+g; t_3=[a-gis(a^2+8R-2b)-d:2]+1; t_4=[a+gis(a^2+8R-2b)+d:2]-1.

The obtained expression for N_R is the numerical sequence, the (R+1)-th term of whose is the total quantity of the pairs of numbers satisfying the conditions of task for the given R. The first 25 terms of this sequence are equal:
1,8,20,32,40,60,64,80,100,108, 120,120,140,168,168,180,180,208,216,236,240,256,276,280,288.

The numerical values of the quantity of pairs N and their distribution along orbits j for R values from 0 to 25 are listed in the Table 3.

Table 3. The Quantity of Pairs N and their Distribution along Orbits j for R Values from 0 to 25.

R	N	j	n_j											
			$n_{j=1}$	$n_{j=2}$	$n_{j=3}$	$n_{j=4}$	$n_{j=5}$	$n_{j=6}$	$n_{j=7}$	$n_{j=8}$	$n_{j=9}$	$n_{j=10}$	$n_{j=11}$	$n_{j=12}$
0	1	1	1											
1	8	2	4	4										
2	20	3	8	8	4									
3	32	3	12	12	8									
4	40	3	12	16	12									
5	60	4	12	20	16	12								
6	64	4	12	16	20	16								
7	80	5	12	16	24	20	8							
8	100	5	12	16	28	24	20							
9	108	6	12	16	24	28	24	4						
10	120	6	12	16	16	32	28	16						
11	120	6	12	16	16	24	32	20						
12	140	7	12	16	16	24	36	24	12					
13	168	7	12	16	16	24	40	36	24					
14	168	8	12	16	16	16	32	40	28	8				
15	180	8	12	16	16	16	24	44	32	20				
16	180	9	12	16	16	16	24	32	36	24	4			
17	208	9	12	16	16	16	24	32	40	36	16			
18	216	9	12	16	16	16	24	24	48	40	28			
19	236	10	12	16	16	16	24	24	40	44	32	12		
20	240	10	12	16	16	16	16	24	28	48	40	24		
21	256	11	12	16	16	16	16	24	32	32	48	36	8	
22	276	11	12	16	16	16	16	24	32	32	52	40	20	
23	280	11	12	16	16	16	16	24	24	24	48	44	40	
24	288	12	12	16	16	16	16	24	24	24	40	48	36	16
25	312	12	12	16	16	16	16	24	24	32	32	48	48	28

Thus, **the quantity has been specified, and all pairs of numbers that satisfy the conditions of Task 1 are listed**.

The totality of quanta whose coordinates are the solutions of Task 1 for R=17, in the form of the circular passage of radius R=17, is shown graphically in Fig.6.

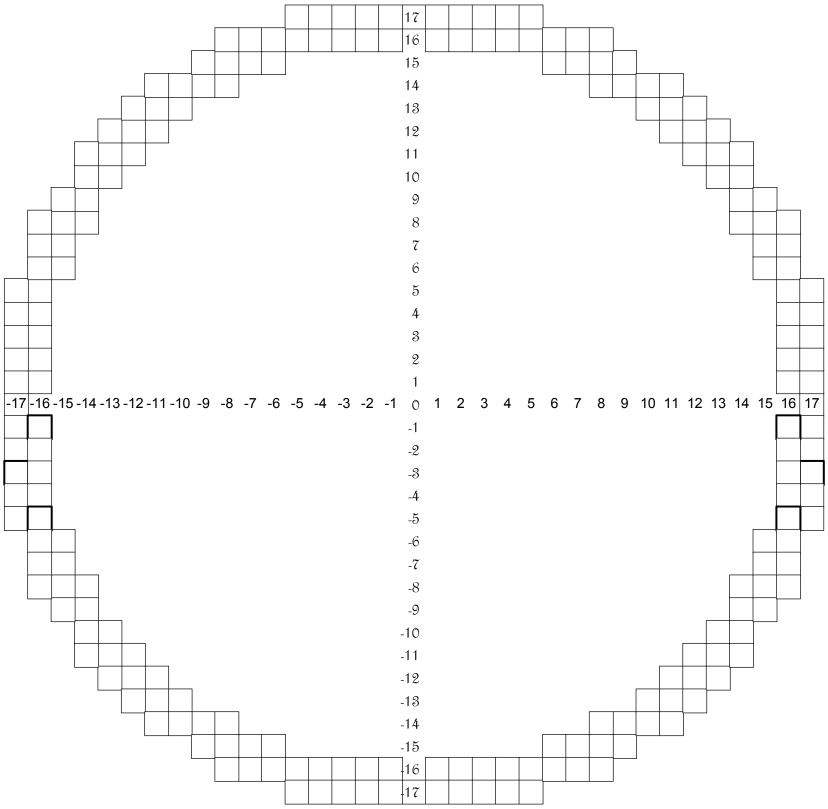

Fig.6. The Circular Passage of Radius R=17.

<u>Definition 21 (the circular passage)</u>: The totality of the quanta of space lying in a given coordinate layer, the coordinates (x_i,y_i), $i=0\div n$, $n \in N$ of whose satisfy the following conditions:

$$\begin{cases} (R-1)^2<x_i^2+y_i^2<(R+1)^2 \\ -R\leqslant x_i\leqslant R \\ -R\leqslant y_i\leqslant R, \text{ where } R\geqslant 0, \end{cases}$$

forms **the circular passage of radius R** with the center at the origin of coordinates, briefly **the C-passage(R)**. The coordinates of the quanta of C-passage(R) can be specified as follows way:

$$\begin{cases} (\cos_a t, \sin_a t) \\ (\mathrm{ver}_a t, \sin_a t),\ t\neq R \text{ for } j=1 \\ (\mathrm{ver}_a t, \mathrm{ops}_a t),\ t\neq 0 \text{ for } j=1 \\ (\cos_a t, \mathrm{ops}_a t),\ t\neq 0;R \text{ for } j=1, \text{ where } 0\leqslant t\leqslant a,\ a=R+j-1,\ R\geqslant 0, \end{cases}$$

j runs through the values from j_1 to j_4, which are determined from the conditions –

$$\begin{cases} j_1=1\leqslant j\leqslant \mathrm{gis}(2(R-1)^2)-R=j_2 \\ j_3=2\leqslant j\leqslant \mathrm{gis}(2(R+1)^2)-R=j_4, \end{cases}$$

$b=j^2+2Rj-2j$, t – the solutions of inequality $-b<2t(t-a)<4R-b$ for the first quarter, which are determined from the conditions –

$$\begin{cases} \begin{cases} 0\leqslant t\leqslant[(a-\mathrm{gis}(a^2-2b)-d:2]-g \\ [a+\mathrm{gis}(a^2-2b)+d:2]+g\leqslant t\leqslant a \\ [a-\mathrm{gis}(a^2+8R-2b)-d:2]+1\leqslant t\leqslant[(a+\mathrm{gis}(a^2+8R-2b)+d:2]-1 \end{cases} \\ 0\leqslant t\leqslant a, \text{ если } a<\mathrm{gis}(a^2+8R-2b). \end{cases}$$

C-passage(R) is planar, two-dimensional-limited, closed passage.

It necessary be noted that the C-passage(R) is the likeness of a circumference in the Quantum Discrete Space. Of course, it is much easier to draw a circumference with the help of compasses in the Euclidean Space of points, lines and planes. But this is an abstraction. Here we are talking about reality.

The obtained solution for a circular "layer", or more precisely for a circular passage, makes it possible, if necessary, to determine the number of quanta that are inside the circular passage, that is, to solve the Task 1 by the replacing of basic condition on following: $x_i^2+y_i^2<(R+1)^2$. But within the framework of this article there is no such necessity, therefore I will only indicate one of the possible approaches to solving this task, which, in essence, is the task about solving the Gauss's problem in the circle.

Here it suffices to note that a circular passage of radius R-2, the quanta's coordinates of whose satisfy the conditions of Task 1, during replacing the basic strict inequality $(R-1)^2<x_i^2+y_i^2<(R+1)^2$ on the non-strict one – $(R-1)^2\leqslant x_i^2+y_i^2\leqslant(R+1)^2$, briefly **C-passag(⩽R-2)**, without overlap, from the inside, closely adjoins to the C-passage (R), and the C-passage(R-4) in the same way without overlap, from the inside, closely adjoins to the C-passage(⩽R-2), etc. Thus, alternating, the corresponding C-passages completely fill all the internal quanta of C-passage(R), and the task reduces to the corresponding summation.

Let $R=2k+d(R)$, where $k>n\geqslant 0$, $n\in N$, $d(R)=0$ (if R is even), $d(R)=1$ (if R is odd), $n=0\div[k:2]+d(k)$, $d(k)=0$ (if k is even), $d(k)=1$(if k is odd), and let N_{R-4n} be the quantity of quanta of the corresponding C-passages(R-4n), and $\overline{N}_{R-2(2n+1)}$ be the quantity of quanta of the corresponding C-passages(⩽R-2(2n+1)). Then the quantity N of the solutions of Task 1 for the condition $x_i^2+y_i^2<(R+1)^2$ is equals:

$$N=\sum_{n=0}^{[k:2]} N_{R-4n}+\sum_{n=0}^{[k:2]+d(k)} \overline{N}_{R-2(2n+1)}. \qquad (11)$$

But there is one feature. If $R=1+4k$, where $k\geqslant 1$, and $n=0\div k-1$, then formula 11 slightly changes:

$$N=\sum_{n=0}^{k-1} N_{R-4n}+\sum_{n=0}^{k-1} \overline{N}_{R-2(2n+1)}+1. \qquad (12)$$

3. The Solution of Task 2

We shall solve the Task 2 in the Cartesian coordinate system. Then the triple of numbers (x_i,y_i,z_i) can be considered as the coordinates of the corresponding quantum of space. The solutions of task will be distributed over the j orbits, which are the T-octahedrons of the radius R+j-1. We will seek the solutions located in the first octant, with using of the coordinate package XY(z=0÷R), so how the other solutions will easily flow out from the obtained solutions. The coordinates of any quantum of the totality of T-octahedrons located in the first octant can be represented in the following form: $(\cos_c t_z, \sin_c t_z, z)$, where $c=R+j-1-z$, $z=0\div R$, $0\leqslant t_z\leqslant c$, $1\leqslant j<R$. Then the main condition of task can be written as follows way:

$$(R-1)^2<\cos^2_c t_z+\sin^2_c t_z+z^2<(R+1)^2 \quad \text{or}$$
$$(R-1)^2<(c-t_z)^2+t_z^2+z^2<(R+1)^2.$$

