The $qq'$-Calculus

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Abstract

We present here a generalisation of the $q$-calculus, the $qq'$-calculus. The calculus is however limited.

1 The $\delta$-derivation

1.1 Definitions

The derivative of a function $f$ at the point $x$ is usually defined as:

$$\frac{df}{dx}(x) = \frac{d_h(f)(x)}{d_h(x)} = \frac{f(x + h) - f(x)}{h}$$

if the limit exists.

Definition 1 Similarly, the $\delta$-derivative of a function is defined as:

$$\frac{\delta f}{\delta x}(x) = \frac{\delta_h(f)(x)}{\delta_h(x)} = \frac{f(x + h) - f(x + h')}{h - h'}$$

If the limit exists, then the derivative of the function exists and we have $\frac{df}{dx}(x) = \frac{\delta f}{\delta x}(x)$.

1.2 A counter-example

The derivative can exist even if the $\delta$-derivative doesn’t. Indeed let be $f$ the function such that $f(x) = x^2$ if $x \in \mathbb{Q}$ and $f(x) = x^3$ if $x \notin \mathbb{Q}$. This function admits a derivative in zero which is zero, but has no $\delta$-derivative as one can verify:

$$\lim_{h,h' \to 0} \frac{h^2 - h'^3}{h - h'} = \lim_{h,h' \to 0} \frac{h + h' + h^2 - h'^3}{h - h'}$$

doesn’t exist because $h - h'$ can be as small as we want.
1.3 The Leibniz rule

The Leibniz rule can be verified:

\[
\frac{f(x + h)g(x + h) - f(x + h')g(x + h')}{h - h'} = \frac{f(x + h) - f(x + h')}{h - h'} g(x + h) + \frac{g(x + h) - g(x + h')}{h - h'} f(x + h')
\]

so that:

Proposition 1

\[
\frac{\delta(fg)}{\delta x} = (\frac{\delta f}{\delta x})g + f(\frac{\delta g}{\delta x})
\]

1.4 Some formulas

The following formulas can be easily verified:

\[
\tilde{(1/f)} = -\frac{1}{f^2} \tilde{f}
\]

\[
\tilde{g} = \tilde{f} g - f \tilde{g}
\]

and also:

\[
(f \circ g) = (\tilde{f} \circ g) \times \tilde{g}
\]

1.5 \(\delta\)-derivative of a function of class \(C^3\)

Theorem 1 If the fonction \(f\) is of class \(C^3\), then the \(\delta\)-derivative exists.

Demonstration 1 As we have:

\[
(1/2) \int_0^h f^{(2)}(t)(h - t)dt = \int_0^h f'(t)dt - f'(0)h = f(h) - f(0) - f'(0)h
\]

so that:

\[
(1/2)h^2 \int_0^1 f^{(2)}(ht)(1-t)dt - (1/2)h^2 \int_0^1 f^{(2)}(h't)(1-t)dt = f(h) - f(h') - f'(0)(h - h') =
\]

\[
= (1/2)(h^2 - h'^2) \int_0^1 f^{(2)}(ht)(1-t)dt + h^2 \int_0^1 [f^{(2)}(ht) - f^{(2)}(h't)](1-t)dt
\]

and

\[
|f^{(2)}(ht) - f^{(2)}(h't)| \leq (\max|f^{(3)}|(h - h')t
\]

by the Taylor’s formula.

So a smooth function is also infinitely \(\delta\)-derivable.

1.6 The \(qq'\)-limit

We have, if the limit exists, for \(x \neq 0\):

\[
\lim_{qq' \to 1} \frac{f(qx) - f(q'x)}{(q - q')x} = \tilde{f}(x)
\]
1.7 Integration and δ-derivation

Theorem 2 If $f$ is continuous over the interval $[a, b]$, it is Riemann integrable and the primitive is δ-derivable, so that we have:

$$\delta \frac{\delta}{\delta x} \int_{a}^{x} f(t)dt = f(x)$$

Demonstration 2

$$\delta \frac{\delta}{\delta x} \int_{a}^{x} f(t)dt = \lim_{h,h' \to 0} \frac{\int_{h}^{h'} f(t + x)dt}{h - h'} = f(x)$$

by the Taylor formula.

2 $qq'$-quantum derivation

2.1 Definitions

Definition 2 Let be two numbers $q, q'$ and let be an arbitrary function $f$, its $qq'$-differential is:

$$d_{qq'}(f)(x) = f(qx) - f(q'x)$$

In particular $d_{qq'}x = (q - q')x$.

