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Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points

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Abstract. Properties on neutrosophic $\in\vee q$-subsets and neutrosophic q-subsets are investigated. Relations between an $(\in, \in\vee q)$-neutrosophic subalgebra and a $(q, \in\vee q)$-neutrosophic subalgebra are considered. Characterization of an $(\in, \in\vee q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets are discussed. Conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra are provided.

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1. Introduction

The concept of neutrosophic set (NS) developed by Smarandache [17, 18, 19] is a more general platform which extends the concepts of the classic set and fuzzy set (see [20], [21]), intuitionistic fuzzy set (see [1]) and interval valued intuitionistic fuzzy set (see [2]). Neutrosophic set theory is applied to various part (see [4], [5], [8], [9], [10], [11], [12], [13], [15], [16]). For further particulars, we refer readers to the site http://fs.gallup.unm.edu/neutrosophy.htm. Barbhuiya [3] introduced and studied the concept of $(\in, \in\vee q)$-intuitionistic fuzzy ideals of BCK/BCI-algebras. Jun [7] introduced the notion of neutrosophic subalgebras in BCK/BCI-algebras with several types. He provided characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in\vee q)$-neutrosophic subalgebra. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets, he considered conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets to be subalgebras. He discussed conditions for a neutrosophic set to be a $(q, \in\vee q)$-neutrosophic subalgebra.

In this paper, we give relations between an $(\in, \in\vee q)$-neutrosophic subalgebra and a $(q, \in\vee q)$-neutrosophic subalgebra. We discuss characterization of an $(\in, \in\vee q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets. We provide conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra. We investigate properties on neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \ \Rightarrow\ x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [6] and [14] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [18]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

3. Neutrosophic subalgebras of several types

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$\begin{aligned}
&T_\in(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha\},\\
&I_\in(A;\beta) := \{x \in X \mid A_I(x) \geq \beta\},\\
&F_\in(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma\},\\
&T_q(A;\alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},\\
&I_q(A;\beta) := \{x \in X \mid A_I(x) + \beta > 1\},\\
&F_q(A;\gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},\\
&T_{\in\vee\, q}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},\\
&I_{\in\vee\, q}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},\\
&F_{\in\vee\, q}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.
\end{aligned}$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic $\in$-subsets*; $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are *neutrosophic q-subsets*; and $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are *neutrosophic $\in\vee\, q$-subsets*. For $\Phi \in \{\in, q, \in\vee\, q\}$, the element of $T_\Phi(A;\alpha)$ (resp., $I_\Phi(A;\beta)$ and $F_\Phi(A;\gamma)$) is called a *neutrosophic T_Φ-point* (resp., *neutrosophic I_Φ-point* and *neutrosophic F_Φ-point*) with value α (resp., β and γ) (see [7]).

It is clear that

$$T_{\in\vee\, q}(A;\alpha) = T_\in(A;\alpha) \cup T_q(A;\alpha), \tag{3.1}$$

$$I_{\in\vee\, q}(A;\beta) = I_\in(A;\beta) \cup I_q(A;\beta), \tag{3.2}$$

$$F_{\in\vee\, q}(A;\gamma) = F_\in(A;\gamma) \cup F_q(A;\gamma). \tag{3.3}$$

Definition 3.1 ([7])**.** Given $\Phi, \Psi \in \{\in, q, \in\vee\, q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called a $(\Phi,\ \Psi)$*-neutrosophic subalgebra* of X if the following assertions are valid.

$$\begin{aligned}
&x \in T_\Phi(A;\alpha_x),\ y \in T_\Phi(A;\alpha_y)\ \Rightarrow\ x * y \in T_\Psi(A;\alpha_x \wedge \alpha_y),\\
&x \in I_\Phi(A;\beta_x),\ y \in I_\Phi(A;\beta_y)\ \Rightarrow\ x * y \in I_\Psi(A;\beta_x \wedge \beta_y),\\
&x \in F_\Phi(A;\gamma_x),\ y \in F_\Phi(A;\gamma_y)\ \Rightarrow\ x * y \in F_\Psi(A;\gamma_x \vee \gamma_y)
\end{aligned} \tag{3.4}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Lemma 3.2 ([7])**.** *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in,\ \in\vee\, q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}\\ A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}\\ A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}\end{array}\right). \tag{3.5}$$

Theorem 3.3. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in,\ \in\vee\, q)$-neutrosophic subalgebra of X if and only if the neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. For any $x, y \in X$, let $\alpha \in (0, 0.5]$ be such that $x, y \in T_\in(A; \alpha)$. Then $A_T(x) \geq \alpha$ and $A_T(y) \geq \alpha$. It follows from (3.5) that

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \geq \alpha \wedge 0.5 = \alpha$$

and so that $x * y \in T_\in(A; \alpha)$. Thus $T_\in(A; \alpha)$ is a subalgebra of X for all $\alpha \in (0, 0.5]$. Similarly, $I_\in(A; \beta)$ is a subalgebra of X for all $\beta \in (0, 0.5]$. Now, let $\gamma \in [0.5, 1)$ be such that $x, y \in F_\in(A; \gamma)$. Then $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. Hence

$$A_F(x * y) \leq \bigvee\{A_F x), A_F(y), 0.5\} \leq \gamma \vee 0.5 = \gamma$$

by (3.5), and so $x * y \in F_\in(A; \gamma)$. Thus $F_\in(A; \gamma)$ is a subalgebra of X for all $\gamma \in [0.5, 1)$.