After the disclosure of brackets and the adduction of like summands, we obtain:

$$2cz-b<2t_z(t_z-c)<4R+2cz-b, \text{ где } c=R+j-1-z=a-z,\ b=j^2+2Rj-2j. \qquad (13)$$

Inequality (13) is equipollent to the following system of inequalities:

$$\begin{cases} 2t_z^2-2ct_z-(2cz-b)>0 & (14) \\ 2t_z^2-2ct_z-(4R+2cz-b)<0. & (15) \end{cases}$$

Inequality (14) is equipollent to the following system of inequalities:

$$\left[\begin{array}{l} 0\le t_z<½(c-\sqrt{c^2-2b+4cz}) \\ ½(c+\sqrt{c^2-2b+4cz}<t_z\le c \end{array}\right. \quad \text{or} \quad \left[\begin{array}{l} 0\leqslant t_z<½(c-\mathrm{gis}(c^2-2b+4cz)+1) \\ ½(c+\mathrm{gis}(c^2-2b+4cz)-1)<t_z\leqslant c. \end{array}\right.$$

Inequality (15) is equipollent to the following double inequality:

$$½\left(c-\sqrt{c^2+4R-2b+4cz}\right)<t_z<½\left(c+\sqrt{c^2+4R-2b+4cz}\right)\qquad \text{or}$$

$$½(c\text{-gis}(c^2+8R-2b))<t_z<½(c+\text{gis}(c^2+8R-2b)).$$

Taking into account the reasoning given during solved of the Task 1, we write the obtained result in the following form:

$$\begin{cases}\left[\begin{array}{l}0\leqslant t_z\leqslant[c-\text{gis}(c^2-2b+4cz)-d:2]-g\\ [c+\text{gis}(c^2-2b+4cz)+d:2]+g\leqslant t_z\leqslant c\end{array}\right.\\ [c-\text{gis}(c^2+8R-2b+4cz)-d:2]+1\leqslant t_z\leqslant [c+\text{gis}(c^2+8R-2b+4cz)+d:2]-1\\ 0\leqslant t_z\leqslant c,\ \text{if } c<\text{gis}(c^2+8R-2b+4cz).\end{cases}\qquad (16)$$

Here $F(x_i)=c\pm\text{gis}(c^2-2b+4cz)$ or $c\pm\text{gis}(c^2+8R-2b+4cz)$, and $f(x_i)=c^2-2b+4cz$, $i=1\div3$.

The conditions from which the values of j can be determined have the follows view:

$$\left[\begin{array}{l}0\leqslant c^2-2b+4cz\leqslant c^2\\ 0\leqslant c^2+8R-2b+4cz\leqslant c^2.\end{array}\right.$$

Substituting in this system the values of c and b, we obtain:

$$\left[\begin{array}{l}\begin{cases}j^2+2(R-1-z)j-(R^2-2R+1+2Rz-2z-3z^2)\leqslant0\\ j^2+2(R-1-z)j+2(z^2-Rz+z)\geqslant0\end{cases}\\ \begin{cases}j^2+2(R-1-z)j-(R^2+6R+1+2Rz-2z-3z^2)\leqslant0\\ j^2+2(R-1-z)j+2(z^2-Rz-2R+z)\geqslant0.\end{cases}\end{array}\right.$$

The solution of the obtained system has the follows view:

$$\left[\begin{array}{l}j_{1z}=\text{gis}((R-1)^2-z^2)-R+z+1+g((R-1)^2-z^2)\leqslant j\leqslant\text{gis}(2(R-1)^2-2z^2)-R+z=j_{2z},\\ j_{3z}=\text{gis}((R+1)^2-z^2)-R+z+1\leqslant j\leqslant\text{gis}(2(R+1)^2-2z^2)-R+z=j_{4z}.\end{array}\right.\qquad (17)$$

The required triples of numbers have the follows view: $(\cos_c t_z, \sin_c t_z, z)$, where $0\leqslant t_z\leqslant c$ are the solutions of system (16), $z=0\div R$, and the corresponding values of j are determined from the system (17).

We will consider the specific example for R=7.

1. $z=0$, $j_1=1\leqslant j\leqslant2=j_2$, $j_3=2\leqslant j\leqslant5=j_4$, $c=6+j$, $b=j^2+14j-2j$. The system (16) takes the form:

$$\begin{cases}\left[\begin{array}{l}0\leqslant t_0\leqslant [c-\text{gis}(c^2-2b)-d:2]-g\\ [c+\text{gis}(c^2-2b)+d:2]+g\leqslant t_0\leqslant c\end{array}\right.\\ [c-\text{gis}(c^2+56-2b)-d:2]+1\leqslant t_0\leqslant [c+\text{gis}(c^2+56-2b)+d:2]-1\\ 0\leqslant t_0\leqslant c,\ \text{if } c<\text{gis}(c^2+56-2b).\end{cases}$$

1.1. $j=1 \Rightarrow c=7, b=13.$

$$\begin{cases}\left[\begin{array}{l}0\leqslant t_0\leqslant [7-\text{gis}23-d:2]+0=[7-5-0:2]=1\\ [7+5+0:2]=6\leqslant t_0\leqslant7\end{array}\right.\\ 7<\text{gis}79=9 \Rightarrow 0\leqslant t_0\leqslant7,\end{cases}\qquad\qquad \begin{cases}\left[\begin{array}{l}0\leqslant t_0\leqslant1\\ 6\leqslant t_0\leqslant7\end{array}\right.\\ 0\leqslant t_0\leqslant7.\end{cases}$$

We have 4 solutions: $t_0=0;1;6;7$, and hence 4 triples:

$(\cos_7 0,\sin_7 0,0)=(7,0,0)$; $(\cos_7 1,\sin_7 1,0)=(6,1,0)$;
$(\cos_7 6,\sin_7 6,0)=(1,6,0)$; $(\cos_7 7,\sin_7 7,0)=(0, 7,0)$.

1.2. $j=2 \Rightarrow c=8, b=28.$

$$\begin{cases}\left[\begin{array}{l}0\leqslant t_0\leqslant [8-\text{gis}8-d:2]-0=[8-3-1:2]=2\\ [8+3+1:2]=6\leqslant t_0\leqslant8\end{array}\right.\\ [8-\text{gis}64-d:2]+1=[8-8-0:2]+1=1\leqslant t_0\leqslant [8+8+0:2]-1=7,\end{cases}\qquad\qquad \begin{cases}\left[\begin{array}{l}0\leqslant t_0\leqslant2\\ 6\leqslant t_0\leqslant8\end{array}\right.\\ 1\leqslant t_0\leqslant7.\end{cases}$$

We have 4 solutions: $t_0=1;2;6;7$, and hence 4 triples:

$(\cos_8 1,\sin_8 1,0)=(7,1,0)$; $(\cos_8 2,\sin_8 2,0)=(6,2,0)$;
$(\cos_8 6,\sin_8 6,0)=(2,6,0)$; $(\cos_8 7,\sin_8 7,0)=(1,7,0)$.

1.3. $j=3 \Rightarrow c=9, b=45.$

$$[9-\text{gis}47-d:2]+1=[9-7-0:2]+1=2\leqslant t_0\leqslant[9+7+0:2]-1=7,\qquad\qquad 2\leqslant t_0\leqslant7.$$

We have 6 solutions: $t_0=2;3;4;5;6;7$, and hence 6 triples:

$(\cos_9 2,\sin_9 2,0)=(7,2,0)$; $(\cos_9 3,\sin_9 3,0)=(6,3,0)$; $(\cos_9 4,\sin_9 4,0)=(5,4,0)$;
$(\cos_9 5,\sin_9 5,0)=(4,5,0)$; $(\cos_9 6,\sin_9 6,0)=(3,6,0)$; $(\cos_9 7,\sin_9 7,0)=(2,7,0)$.

1.4. $j=4 \Rightarrow c=10,\ b=64.$

$[10-gis28-d:2]+1=[10-6-0:2]+1=3\leqslant t_0\leqslant [10+6+0:2]-1=7, \qquad 3\leqslant t_0\leqslant 7.$

We have 5 solutions: $t_0=3;4;5;6;7$, and hence 5 triples:

$(\cos_{10}3,\sin_{10}3,0)=(7,3,0)$; $(\cos_{10}4,\sin_{10}4,0)=(6,4,0)$; $(\cos_{10}5,\sin_{10}5,0)=(5,5,0)$;
$(\cos_{10}6,\sin_{10}6,0)=(4,6,0)$; $(\cos_{10}7,\sin_{10}7,0)=(3,7,0)$.

1.5. $j=5 \Rightarrow c=11,\ b=85.$

$[11-gis7-d:2]+1=[11-3-0:2]+1=5\leqslant t_0\leqslant [11+3+0:2]-1=6, \qquad 5\leqslant t_0\leqslant 6.$

We have 2 solutions: $t_0=5;6$, and hence 2 triples.

$(\cos_{11}5,\sin_{11}5,0)=(6,5,0)$; $(\cos_{11}6,\sin_{11}6,0)=(5,6,0)$.

Total, we have 21 triples for z=0.