We have the following Leibniz rule:

$$d_{qq'}(fg)(x) = f(qx)g(qx) - f(q'x)g(q'x) = (f(qx) - f(q'x))g(qx) + f(q'x)(g(qx) - g(q'x))$$

Proposition 2

$$d_{qq'}(fg)(x) = d_{qq'}(f)(x).g(qx) + f(q'x).d_{qq'}(g)(x)$$

Definition 3 The following formula:

$$D_{qq'}f(x) = \frac{d_{qq'}(f)(x)}{d_{qq'}(x)} = \frac{f(qx) - f(q'x)}{(q - q')x}$$

is called the $qq'$-derivative of the function $f$

2.2 The Leibniz rule

The Leibniz rule is:

Proposition 3

$$D_{qq'}(fg)(x) = D_{qq'}(f)(x).g(qx) + f(q'x).D_{qq'}(g)(x)$$
2.3 Some formulas

The $qq'$-derivative is a linear operator as we can verify:

$$D_{qq'}(af + bg) = aD_{qq'}(f) + bD_{qq'}(g)$$

for any scalars $a, b$ and functions $f, g$.

**Example 1**

$$D_{qq'}(x^n) = \left[n\right]_{qq'} x^{n-1}$$

The number $[n]_{qq'}$ is called the $qq'$-analog of $n$ as $\lim_{qq' \to 1} [n]_{qq'} = n$. We obtain also:

**Proposition 4**

$$D_{qq'}(f \circ u)(x) = (D_{qq'}(f \circ u))(x) \times D_{qq'}(u)(x)$$

3 $qq'$-analogue of $(x - a)^n$

**3.1 Definition**

Definition 4

$$[0]_{qq'}! = 1$$

$$[n]_{qq'}! = [n]_{qq'} \times [n - 1]_{qq'} \times \ldots \times [1]_{qq'} \text{ if } n \neq 0$$

**3.2 The exponential**

Definition 5

$$\exp_{qq'}(x) = \sum_{n \geq 0} \frac{x^n}{[n]_{qq'}!}$$

The derivative is:

$$D_{qq'}(\exp_{qq'})(x) = \exp_{qq'}(x)$$

**3.3 The $qq'$-analogue of $(x - a)^n$**

Definition 6 

The $qq'$-analogue of $(x - a)^n$ is:

$$(x - a)^n_{qq'} = \prod_{k,l,k+l=n-1} (x - q^k q^l a)$$

We have the following theorem:
Theorem 3
\[ D_{qq'}(x-a)^n_{qq'} = [n]_{qq'}(x-a)^{n-1}_{qq'} \]

Demonstration 3
\[ (x-a)^n_{qq'} = (x-qa)^{n-1}_{qq'}(x-q^n a) \]
so that, by induction on \( n \), using Leibniz rule:
\[ D_{qq'}(x-a)^n_{qq'} = D_{qq'}(x-a)^{n-1}_{qq'}(q'x-q^{n-1} a) + (qx-qa)^{n-1}_{qq'} = [n]_{qq'}(x-a)^{n-1}_{qq'} \]
We have also:
\[ (x-a)^{n+m}_{qq'} = (x-q^m a)^n_{qq'}(x-q^n a)^m_{qq'} \]

4 \( qq' \)-Taylor’s Formula for polynomials

4.1 The Taylor’s expansion

Theorem 4 For any polynomial \( P(X) \) of degree \( n \), and any number \( a \), we have the following \( qq' \)-Taylor expansion:
\[ P(x) = \sum_{j=0}^{n} (D_j^{qq'} P)(a) \frac{(x-a)^j_{qq'}}{[j]_{qq'}!} \]

Demonstration 4 Due to the degree, we can write:
\[ P(x) = \sum_{j=0}^{n} c_j \frac{(x-a)^j_{qq'}}{[j]_{qq'}!} \]
and now, by derivation, we have inductively on the degree of \( P \):
\[ c_k = (D_k^{qq'} P)(a) \]

4.2 A formula

The \( qq' \)-Taylor formula for \( x^n \) about \( x=1 \) then gives:
\[ x^n = \sum_{j=0}^{n} [n]_{qq'} \ldots [n-j+1]_{qq'} \frac{(x-a)^j_{qq'}}{[j]_{qq'}!} \]

Formula 1
\[ x^n = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right]_{qq'} (x-a)^j_{qq'} \]
with \[ \left[ \begin{array}{c} n \\ j \end{array} \right]_{qq'} = \frac{[n]_{qq'}!}{[j]_{qq'}! [n-j]_{qq'}!} . \]

References