Conversely, let $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ be such that $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are subalgebras of X. If there exist $a, b \in X$ such that

$$A_I(a * b) < \bigwedge\{A_I(a), A_I(b), 0.5\},$$

then we can take $\beta \in (0, 1)$ such that

$$A_I(a * b) < \beta < \bigwedge\{A_I(a), A_I(b), 0.5\}. \tag{3.6}$$

Thus $a, b \in I_\in(A; \beta)$ and $\beta < 0.5$, and so $a * b \in I_\in(A; \beta)$. But, the left inequality in (3.6) induces $a * b \notin I_\in(A; \beta)$, a contradiction. Hence

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$$

for all $x, y \in X$. Similarly, we can show that

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$$

for all $x, y \in X$. Now suppose that

$$A_F(a * b) > \bigvee\{A_F(a), A_F(b), 0.5\}$$

for some $a, b \in X$. Then there exists $\gamma \in (0, 1)$ such that

$$A_F(a * b) > \gamma > \bigvee\{A_F(a), A_F(b), 0.5\}.$$

It follows that $\gamma \in (0.5, 1)$ and $a, b \in F_\in(A; \gamma)$. Since $F_\in(A; \gamma)$ is a subalgebra of X, we have $a * b \in F_\in(A; \gamma)$ and so $A_F(a * b) \leq \gamma$. This is a contradiction, and thus

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$$

for all $x, y \in X$. Using Lemma 3.2, $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

Using Theorem 3.3 and [7, Theorem 3.8], we have the following corollary.

Corollary 3.4. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee q$-subsets $T_{\in\vee q}(A; \alpha)$, $I_{\in\vee q}(A; \beta)$ and $F_{\in\vee q}(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then the neutrosophic $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Theorem 3.5. *Given neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ if and only if the following assertion is valid.*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x * y) \vee 0.5 \geq A_T(x) \wedge A_T(y) \\ A_I(x * y) \vee 0.5 \geq A_I(x) \wedge A_I(y) \\ A_F(x * y) \wedge 0.5 \leq A_F(x) \vee A_F(y) \end{array}\right). \tag{3.7}$$

Proof. Assume that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. Suppose that there are $a, b \in X$ such that $A_T(a * b) \vee 0.5 < A_T(a) \wedge A_T(b) := \alpha$. Then $\alpha \in (0.5, 1]$ and $a, b \in T_\in(A;\alpha)$. Since $T_\in(A;\alpha)$ is a subalgebra of X, it follows that $a * b \in T_\in(A;\alpha)$, that is, $A_T(a * b) \geq \alpha$ which is a contradiction. Thus

$$A_T(x * y) \vee 0.5 \geq A_T(x) \wedge A_T(y)$$

for all $x, y \in X$. Similarly, we know that $A_I(x * y) \vee 0.5 \geq A_I(x) \wedge A_I(y)$ for all $x, y \in X$. Now, if $A_F(x * y) \wedge 0.5 > A_F(x) \vee A_F(y)$ for some $x, y \in X$, then $x, y \in F_\in(A;\gamma)$ and $\gamma \in [0, 0.5)$ where $\gamma = A_F(x) \vee A_F(y)$. But, $x * y \notin F_\in(A;\gamma)$ which is a contradiction. Hence $A_F(x * y) \wedge 0.5 \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying the condition (3.7). Let $x, y, a, b \in X$ and $\alpha, \beta \in (0.5, 1]$ be such that $x, y \in T_\in(A;\alpha)$ and $a, b \in I_\in(A;\beta)$. Then

$$A_T(x * y) \vee 0.5 \geq A_T(x) \wedge A_T(y) \geq \alpha > 0.5,$$
$$A_I(a * b) \vee 0.5 \geq A_I(a) \wedge A_I(b) \geq \beta > 0.5.$$

It follows that $A_T(x * y) \geq \alpha$ and $A_I(a * b) \geq \beta$, that is, $x * y \in T_\in(A;\alpha)$ and $a * b \in I_\in(A;\beta)$. Now, let $x, y \in X$ and $\gamma \in [0, 0.5)$ be such that $x, y \in F_\in(A;\gamma)$. Then $A_F(x * y) \wedge 0.5 \leq A_F(x) \vee A_F(y) \leq \gamma < 0.5$ and so $A_F(x * y) \leq \gamma$, i.e., $x * y \in F_\in(A;\gamma)$. This completes the proof. □

We consider relations between a $(q, \in\vee\, q)$-neutrosophic subalgebra and an $(\in, \in\vee\, q)$-neutrosophic subalgebra.