2. $z=1$, $j_1=1\leqslant j\leqslant 3=j_2$, $j_3=3\leqslant j\leqslant 6=j_4$, $c=5+j$, $b=j^2+14j-2j$. The system (16) takes the form:

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t_1\leqslant [c-gis(c^2-2b+4a)-d:2]-g \\ [c+gis(c^2-2b+4c)+d:2]+g\leqslant t_1\leqslant c \end{array}\right. \\ [c-gis(c^2+56-2b+4c)-d:2]+1\leqslant t_1\leqslant [c+gis(c^2+56-2b+4c)+d:2]-1 \\ 0\leqslant t_1\leqslant c, \text{ if } c<gis(c^2+56-2b+4c). \end{cases}$$

2.1. $j=1 \Rightarrow c=6,\ b=13.$

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t_1\leqslant [6-gis34-d:2]-0=[6-6-0:2]=0 \\ [6+6+0:2]=6\leqslant t_1\leqslant 6 \end{array}\right. \\ 6<gis90=10 \Rightarrow 0\leqslant t_1\leqslant 6, \end{cases} \qquad \begin{cases} \left[\begin{array}{l} 0\leqslant t_1\leqslant 0 \\ 6\leqslant t_1\leqslant 6 \end{array}\right. \\ 0\leqslant t_1\leqslant 6. \end{cases}$$

We have 2 solutions: $t_0=0;6$, and hence 2 triples:

$(\cos_6 0,\sin_6 0,1)=(6,0,1)$; $(\cos_6 6,\sin_6 6,1)=(0,6,1)$.

2.2. $j=2 \Rightarrow c=7,\ b=28.$

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t_1\leqslant [7-gis21-d:2]-0=[7-5-0:2]=1 \\ [7+5+0:2]=6\leqslant t_1\leqslant 7 \end{array}\right. \\ 7<gis77=9 \Rightarrow 0\leqslant t_1\leqslant 7, \end{cases} \qquad \begin{cases} \left[\begin{array}{l} 0\leqslant t_1\leqslant 1 \\ 6\leqslant t_1\leqslant 7 \end{array}\right. \\ 0\leqslant t_1\leqslant 7. \end{cases}$$

We have 4 solutions: $t_0=0;1;6;7$, and hence 4 triples:

$(\cos_7 0,\sin_7 0,1)=(7,0,1)$; $(\cos_7 1,\sin_7 1,1)=(6,1,1)$;
$(\cos_7 6,\sin_7 6,1)=(1,6,1)$; $(\cos_7 7,\sin_7 7,1)=(0,7,1)$.

2.3. $j=3 \Rightarrow c=8,\ b=45.$

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t_1\leqslant [8-gis6-d:2]-0=[8-3-1:2]=2 \\ [8+3+1:2]=6\leqslant t_1\leqslant 8 \end{array}\right. \\ [8-gis62-d:2]+1=[8-8-0:2]+1=1\leqslant t_1\leqslant [8+8+0:2]-1=7, \end{cases} \qquad \begin{cases} \left[\begin{array}{l} 0\leqslant t_1\leqslant 2 \\ 6\leqslant t_1\leqslant 8 \end{array}\right. \\ 1\leqslant t_1\leqslant 7. \end{cases}$$

We have 4 solutions: $t_0=1;2;6;7$, and hence 4 triples:

$(\cos_8 1,\sin_8 1,1)=(7,1,1)$; $(\cos_8 2,\sin_8 2,1)=(6,2,1)$;
$(\cos_8 6,\sin_8 6,1)=(2,6,1)$; $(\cos_8 7,\sin_8 7,1)=(1,7,1)$.

2.4. $j=4 \Rightarrow c=9,\ b=64.$

$[9-gis45-d:2]+1=[9-7-0:2]+1=2\leqslant t_1\leqslant [9+7+0:2]-1=7, \qquad 2\leqslant t_1\leqslant 7.$

We have 6 solutions: $t_1=2;3;4;5;6;7$, and hence 6 triples:

$(\cos_9 2,\sin_9 2,1)=(7,2,1)$; $(\cos_9 3,\sin_9 3,1)=(6,3,1)$; $(\cos_9 4,\sin_9 4,1)=(5,4,1)$;
$(\cos_9 5,\sin_9 5,1)=(4,5,1)$; $(\cos_9 6,\sin_9 6,1)=(3,6,1)$; $(\cos_9 7,\sin_9 7,1)=(2,7,1)$.

2.5. $j=5 \Rightarrow c=10,\ b=85.$

$[10-gis26-d:2]+1=[10-6-0:2]+1=3\leqslant t_1\leqslant [10+6+0:2]-1=7, \qquad 3\leqslant t_1\leqslant 7.$

We have 5 solutions: $t_1=3;4;5;6;7$, and hence 5 triples:

$(\cos_{10}3,\sin_{10}3,1)=(7,3,1)$; $(\cos_{10}4,\sin_{10}4,1)=(6,4,1)$; $(\cos_{10}5,\sin_{10}5,1)=(5,5,1)$;
$(\cos_{10}6,\sin_{10}6,1)=(4,6,1)$; $(\cos_{10}7,\sin_{10}7,1)=(3,7,1)$.

2.6. $j=6 \Rightarrow c=11,\ b=108.$

$[11-gis5-d:2]+1=[11-3-0:2]+1=5\leqslant t_1\leqslant [11+3+0:2]-1=6, \qquad 5\leqslant t_1\leqslant 6.$

We have 2 solutions: $t_1=5;6$, and hence 2 triples:

$(\cos_{11}5,\sin_{11}5,1)=(6,5,1)$; $(\cos_{11}6,\sin_{11}6,1)=(5,6,1)$.

Total, we have 23 triples for z=1.

3. $z=2$, $j_1=2\leqslant j\leqslant 4=j_2$, $j_3=4\leqslant j\leqslant 6=j_4$, $c=4+j$, $b=j^2+14j-2j$. The system (16) takes the form:

$$\begin{cases} \left[\begin{array}{l} 0\leqslant t_2\leqslant [c-gis(c^2-2b+8c)-d:2]-g \\ [c+gis(c^2-2b+8c)+d:2]+g\leqslant t_2\leqslant c \end{array}\right. \\ [c-gis(c^2+56-2b+8c)-d:2]+1\leqslant t_2\leqslant [c+gis(c^2+56-2b+8c)+d:2]-1 \\ 0\leqslant t_2\leqslant c, \text{ if } c<gis(c^2+56-2b+8c). \end{cases}$$

3.1. $j=1 \Rightarrow c=5,\ b=13.$

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_2\leqslant [5-\mathrm{gis}39-d:2]-0=[5-7-d:2] \text{ – not determined}\\ [5+7+0:2]=6\leqslant t_2\leqslant 5 \text{ – no solutions}\end{array}\right.\\ 5<\mathrm{gis}95=10 \Rightarrow 0\leqslant t_2\leqslant 5,\end{array}\right. \qquad \left\{\begin{array}{l}\varnothing\\ 0\leqslant t_2\leqslant 5.\end{array}\right.$$

There are no solutions.

3.2. $j=2 \Rightarrow c=6,\ b=28.$

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_2\leqslant [6-\mathrm{gis}28-d:2]-0=[6-6-0:2]=0\\ [6+6+0:2]=6\leqslant t_2\leqslant 6\end{array}\right.\\ 6<\mathrm{gis}84=10 \Rightarrow 0\leqslant t_2\leqslant 6,\end{array}\right. \qquad \left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_2\leqslant 0\\ 6\leqslant t_2\leqslant 6\end{array}\right.\\ 0\leqslant t_2\leqslant 6.\end{array}\right.$$

We have 2 solutions: $t_2=0;6$, and hence 2 triples:
$(\cos_6 0,\sin_6 0,2)=(6,0,2)$; $(\cos_6 6,\sin_6 6,2)=(0,6,2)$.

3.3. $j=3 \Rightarrow c=7,\ b=45.$

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_2\leqslant [7-\mathrm{gis}15-d:2]-0=[7-4-1:2]=1\\ [7+4+1:2]=6\leqslant t_2\leqslant 7\end{array}\right.\\ [7<\mathrm{gis}71=9 \Rightarrow 0\leqslant t_2\leqslant 7,\end{array}\right. \qquad \left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_2\leqslant 1\\ 6\leqslant t_2\leqslant 7\end{array}\right.\\ 0\leqslant t_2\leqslant 7.\end{array}\right.$$

We have 4 solutions: $t_2=0;1;6;7$, and hence 4 triples:
$(\cos_7 0,\sin_7 0,2)=(7,0,2)$; $(\cos_7 1,\sin_7 1,2)=(6,1,2)$;
$(\cos_7 6,\sin_7 6,2)=(1,6,2)$; $(\cos_7 7,\sin_7 7,2)=(0,7,2)$.

3.4. $j=4 \Rightarrow c=8,\ b=64.$

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_2\leqslant [8-\mathrm{gis}0-d:2]-1=[8-0-0:2]-1=3\\ [8+0+0:2]+1=5\leqslant t_2\leqslant 8\end{array}\right.\\ [8-\mathrm{gis}56-d:2]+1=[8-8-0:2]+1=1\leqslant t_2\leqslant [8+8+0:2]-1=7,\end{array}\right. \qquad \left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_2\leqslant 3\\ 5\leqslant t_2\leqslant 8\end{array}\right.\\ 1\leqslant t_2\leqslant 7.\end{array}\right.$$

We have 6 solutions: $t_2=1;2;3;5;6;7$, and hence 6 triples:
$(\cos_8 1,\sin_8 1,2)=(7,1,2)$; $(\cos_8 2,\sin_8 2,2)=(6,2,2)$; $(\cos_8 3,\sin_8 3,2)=(5,3,2)$;
$(\cos_8 5,\sin_8 5,2)=(3,5,2)$; $(\cos_8 6,\sin_8 6,2)=(2,6,2)$. $(\cos_8 7,\sin_8 7,2)=(1,7,2)$.

3.5. $j=5 \Rightarrow c=9,\ b=85.$

$$[9-\mathrm{gis}39-d:2]+1=[9-7-0:2]+1=2\leqslant t_2\leqslant [9+7+0:2]-1=7, \qquad 2\leqslant t_2\leqslant 7.$$

We have 6 solutions: $t_2=2;3;4;5;6;7$, and hence 6 triples:
$(\cos_9 2,\sin_9 2,2)=(7,2,2)$; $(\cos_9 3,\sin_9 3,2)=(6,3,2)$; $(\cos_9 4,\sin_9 4,2)=(5,4,2)$;
$(\cos_9 5,\sin_9 5,2)=(4,5,2)$; $(\cos_9 6,\sin_9 6,2)=(3,6,2)$; $(\cos_9 7,\sin_9 7,2)=(2,7,2)$.

3.6. $j=6 \Rightarrow c=10,\ b=108.$

$$[10-\mathrm{gis}20-d:2]+1=[10-5-1:2]+1=3\leqslant t_2\leqslant [10+5+1:2]-1=7, \qquad 3\leqslant t_2\leqslant 7.$$

We have 5 solutions: $t_2=3;4;5;6;7$, and hence 5 triples:
$(\cos_{10} 3,\sin_{10} 3,2)=(7,3,2)$; $(\cos_{10} 4,\sin_{10} 4,2)=(6,4,2)$; $(\cos_{10} 5,\sin_{10} 5,2)=(5,5,2)$;
$(\cos_{10} 6,\sin_{10} 6,2)=(4,6,2)$; $(\cos_{10} 7,\sin_{10} 7,2)=(3,7,2)$.
Total, we have 23 triples for $z=2$.

4. $z=3$, $j_1=3\leqslant j\leqslant 4=j_2$, $j_3=5\leqslant j\leqslant 7=j_4$, $c=3+j$, $b=j^2+14j-2j$. The system (16) takes the form:

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_3\leqslant [c-\mathrm{gis}(c^2-2b+12c)-d:2]-g\\ [c+\mathrm{gis}(c^2-2b+12c)+d:2]+g\leqslant t_3\leqslant c\end{array}\right.\\ [c-\mathrm{gis}(c^2+56-2b+12c)-d:2]+1\leqslant t_3\leqslant [c+\mathrm{gis}(c^2+56-2b+12c)+d:2]-1\\ 0\leqslant t_3\leqslant c, \text{ if } c<\mathrm{gis}(c^2+56-2b+12c).\end{array}\right.$$

4.1. $j=1 \Rightarrow c=4,\ b=13.$

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_3\leqslant [4-\mathrm{gis}38-d:2]-0=[4-7-d:2] \text{ – not determined}\\ [4+7+1:2]=6\leqslant t_3\leqslant 4 \text{ – no solutions}\end{array}\right.\\ 4<\mathrm{gis}94=10 \Rightarrow 0\leqslant t_3\leqslant 4,\end{array}\right. \qquad \left\{\begin{array}{l}\varnothing\\ 0\leqslant t_3\leqslant 4.\end{array}\right.$$

There are no solutions.

4.2. $j=2 \Rightarrow c=5,\ b=28.$

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_3\leqslant [5-\mathrm{gis}29-d:2]-0=[5-6-d:2] \text{ – not determined}\\ [5+6+1:2]=6\leqslant t_3\leqslant 5 \text{ – no solutions}\end{array}\right.\\ 5<\mathrm{gis}85=10\Rightarrow 0\leqslant t_3\leqslant 5,\end{array}\right. \qquad \left\{\begin{array}{l}\varnothing\\ 0\leqslant t_3\leqslant 5.\end{array}\right.$$

There are no solutions.

4.3. $j=3\Rightarrow c=6,\ b=45$.

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_3\leqslant [6\text{-gis}18\text{-d}:2]-0=[6-5-1:2]=0\\ {[6+5+1:2]=6\leqslant t_3\leqslant 6}\end{array}\right.\\ 6<\text{gis}74=8\Rightarrow 0\leqslant t_3\leqslant 6,\end{array}\right. \qquad \left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_3\leqslant 0\\ 6\leqslant t_3\leqslant 6\end{array}\right.\\ 0\leqslant t_3\leqslant 6.\end{array}\right.$$

We have 2 solutions: $t_3=0;6$, and hence 2 triples:
$(\cos_6 0,\sin_6 0,3)=(6,0,3)$; $(\cos_6 6,\sin_6 6,3)=(0,6,3)$.

4.4. $j=4\Rightarrow c=7,\ b=64$.

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_3\leqslant [7\text{-gis}5\text{-d}:2]-0=[7-3-0:2]=2\\ {[7+3+0:2]=5\leqslant t_3\leqslant 7}\end{array}\right.\\ 7<\text{gis}61=8\Rightarrow 0\leqslant t_3\leqslant 7,\end{array}\right. \qquad \left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_3\leqslant 2\\ 5\leqslant t_3\leqslant 7\end{array}\right.\\ 0\leqslant t_3\leqslant 7.\end{array}\right.$$

We have 6 solutions: $t_3=0;1;2;5;6;7$, and hence 6 triples:
$(\cos_7 0,\sin_7 0,3)=(7,0,3)$; $(\cos_7 1,\sin_7 1,3)=(6,1,3)$; $(\cos_7 2,\sin 2,3)=(5,2,3)$;
$(\cos_7 5,\sin_7 5,3)=(2,5,3)$; $(\cos_7 6,\sin_7 6,3)=(1,6,3)$; $(\cos_7 7,\sin_7 7,3)=(0,7,3)$.

4.5. $j=5\Rightarrow c=8,\ b=85$.

$[8\text{-gis}46\text{-d}:2]+1=[8-7-1:2]+1=1\leqslant t_3\leqslant [8+7+1:2]-1=7,\qquad 1\leqslant t_3\leqslant 7.$

We have 7 solutions: $t_3=1;2;3;4;5;6;7$, and hence 7 triples:
$(\cos_8 1,\sin_8 1,3)=(7,1,3)$; $(\cos_8 2,\sin_8 2,3)=(6,2,3)$; $(\cos_8 3,\sin_8 3,3)=(5,3,3)$;
$(\cos_8 4,\sin_8 4,3)=(4,4,3)$; $(\cos_8 5,\sin_8 5,3)=(3,5,3)$; $(\cos_8 6,\sin_8 6,3)=(2,6,3)$;
$(\cos_8 7,\sin_8 7,3)=(1,7,3)$.

4.6. $j=6\Rightarrow c=9,\ b=108$.

$[9\text{-gis}29\text{-d}:2]+1=[9-6-1:2]+1=2\leqslant t_3\leqslant [9+6+1:2]-1=7,\qquad 2\leqslant t_3\leqslant 7.$

We have 6 solutions: $t_3=2;3;4;5;6;7$, and hence 6 triples:
$(\cos_9 2,\sin_9 2,3)=(7,2,3)$; $(\cos_9 3,\sin_9 3,3)=(6,3,3)$; $(\cos_9 4,\sin_9 4,3)=(5,4,3)$;
$(\cos_9 5,\sin_9 5,3)=(4,5,3)$; $(\cos_9 6,\sin_9 6,3)=(3,6,3)$; $(\cos_9 7,\sin_9 7,3)=(2,7,3)$.

4.7. $j=7\Rightarrow c=10,\ b=133$.

$[10\text{-gis}10\text{-d}:2]+1=[10-4-0:2]+1=4\leqslant t_3\leqslant [10+4+0:2]-1=6,\qquad 4\leqslant t_3\leqslant 6.$

We have 3 solutions: $t_3=4;5;6$, and hence 3 triples:
$(\cos_{10} 4,\sin_{10} 4,3)=(6,4,3)$; $(\cos_{10} 5,\sin_{10} 5,3)=(5,5,3)$; $(\cos_{10} 6,\sin_{10} 6,3)=(4,6,3)$.
Total, we have 24 triples for $z=3$.

5. $z=4$, $j_1=3\leqslant j\leqslant 4=j_2$, $j_3=5\leqslant j\leqslant 7=j_4$, $c=2+j$, $b=j^2+14j-2j$. The system (16) takes the form:

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_4\leqslant [c\text{-gis}(c^2-2b+16c)\text{-d}:2]-g\\ {[c+\text{gis}(c^2-2b+16c)+d:2]+g\leqslant t_4\leqslant c}\end{array}\right.\\ {[c\text{-gis}(c^2+56-2b+16c)\text{-d}:2]+1\leqslant t_4\leqslant [c+\text{gis}(c^2+56-2b+16c)+d:2]-1}\\ 0\leqslant t_4\leqslant c,\ \text{if } c<\text{gis}(c^2+56-2b+16c).\end{array}\right.$$

5.1. $j=1\Rightarrow c=3,\ b=13$.

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_4\leqslant [3\text{-gis}31\text{-d}:2]-0=[3-6\text{-d}:2] \text{ – not determined}\\ {[3+6+1:2]=5\leqslant t_4\leqslant 3} \text{ – no solutions}\end{array}\right.\\ 3<\text{gis}87=10\Rightarrow 0\leqslant t_4\leqslant 3,\end{array}\right. \qquad \left\{\begin{array}{l}\varnothing\\ 0\leqslant t_4\leqslant 3.\end{array}\right.$$

There are no solutions.

5.2. $j=2\Rightarrow c=4,\ b=28$.

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_4\leqslant [4\text{-gis}24\text{-d}:2]-0=[4-5\text{-d}:2] \text{ – not determined}\\ {[4+5+1:2]=5\leqslant t_4\leqslant 4} \text{ – no solutions}\end{array}\right.\\ 4<\text{gis}80=9\Rightarrow 0\leqslant t_4\leqslant 4,\end{array}\right. \qquad \left\{\begin{array}{l}\varnothing\\ 0\leqslant t_4\leqslant\leqslant 4.\end{array}\right.$$

There are no solutions.

5.3. $j=3\Rightarrow c=5,\ b=45$.

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_4\leqslant [5\text{-gis}15\text{-d}:2]-0=[5-4-1:2]=0\\ {[5+4+1:2]=5\leqslant t_4\leqslant 5}\end{array}\right.\\ 5<\text{gis}71=9\Rightarrow 0\leqslant t_4\leqslant 5,\end{array}\right. \qquad \left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_4\leqslant 0\\ 5\leqslant t_4\leqslant 5\end{array}\right.\\ 0\leqslant t_4\leqslant 5.\end{array}\right.$$

We have 2 solutions: $t_4=0;5$, and hence 2 triples:
$(\cos_5 0,\sin_5 0,4)=(5,0,4)$; $(\cos_5 5,\sin_5 5,4)=(0,5,4)$.

5.4. $j=4 \Rightarrow c=6,\ b=64$.

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_4\leqslant[6-\mathrm{gis}4-d:2]-1=[6-2-0:2]-1=1\\ {[}6+2+0:2]+1=5\leqslant t_4\leqslant 6\end{array}\right.\\ 6<\mathrm{gis}60=8 \Rightarrow 0\leqslant t_4\leqslant 6,\end{array}\right. \qquad \left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_4\leqslant 1\\ 5\leqslant t_4\leqslant 6\end{array}\right.\\ 0\leqslant t_4\leqslant 6.\end{array}\right.$$

We have 4 solutions: $t_4=0;1;5;6$, and hence 4 triples:
$(\cos_6 0,\sin_6 0,4)=(6,0,4)$; $(\cos_6 1,\sin_6 1,4)=(5,1,4)$;
$(\cos_6 5,\sin_6 5,4)=(1,5,4)$; $(\cos_6 6,\sin_6 6,4)=(0,6,4)$.

5.5. $j=5 \Rightarrow c=7,\ b=85$.

$[7-\mathrm{gis}47-d:2]+1=[7-7-0:2]+1=1\leqslant t_4\leqslant[7+7+0:2]-1=6,$ $\qquad 1\leqslant t_4\leqslant 6.$

We have 6 solutions: $t_3=1;2;3;4;5;6$, and hence 6 triples:
$(\cos_7 1,\sin_7 1,4)=(6,1,4)$; $(\cos_7 2,\sin_7 2,4)=(5,2,4)$; $(\cos_7 3,\sin_7 3,4)=(4,3,4)$;
$(\cos_7 4,\sin_7 4,4)=(3,4,4)$; $\cos_7 5,\sin_7 5,4)=(2,5,4)$; $(\cos_7 6,\sin_7 6,4)=(1,6,4)$.

5.6. $j=6 \Rightarrow c=8,\ b=108$.

$[8-\mathrm{gis}32-d:2]+1=[8-6-0:2]+1=2\leqslant t_4\leqslant[8+6+0:2]-1=6,$ $\qquad 2\leqslant t_4\leqslant 6.$

We have 5 solutions: $t_4=2;3;4;5;6$, and hence 5 triples:
$(\cos_8 2,\sin_8 2,4)=(6,2,4)$; $(\cos_8 3,\sin_8 3,4)=(5,3,4)$; $(\cos_8 4,\sin_8 4,4)=(4,4,4)$;
$(\cos_8 5,\sin_8 5,4)=(3,5,4)$; $(\cos_8 6,\sin_8 6,4)=(2,6,4)$.

5.7. $j=7 \Rightarrow c=9,\ b=133$.

$[9-\mathrm{gis}15-d:2]+1=[9-4-1:2]+1=3\leqslant t_4\leqslant [9+4+1:2]-1=6,$ $\qquad 3\leqslant t_4\leqslant 6.$

We have 4 solutions: $t_4=3;4;5;6$, and hence 4 triples:
$(\cos_9 3,\sin_9 3,4)=(6,3,4)$; $(\cos_9 4,\sin_9 4,4)=(5,4,4)$;
$(\cos_9 5,\sin_9 5,4)=(4,5,4)$; $(\cos_9 6,\sin_9 6,4)=(3,6,4)$.

Total, we have 21 triples for $z=4$.

6. $z=5$, $j_1=3\leqslant j\leqslant 3=j_2$, $j_3=6\leqslant j\leqslant 7=j_4$, $c=1+j$, $b=j^2+14j-2j$. The system (16) takes the form:

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_5\leqslant[c-\mathrm{gis}(c^2-2b+20c)-d:2]-g\\ {[}c+\mathrm{gis}(c^2-2b+20c)+d:2]+g\leqslant t_5\leqslant c\end{array}\right.\\ {[}c-\mathrm{gis}(c^2+8R-2b+20c)-d:2]+1\leqslant t_5\leqslant[c+\mathrm{gis}(c^2+8R-2b+20c)+d:2]-1\\ 0\leqslant t_5\leqslant c,\ \text{if } a<\mathrm{gis}(c^2+56-2b+20c).\end{array}\right.$$

6.1. $j=1 \Rightarrow c=2,\ b=13$.

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_5\leqslant[2-\mathrm{gis}18-d:2]-0=[2-5-d:2] \text{ – not determined}\\ {[}2+5+1:2]=4\leqslant t_5\leqslant 2 \text{ – no solutions}\end{array}\right.\\ 2<\mathrm{gis}74=9 \Rightarrow 0\leqslant t_5\leqslant 2,\end{array}\right. \qquad \left\{\begin{array}{l}\varnothing\\ 0\leqslant t_5\leqslant 2.\end{array}\right.$$

There are no solutions.

6.2. $j=2 \Rightarrow c=3,\ b=28$.

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_5\leqslant[3-\mathrm{gis}13-d:2]-0=[3-4-d:2] \text{ – not determined}\\ {[}3+4+1:2]=4\leqslant t_5\leqslant 3 \text{ – no solutions}\end{array}\right.\\ 3<\mathrm{gis}69=9 \Rightarrow 0\leqslant t_5\leqslant 3,\end{array}\right. \qquad \left\{\begin{array}{l}\varnothing\\ 0\leqslant t_5\leqslant 3.\end{array}\right.$$

There are no solutions.

6.3. $j=3 \Rightarrow c=4,\ b=45$.

$$\left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_5\leqslant[4-\mathrm{gis}6-d:2]-0=[4-3-1:2]=0\\ {[}4+3+1:2]=4\leqslant t_5\leqslant 4\end{array}\right.\\ 4<\mathrm{gis}62=9 \Rightarrow 0\leqslant t_5\leqslant 4,\end{array}\right. \qquad \left\{\begin{array}{l}\left[\begin{array}{l}0\leqslant t_5\leqslant 0\\ 4\leqslant t_5\leqslant 4\end{array}\right.\\ 0\leqslant t_5\leqslant 4.\end{array}\right.$$

We have 2 solutions: $t_5=0;4$, and hence 4 triples:
$(\cos_4 0,\sin_4 0,5)=(4,0,5)$; $(\cos_4 4,\sin_4 4,5)=(0,4,5)$.

6.4. $j=4 \Rightarrow c=5,\ b=64$.

$5<\mathrm{gis}53=8 \Rightarrow 0\leqslant t_5\leqslant 5,$ $\qquad 0\leqslant t_5\leqslant 5.$

We have 6 solutions: $t_5=0;1;2;3;4;5;6$, and hence 6 triples:
$(\cos_5 0,\sin_5 0,5)=(5,0,5)$; $(\cos_5 1,\sin_5 1,5)=(4,1,5)$; $(\cos_5 2,\sin_5 2,5)=(3,2,5)$;
$(\cos_5 3,\sin_5 3,5)=(2,3,5)$; $(\cos_5 4,\sin_5 4,5)=(1,4,5)$; $(\cos_5 5,\sin_5 5,5)=(0,5,5)$.

6.5. $j=5 \Rightarrow c=6,\ b=85$.

$6<\mathrm{gis}42=7 \Rightarrow 0\leqslant t_5\leqslant 6,$ $\qquad 0\leqslant t_5\leqslant 6.$

We have 7 solutions: $t_5=0;1;2;3;4;5;6$, and hence 7 triples:
$(\cos_6 0,\sin_6 0,5)=(6,0,5)$; $(\cos_6 1,\sin_6 1,5)=(5,1,5)$; $(\cos_6 2,\sin_6 2,5)=(4,2,5)$; $(\cos_6 3,\sin_6 3,5)=(3,3,5)$;
$(\cos_6 4,\sin_6 4,5)=(2,4,5)$; $(\cos_6 5,\sin_6 5,5)=(1,5,5)$; $(\cos_6 6,\sin_6 6,5)=(0,6,5)$.

6.6. $j=6 \Rightarrow c=7$, $b=108$.
$[7\text{-gis}(7^2+8x7-2x108+20x7):2]+1=[7-6:2]+1=1\leqslant t_5\leqslant[7+6-1:2]=6$, $1\leqslant t_5\leqslant 6$.
We have 6 solutions: $t_5=1;2;3;4;5;6$, and hence 6 triples:
$(\cos_7 1,\sin_7 1,5)=(6,1,5)$; $(\cos_7 2,\sin_7 2,5)=(5,2,5)$; $(\cos_7 3,\sin_7 3,5)=(4,3,5)$;
$(\cos_7 4,\sin_7 4,5)=(3,4,5)$; $(\cos_7 5,\sin_7 5,5)=(2,5,5)$; $(\cos_7 6,\sin_7 6,5)=(1,6,5)$.
6.7. $j=7 \Rightarrow c=8$, $b=133$.
$[8\text{-gis}14\text{-}d:2]+1=[8-4-0:2]+1=3\leqslant t_5\leqslant[8+4+0:2]-1=5$, $3\leqslant t_5\leqslant 5$.
We have 3 solutions: $t_5=3;4;5$, and hence 3 triples:
$(\cos_8 3,\sin_8 3,5)=(5,3,5)$; $(\cos_8 4,\sin_8 4,5)=(4,4,5)$; $(\cos_8 5,\sin_8 5,5)=(3,5,5)$.
Total, we have 24 triples for z=5.

7. $z=6$, j_1и j_2 – not determined, $j_3=6\leqslant j\leqslant 7=j_4$, $c=j$, $b=j^2+14j-2j$. The system (16) takes the form:
$$\begin{cases}[c\text{-gis}(c^2+8R-2b+24c)\text{-}d:2]+1\leqslant t_6\leqslant[c+\text{gis}(c^2+8R-2b+24c)+d:2]-1\\ 0\leqslant t_5\leqslant c, \text{ if } c<\text{gis}(c^2+56-2b+24c).\end{cases}$$
7.1. $j=1 \Rightarrow c=1$, $b=13$.
$1<\text{gis}55=8 \Rightarrow 0\leqslant t_6\leqslant 1$, $0\leqslant t_6\leqslant 1$.
We have 2 solutions: $t_6=0;1$, and hence 2 triples:
$(\cos_1 0,\sin_1 0,6)=(1,0,6)$; $(\cos_1 1,\sin_1 1,6)=(0,1,6)$.
7.2. $j=2 \Rightarrow c=2$, $b=28$.
$2<\text{gis}52=8 \Rightarrow 0\leqslant t_6\leqslant 2$, $0\leqslant t_6\leqslant 2$.
We have 3 solutions: $t_6=0;1;2$, and hence 3 triples:
$(\cos_2 0,\sin_2 0,6)=(2,0,6)$; $(\cos_2 1,\sin_2 1,6)=(1,1,6)$; $(\cos_2 2,\sin_2 2,6)=(0,2,6)$.
7.3. $j=3 \Rightarrow c=3$, $b=45$.
$3<\text{gis}47=7 \Rightarrow 0\leqslant t_6\leqslant 3$, $0\leqslant t_6\leqslant 3$.
We have 4 solutions: $t_6=0;1;2;3$, and hence 4 triples:
$(\cos_3 0,\sin_3 0,6)=(3,0,6)$; $(\cos_3 1,\sin_3 1,6)=(2,1,6)$;
$(\cos_3 2,\sin_3 2,6)=(1,2,6)$; $(\cos_3 3,\sin_3 3,6)=(0,3,6)$.
7.4. $j=4 \Rightarrow c=4$, $b=64$.
$4<\text{gis}40=7 \Rightarrow 0\leqslant t_6\leqslant 4$, $0\leqslant t_6\leqslant 4$.
We have 5 solutions: $t_6=0;1;2;3;4;5$, and hence 5 triples:
$(\cos_4 0,\sin_4 0,6)=(4,0,6)$; $(\cos_4 1,\sin_4 1,6)=(3,1,6)$; $(\cos_4 2,\sin_4 2,6)=(2,2,6)$;
$(\cos_4 3,\sin_4 3,6)=(1,3,6)$; $(\cos_4 4,\sin_4 4,6)=(0,4,6)$.
7.5. $j=5 \Rightarrow c=5$, $b=85$.
$5<\text{gis}31=6 \Rightarrow 0\leqslant t_6\leqslant 5$, $0\leqslant t_6\leqslant 5$.
We have 6 solutions: $t_6=0;1;2;3;4;5$, and hence 6 triples:
$(\cos_5 0,\sin_5 0,6)=(5,0,6)$; $(\cos_5 1,\sin_5 1,6)=(4,1,6)$; $(\cos_5 2,\sin_5 2,6)=(3,2,6)$;
$(\cos_5 3,\sin_5 3,6)=(2,3,6)$; $(\cos_5 4,\sin_5 4,6)=(1,4,6)$; $(\cos_5 5,\sin_5 5,6)=(0,5,6)$.
7.6. $j=6 \Rightarrow c=6$, $b=108$.
$[6\text{-gis}20\text{-}d:2]+1=[6-5-1:2]+1=1\leqslant t_6\leqslant [6+5+1:2]-1=5$, $1\leqslant t_6\leqslant 5$.
We have 5 solutions: $t_6=1;2;3;4;5$, and hence 5 triples:
$(\cos_6 1,\sin_6 1,6)=(5,1,6)$; $(\cos_6 2,\sin_6 2,6)=(4,2,6)$; $(\cos_6 3,\sin_6 3,6)=(3,3,6)$;
$(\cos_6 4,\sin_6 4,6)=(2,4,6)$; $(\cos_6 5,\sin_6 5,6)=(1,5,6)$.
7.7. $j=7 \Rightarrow c=7$, $b=133$.
$[7\text{-gis}7\text{-}d:2]+1=[7-3-0:2]+1=3\leqslant t_6\leqslant [7+3+0:2]-1=4$, $3\leqslant t_6\leqslant 4$.
We have 2 solutions: $t_6=3;4$, and hence 2 triples:
$(\cos_7 3,\sin_7 3,6)=(4,3,6)$; $(\cos_7 4,\sin_7 4,6)=(3,4,6)$.
Total, we have 27 triples for z=6.

8. $z=7$, j_1 и j_2 – not determined, $j_3=2\leqslant j\leqslant 5=j_4$, $c=j-1$, $b=j^2+14j-2j$. The system (16) takes the form:
$$\begin{cases}[c\text{-gis}(c^2+8R-2b+28c)\text{-}d:2]+1\leqslant t_7\leqslant[c+\text{gis}(c^2+8R-2b+24c)+d:2]-1\\ 0\leqslant t_5\leqslant c, \text{ if } c<\text{gis}(c^2+56-2b+24c).\end{cases}$$
8.1. $j=1 \Rightarrow c=0$, $b=13$.
$0<\text{gis}26=6 \Rightarrow 0\leqslant t_7\leqslant 0$, $0\leqslant t_7\leqslant 0$.
We have 1 solutions: $t_7=0$, and hence 1 triples:
$(\cos_0 0,\sin_0 0,7)=(0,0,7)$.

8.2. $j=2 \Rightarrow c=1, b=28$.
$1<gis29=6 \Rightarrow 0\leqslant t_7\leqslant 1$, $\quad 0\leqslant t_7\leqslant 1$.
We have 2 solutions: $t_7=0;1$, and hence 2 triples:
$(\cos_1 0,\sin_1 0,7)=(1,0,7)$; $(\cos_1 1,\sin_1 1,7)=(0,1,7)$.
8.3. $j=3 \Rightarrow c=2, b=45$.
$2<gis54=8 \Rightarrow 0\leqslant t_7\leqslant 2$, $\quad 0\leqslant t_7\leqslant 2$.
We have 3 solutions: $t_7=0;1;2$, and hence 3 triples:
$(\cos_2 0,\sin_2 0,7)=(2,0,7)$; $(\cos_2 1,\sin_2 1,7)=(1,1,7)$; $(\cos_2 2,\sin_2 2,7)=(0,2,7)$.
8.4. $j=4 \Rightarrow c=3, b=64$.
$3<gis21=5 \Rightarrow 0\leqslant t_7\leqslant 3$, $\quad 0\leqslant t_7\leqslant 3$.
We have 4 solutions: $t_7=0;1;2;3;4$, and hence 4 triples:
$(\cos_3 0,\sin_3 0,7)=(3,0,7)$; $(\cos_3 1,\sin_3 1,7)=(2,1,7)$;
$(\cos_3 2,\sin_3 2,7)=(1,2,7)$; $(\cos_3 3,\sin_3 3,7)=(0,3,7)$.
8.5. $j=5 \Rightarrow c=4, b=85$.
$[4-gis14-d:2]+1=[4-4-0:2]+1=1\leqslant t_7\leqslant[4+4+0:2]-1=3$, $\quad 1\leqslant t_7\leqslant 3$.
We have 3 solutions: $t_7=1;2;3$, and hence 3 triples:
$(\cos_4 1,\sin_4 1,7)=(3,1,7)$; $(\cos_4 2,\sin_4 2,7)=(2,2,7)$; $(\cos_4 3,\sin_4 3,7)=(1,3,7)$.
8.6. $j=6 \Rightarrow c=5, b=108$.
$[5-gis5-d:2]+1=[5-3-0:2]+1=2\leqslant t_7\leqslant [5+3+0:2]-1=3$, $\quad 2\leqslant t_7\leqslant 3$.
We have 2 solutions: $t_7=2;3$, and hence 2 triples:
$(\cos_5 2,\sin_5 2,7)=(3,2,7)$; $(\cos_5 3,\sin_5 3,7)=(2,3,7)$.
Total, we have 15 triples for $z=7$.

All solutions for the first octant are found. Total – 178 triples. Then the quantity of triples for the entire space at $R=7$, after deduction of overlaps, is equals: $N_{R=7}=178x8-(80x3-6)=1190$.

The general solution of Task 2 for the space, in the part of enumeration of triples, is looks as follows way:

$$\left\{\begin{array}{l}
\mathbf{(\cos_c t_z, \sin_c t_z, z), z=0\div R, 0\leqslant t_z\leqslant c} \\
\mathbf{(ver_c t_z, \sin_c t_z, z), z=0\div R, 0\leqslant t_z\leqslant c-1} \\
\mathbf{(ver_c t_z, ops_c t_z, z), z=0\div R, 1\leqslant t_z\leqslant c} \\
\mathbf{(\cos_c t_z, ops_c t_z, z), z=0\div R, 1\leqslant t_z\leqslant c-1} \\
\mathbf{(\cos_c t_z, \sin_c t_z, -z), z=1\div R, 0\leqslant t_z\leqslant c} \\
\mathbf{(ver_c t_z, \sin_c t_z, -z), z=1\div R, 0\leqslant t_z\leqslant c-1} \\
\mathbf{(ver_c t_z, ops_c t_z, -z), z=1\div R, 1\leqslant t_z\leqslant c} \\
\mathbf{(\cos_c t_z, ops_c t_z, -z), z=1\div R, 1\leqslant t_z\leqslant c-1,}
\end{array}\right. \qquad (18)$$

where $R\geqslant 0$; for each z, j runs through values from j_{1z} to j_{4z}, which are determined from the conditions –

$$\left\{\begin{array}{l}
\mathbf{j_{1z}=gis\ ((R-1)^2-z^2)-R+z+1+g((R-1)^2-z^2)\leqslant j\leqslant gis(2(R-1)^2-2z^2)-R+z=j_{2z}} \\
\mathbf{j_{3z}=gis((R+1)^2-z^2)-R+z+1\leqslant j\leqslant gis(2(R+1)^2-2z^2)-R+z=j_{4z},}
\end{array}\right.$$

$c=R+j-1-z$, $b=j^2+2Rj-2j$, t_z – the solutions of inequality $2cz-b<2t_z(t_z-c)<4R+2cz-b$ for the first octant, which are determined from the conditions –

$$\left\{\begin{array}{l}
\left[\begin{array}{l}
\mathbf{0\leqslant t_z\leqslant[c-gis(c^2-2b+4cz)-d:2]-g} \\
\mathbf{[c+gis(c^2-2b+4cz)+d:2]+g\leqslant t_z\leqslant c}
\end{array}\right. \\
\mathbf{[c-gis(c^2+8R-2b+4cz)-d:2]+1\leqslant t_z\leqslant[c+gis(c^2+8R-2b+4cz)+d:2]-1} \\
\mathbf{0\leqslant t_z\leqslant c, if\ c<gis(c^2+8R-2b+4cz).}
\end{array}\right.$$

If the value $c=a-z$ to substitute into inequality (13), then we obtain an equisignificant inequality: $-b<2t(t+z-a)+2z(z-a)<4R-b$. Then, the solutions of system (16) can be visualized with the help of function $(y_{ijk})=2t_i(t_i+z_k-a_j)+2z_k(z_k-a_j)$, by the graph of which is **a cascade of paraboloids.** The frontal slices of 3F-matrix(y_{ijk}) of the indicated function, considered above for the case of $R=7$, are presented in Tables 4-10. Partially, the graph of the indicated function, corresponding to the 3F-matrix(y_{ijk}) for $j=1$, is represented in Fig. 7 in the form of paraboloid's mockup. The entire graph will consist of seven similar paraboloids, which are the graphic reflection of represented frontal slices. All lines and surfaces are for illustration only.
In contrast to the planar case, the solutions of inequality (13) will be not between two channels, but between two corresponding layers.

Table 4. The Frontal Slice of 3F-matrix(y_{ijk}) for j=1.

0	i	1	2	3	4	5	6	7	8
k	$z_k \backslash t_i$	0	1	2	3	4	5	6	7
1	0	0	-12	-20	-24	-24	-20	-12	0
2	1	-12	-22	-28	-30	-28	-22	-12	2
3	2	-20	-28	-32	-32	-28	-20	-8	8
4	3	-24	-30	-32	-30	-24	-14	0	18
5	4	-24	-28	-28	-24	-16	-4	12	32
5	5	-20	-22	-20	-14	-4	10	28	50
7	6	-12	-12	-8	0	12	28	48	72
8	7	0	2	8	18	32	50	72	98

Table 5. The Frontal Slice of 3F-matrix(y_{ijk}) for j=2.

0	i	1	2	3	4	5	6	7	8	9
k	$z_k \backslash t$	0	1	2	3	4	5	6	7	8
1	0	0	-14	-24	-30	-32	-30	-24	-14	0
2	1	-14	-26	-34	-38	-38	-34	-26	-14	2
3	2	-24	-34	-40	-42	-40	-34	-24	-10	8
4	3	-30	-38	-42	-42	-38	-30	-18	-2	18
5	4	-32	-38	-40	-38	-32	-22	-8	10	32
6	5	-30	-34	-34	-30	-22	-10	6	26	50
7	6	-24	-26	-24	-18	-8	6	24	46	72
8	7	-14	-14	-10	-2	10	26	46	70	98
9	8	0	2	8	18	32	50	72	98	128

Table 6. The Frontal Slice of 3F-matrix(y_{ijk}) for j=3.

0	i	1	2	3	4	5	6	7	8	9	10
k	$z_k \backslash t_i$	0	1	2	3	4	5	6	7	8	9
1	0	**0**	-16	-28	-36	-40	-40	-36	-28	-16	0
2	1	-16	-30	-40	-46	-48	-46	-40	-30	-16	2
3	2	-28	-40	-48	-52	-52	-48	-40	-28	-12	8
4	3	-36	-46	-52	-54	-52	-46	-36	-22	-4	18
5	4	-40	-48	-52	-52	-48	-40	-28	-12	8	32
6	5	-40	-46	-48	-46	-40	-30	-16	2	24	50
7	6	-36	-40	-40	-36	-28	-16	0	20	44	72
8	7	-28	-30	-28	-22	-12	2	20	42	68	98
9	8	-16	-16	-12	-4	8	24	44	68	96	128
10	9	0	2	8	18	32	50	72	98	128	162

Table 7. The Frontal Slice of 3F-matrix(y_{ijk}) for j=4.

0	i	1	2	3	4	5	6	7	8	9	10	11
k	$z_k \backslash t_i$	0	1	2	3	4	5	6	7	8	9	10
1	0	0	-18	-32	-42	-48	-50	-48	-42	-32	-18	0
2	1	-18	-34	-46	-54	-58	-58	-54	-46	-34	-18	2
3	2	-32	-46	-56	-62	-64	-62	-56	-46	-32	-14	8
4	3	-42	-54	-62	-66	-66	-62	-54	-42	-26	-6	18
5	4	-48	-58	-64	-66	-64	-58	-48	-34	-16	6	32
6	5	-50	-58	-62	-62	-58	-50	-38	-22	-2	22	50
7	6	-48	-54	-56	-54	-48	-38	-12	-6	16	42	72
8	7	-42	-46	-46	-42	-34	-22	-6	14	38	66	98
9	8	-32	-34	-32	-26	-16	-2	16	38	64	94	128
10	9	-18	-18	-14	-6	6	22	42	66	94	126	162
11	10	0	2	8	18	32	50	72	98	128	162	200

Table 8. The Frontal Slice of 3F-matrix(y_{ijk}) for j=5.

0	i	1	2	3	4	5	6	7	8	9	10	11	12
k	$z_k \backslash t_i$	0	1	2	3	4	5	6	7	8	9	10	11
1	0	0	-20	-36	-48	-56	-60	-60	-56	-48	-36	-20	0
2	1	-20	-38	-52	-62	-68	-70	-68	-62	-52	-38	-20	2
3	2	-36	-52	-64	-72	-76	-76	-72	-64	-52	-36	-16	8
4	3	-48	-62	-72	-78	-80	-78	-72	-62	-48	-30	-8	18
5	4	-56	-68	-76	-80	-80	-76	-68	-56	-40	-20	4	32
6	5	-60	-70	-76	-78	-76	-70	-60	-46	-28	-6	20	50
7	6	-60	-68	-72	-72	-68	-60	-36	-32	-12	12	40	72
8	7	-56	-62	-64	-62	-56	-46	-32	-14	8	34	64	98
9	8	-48	-52	-52	-48	-40	-28	-12	8	32	60	92	128
10	9	-36	-38	-36	-30	-20	-6	12	34	60	90	124	162
11	10	-20	-20	-16	-8	4	20	40	64	92	124	160	200
12	11	0	2	8	18	32	50	72	98	128	162	200	242

Table 9. The Frontal Slice of 3F-matrix(y_{ijk}) for j=6.

0	i	1	2	3	4	5	6	7	8	9	10	11	12	13
k	$z_k \backslash t_i$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	-22	-40	-54	-64	-70	-72	-70	-64	-54	-40	-22	0
2	1	-22	-42	-58	-70	-78	-82	-82	-78	-70	-58	-42	-22	2
3	2	-40	-58	-72	-82	-88	-90	-88	-82	-72	-58	-40	-18	8
4	3	-54	-70	-82	-90	-94	-94	-90	-82	-70	-54	-34	-10	18
5	4	-64	-78	-88	-94	-96	-94	-88	-78	-64	-46	-24	2	32
6	5	-70	-82	-90	-94	-94	-90	-82	-70	-54	-34	-10	18	50
7	6	-72	-82	-88	-90	-88	-82	-72	-58	-40	-18	8	38	72
8	7	-70	-78	-82	-82	-78	-70	-58	-42	-22	2	30	62	98
9	8	-64	-70	-72	-70	-64	-54	-40	-22	0	26	56	90	128
10	9	-54	-58	-58	-54	-46	-34	-18	2	26	54	86	122	162
11	10	-40	-42	-40	-34	-24	-10	8	30	56	86	120	158	200
12	11	-22	-22	-18	-10	2	18	38	62	90	122	158	198	242
13	12	0	2	8	18	32	50	72	98	128	162	200	242	288

Table 10. The Frontal Slice of 3F-matrix(y_{ijk}) for j=7.

0	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
k	z_k \ t_i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	0	-24	-44	-60	-72	-80	-84	-84	-80	-72	-60	-44	-24	0
2	1	-24	-46	-64	-78	-88	-94	-96	-94	-88	-78	-64	-46	-24	2
3	2	-44	-64	-80	-92	-100	-104	-104	-100	-92	-80	-64	-44	-20	8
4	3	-60	-78	-92	-102	-108	-110	-108	-102	-92	-78	-60	-38	-12	18
5	4	-72	-88	-100	-108	-112	-112	-108	-100	-88	-72	-52	-28	0	32
6	5	-80	-94	-104	-110	-112	-110	-104	-94	-80	-62	-40	-14	16	50
7	6	-84	-96	-104	-108	-108	-104	-96	-84	-68	-48	-24	4	36	72
8	7	-84	-94	-100	-102	-100	-94	-84	-70	-52	-30	-4	26	60	98
9	8	-80	-88	-92	-92	-88	-80	-68	-52	-32	-8	20	52	88	128
10	9	-72	-78	-80	-78	-72	-62	-48	-30	-8	18	48	82	120	162
11	10	-60	-64	-64	-60	-52	-450	-24	-4	20	48	80	116	156	200
12	11	-44	-46	-44	-38	-28	-14	4	36	52	82	116	154	196	242
13	12	-24	-24	-18	-10	2	18	38	62	90	122	158	198	242	288
14	13	0	2	8	18	32	50	72	98	128	162	200	242	288	338

For indication the quantity of triples satisfying the conditions of task, we will make use the following considerations relatively for the first octant:

1. For z=0, the formula (10) is acting;
2. For z=1÷R-2, depending on j, we have:

a) If $j_{1z} \leqslant j \leqslant j_{2z}$, then the conditions 1, 2 and 4 of system (16) are satisfied, and hence, it is acting the following condition: n_{j1}=[c-gis(c^2-2b+4cz)-d:2]+c-[c+gis(c^2-2b+4cz)+d:2]-2g;

b) If $j_{2z}+1 \leqslant j \leqslant j_{3z}-1$, then only the fourth condition of system (16) is satisfied, and hence, it is acting the following condition: n_{j2}=c+1;

c) If $j_{3z} \leqslant j \leqslant j_{4z}$, then only the third condition of system (16) is satisfied, and hence, the formula (9) is acting;

3. For z=R-1 and z=R, j_1 and j_2 are not defined, and depending on j, we have:

a) If $1 \leqslant j \leqslant j_{3z}-1$, then only the fourth condition of system (16) is satisfied, and hence, it is acting the condition: n_{j2}=c+1;

b) If $j_{3z} \leqslant j \leqslant j_{4z}$, then only the third condition of system (16) is satisfied, and hence, the formula (9) is acting.

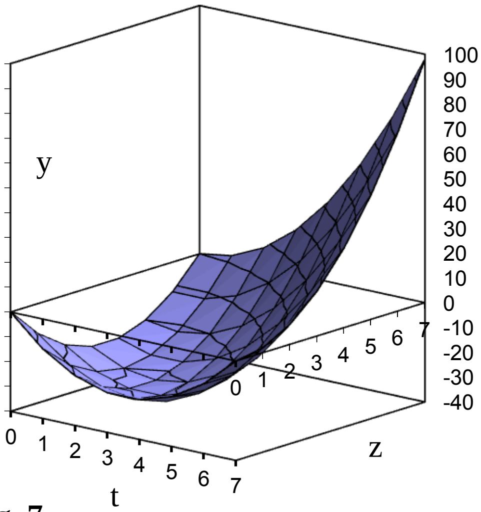

Fig. 7.
The Mockup of the Graph of Function (y_{ijk})=$2t_i(t_i+z_k-a_j)+2z_k(z_k-a_j)$
for R=7 and j=1.

The quantity of triples for the whole space will be equals to the eight times the number of triples of the first octant without repeats, the quantity of which is equals to the three times quantity of triples of the layer z=0 without six repeats.

What has been said can be reflected with the help of following formula:

$$N_R = 8\left(4+\sum_{j=2}^{j_{20}} n_{j0} + \sum_{j=j_{20}+1}^{j_{40}} n_{j3} + \sum_{z=1}^{R-2}\left(\sum_{j=j_{1z}}^{j_{2z}} n_{j1} + \sum_{j=j_{2z}+1}^{j_{3z}-1} n_{j2} + \sum_{j=j_{3z}}^{j_{4z}} n_{j3}\right) + \right.$$
$$\left. + \sum_{z=R-1}^{R}\left(\sum_{j=1}^{j_{3z}-1} n_{j2} + \sum_{j=j_{3z}}^{j_{4z}} n_{j3}\right)\right) - \left(3\left(4\left(4+\sum_{j=2}^{j_{20}} n_{j0} + \sum_{j=j_{20}+1}^{j_{40}} n_{j3}\right)-4\right)-6\right).$$

After little transformations, we finally obtain:

$$N_R = 8\left(\sum_{z=1}^{R-2}\left(\sum_{j=j_{1z}}^{j_{2z}} n_{j1}+\sum_{j=j_{2z}+1}^{j_{3z}-1} n_{j2}+\sum_{j=j_{3z}}^{j_{4z}} n_{j3}\right)+\sum_{z=R-1}^{R}\left(\sum_{j=1}^{j_{3z}-1} n_{j2}+\sum_{j=j_{3z}}^{j_{4z}} n_{j3}\right)\right)-4\left(\sum_{j=2}^{j_{20}} n_{j0}+\sum_{j=j_{20}+1}^{j_{40}} n_{j3}\right)+2, \quad (19)$$

where $R\geqslant 0$, $\mathbf{n_{j0}=[c\text{-}gis(c^2\text{-}2b)\text{-}d:2]\text{-}[c\text{-}gis(c^2\text{+}8R\text{-}2b)\text{-}d:2]\text{+}[c\text{+}gis(c^2\text{+}8R\text{-}2b)\text{+}d:2]\text{-}[c\text{+}gis(c^2\text{-}2b)\text{+}d:2]\text{-}2g}$;
$\mathbf{n_{j1}=[c\text{-}gis(c^2\text{-}2b\text{+}4cz)\text{-}d:2]\text{+}c\text{-}[c\text{+}gis(c^2\text{-}2b\text{+}4cz)\text{+}d:2]\text{-}2g}$;
$\mathbf{n_{j2}=c+1}$; $\mathbf{n_{j3}=[c\text{+}gis(c^2\text{+}8R\text{-}2b\text{+}4cz)\text{+}d:2]\text{-}[c\text{-}gis(c^2\text{+}8R\text{-}2b\text{+}4cz)\text{-}d:2]\text{-}1}$; $\mathbf{c=R+j-1-z,\ b=j^2+2Rj-2j}$;
$\mathbf{j_{1z}=gis((R\text{-}1)^2\text{-}z^2)\text{-}R\text{+}z\text{+}1\text{+}g}$; $\mathbf{j_{2z}=gis(2(R\text{-}1)^2\text{-}2z^2)\text{-}R\text{+}z\text{+}g}$;
$\mathbf{j_{3z}=gis((R\text{+}1)^2\text{-}z^2)\text{-}R\text{+}z\text{+}1}$; $\mathbf{j_{4z}=gis(2(R\text{+}1)^2\text{-}2z^2)\text{-}R\text{+}z}$.

Thus, **the quantity has been specified, and all triples of numbers that satisfy the conditions of Task 2 are listed**.

The part of quanta, whose coordinates are the solutions of Task 2, is represented in Fig. 8 in the form of the model of upper half of the spherical passage of radius R=7 (z_i=0÷R).

Definition 22 (spherical passage): The totality of the quanta of space whose coordinates (x_i,y_i,z_i), where i=0÷n, n∈N, satisfy the following conditions:

$$\begin{cases}(R-1)^2<x_i^2+y_i^2+z_i^2<(R+1)^2\\ -R\leqslant x_i\leqslant R\\ -R\leqslant y_i\leqslant R\\ -R\leqslant z_i\leqslant R,\ \text{where } R\geqslant 0,\end{cases}$$

forms **the spherical passage of radius R** with the center at the origin of coordinates, briefly **the S-passage(R)**.

The coordinates of the quanta of S-passage(R) can be specified as follows way:

$$\begin{cases}(\cos_c t_z,\ \sin_c t_z,\ z),\ z=0\div R,\ 0\leqslant t_z\leqslant c\\ (\mathrm{ver}_c t_z,\ \sin_c t_z,\ z),\ z=0\div R,\ 0\leqslant t_z\leqslant c-1\\ (\mathrm{ver}_c t_z,\ \mathrm{ops}_c t_z,\ z),\ z=0\div R,\ 1\leqslant t_z\leqslant c\\ (\cos_c t_z,\ \mathrm{ops}_c t_z,\ z),\ z=0\div R,\ 1\leqslant t_z\leqslant c-1\\ (\cos_c t_z,\ \sin_c t_z,\ -z),\ z=1\div R,\ 0\leqslant t_z\leqslant c\\ (\mathrm{ver}_c t_z,\ \sin_c t_z,\ -z),\ z=1\div R,\ 0\leqslant t_z\leqslant c-1\\ (\mathrm{ver}_c t_z,\ \mathrm{ops}_c t_z,\ -z),\ z=1\div R,\ 1\leqslant t_z\leqslant c\\ (\cos_c t_z,\ \mathrm{ops}_c t_z,\ -z),\ z=1\div R,\ 1\leqslant t_z\leqslant c-1,\end{cases}$$

where c=R+j-1-z, R⩾0; for each z, j runs through values from j_{1z} to j_{4z}, which are determined from the conditions –

$$\begin{cases}j_{1z}=gis((R-1)^2-z^2)-R+z+1+g((R-1)^2-z^2)\leqslant j\leqslant gis(2(R-1)^2-2z^2)-R+z=j_{2z}\\ j_{3z}=gis((R+1)^2-z^2)-R+z+1\leqslant j\leqslant gis(2(R+1)^2-2z^2)-R+z=j_{4z},\end{cases}$$

$b=j^2+2Rj-2j$, t_z – the solutions of inequality $2cz-b<2t_z(t_z-c)<4R+2cz-b$ for the first octant, which are determined from the conditions –

$$\begin{cases}\begin{cases}0\leqslant t_z\leqslant [c-gis(c^2-2b+4cz)-d:2]-g\\ [c+gis(c^2-2b+4cz)+d:2]+g\leqslant t_z\leqslant c\end{cases}\\ [c-gis(c^2+8R-2b+4cz)-d:2]+1\leqslant t_z\leqslant[c+gis(c^2+8R-2b+4cz)+d:2]-1\\ \quad 0\leqslant t_z\leqslant c,\ \text{if } c<gis(c^2+8R-2b+4cz).\end{cases}$$

S-passage(R) is spatial, three-dimensional-limited, closed passage.

Fig. 8. Spherical Passage of Radius R=7 (exterior view for z_i=0÷R).
The model has been composed up of wooden cubes with the edge of 2 cm.

S-passage(R) is the likeness of a sphere in the Quantum Discrete Space.

The obtained solution for a spherical "layer", or more precisely for a spherical passage, makes it possible, if necessary, to determine the number of quanta that are inside the circular passage, that is, to solve the Task 2 by the replacing of basic condition on following: $x_i^2+y_i^2+z_i^2<(R+1)^2$. But within the framework of this article there is no such necessity, therefore it can be noted only that the possible approach to solving this task, which, in essence, is the task about solving the Gauss's problem in a ball, is analogous to the approach that was proposed to solve the Gauss's problem in the circle.

Conclusion

The above given definitions of circular and spherical passages are the best proof that the exact solution of Gauss's problem exists, and this solution is possible only within the framework of the Quantum Discrete Space. Indeed, if you are looking for a specific book in one of the two bookcases, and having carefully sorted all the books in one of them, you did not find the book that you are looking for, then, consequently, it is in the other. Exactly in the same way in the case with the Gauss's problem. Many remarkable mathematicians have far and wide ploughed the continuous abstract-analytic mathematical field, and found no exact solution to the Gauss's problem. The maximum, what they have achieved, it's of the unimprovable asymptotic estimates. Consequently, it's need to move on another field. It seems to me that the Gauss's problem is not an accident, but it is some unconscious Gauss's message to future mathematicians concerning the existence of an alternative mathematical field. I think that the Quantum Discrete Space, the general representation about of which is briefly reflected above, is a huge unploughed mathematical field. And on this field, we have already managed to construct the likenesses of circumference and sphere in the form of C-passage(R) and S-passage(R), respectively. And after all there is else an ellipse, cone, cylinder, ellipsoid, hyperboloid and many other interesting mathematical objects. It remains only to wish good luck to " ploughers" and "sowers".

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