Theorem 3.6. *In a BCK/BCI-algebra, every $(q, \in\vee\, q)$-neutrosophic subalgebra is an $(\in, \in\vee\, q)$-neutrosophic subalgebra.*

Proof. Let $A = (A_T, A_I, A_F)$ be a $(q, \in\vee\, q)$-neutrosophic subalgebra of a BCK/BCI-algebra X and let $x, y \in X$. Let $\alpha_x, \alpha_y \in (0, 1]$ be such that $x \in T_\in(A;\alpha_x)$ and $y \in T_\in(A;\alpha_y)$. Then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$. Suppose $x*y \notin T_{\in\vee\, q}(A;\alpha_x \wedge \alpha_y)$. Then

$$A_T(x * y) < \alpha_x \wedge \alpha_y, \tag{3.8}$$

$$A_T(x * y) + (\alpha_x \wedge \alpha_y) \leq 1. \tag{3.9}$$

It follows that

$$A_T(x * y) < 0.5. \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$A_T(x*y) < \bigwedge\{\alpha_x, \alpha_y, 0.5\}$$

and so

$$\begin{aligned} 1 - A_T(x*y) &> 1 - \bigwedge\{\alpha_x, \alpha_y, 0.5\} \\ &= \bigvee\{1-\alpha_x, 1-\alpha_y, 0.5\} \\ &\geq \bigvee\{1-A_T(x), 1-A_T(y), 0.5\}. \end{aligned}$$

Hence there exists $\alpha \in (0,1]$ such that

$$1 - A_T(x*y) \geq \alpha > \bigvee\{1-A_T(x), 1-A_T(y), 0.5\}. \tag{3.11}$$

The right inequality in (3.11) induces $A_T(x) + \alpha > 1$ and $A_T(y) + \alpha > 1$, that is, $x, y \in T_q(A;\alpha)$. Since $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic subalgebra of X, we have $x*y \in T_{\in\vee q}(A;\alpha)$. But, the left inequality in (3.11) implies that $A_T(x*y)+\alpha \leq 1$, i.e., $x*y \notin T_q(A;\alpha)$, and $A_T(x*y) \leq 1-\alpha < 1-0.5 = 0.5 < \alpha$, i.e., $x*y \notin T_\in(A;\alpha)$. Hence $x*y \notin T_{\in\vee q}(A;\alpha)$, a contradiction. Thus $x*y \in T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$. Similarly, we can show that if $x \in I_\in(A;\beta_x)$ and $y \in I_\in(A;\beta_y)$ for $\beta_x, \beta_y \in (0,1]$, then $x*y \in I_{\in\vee q}(A;\beta_x \wedge \beta_y)$. Now, let $\gamma_x, \gamma_y \in [0,1)$ be such that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$. If $x*y \notin F_{\in\vee q}(A;\gamma_x \vee \gamma_y)$, then

$$A_F(x*y) > \gamma_x \vee \gamma_y, \tag{3.12}$$

$$A_F(x*y) + (\gamma_x \vee \gamma_y) \geq 1. \tag{3.13}$$

It follows that

$$A_F(x*y) > \bigvee\{\gamma_x, \gamma_y, 0.5\}$$

and so that

$$\begin{aligned} 1 - A_F(x*y) &< 1 - \bigvee\{\gamma_x, \gamma_y, 0.5\} \\ &= \bigwedge\{1-\gamma_x, 1-\gamma_y, 0.5\} \\ &\leq \bigwedge\{1-A_F(x), 1-A_F(y), 0.5\}. \end{aligned}$$

Thus there exists $\gamma \in [0,1)$ such that

$$1 - A_F(x*y) \leq \gamma < \bigwedge\{1-A_F(x), 1-A_F(y), 0.5\}. \tag{3.14}$$

It follows from the right inequality in (3.14) that $A_F(x)+\gamma < 1$ and $A_F(y)+\gamma < 1$, that is, $x, y \in F_q(A;\gamma)$, which implies that $x*y \in F_{\in\vee q}(A;\gamma)$. But, we have $x*y \notin F_{\in\vee q}(A;\gamma)$ by the left inequality in (3.14). This is a contradiction, and so $x*y \in F_{\in\vee q}(A;\gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

The following example shows that the converse of Theorem 3.6 is not true.

TABLE 1. Cayley table of the operation $*$

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.8	0.3
1	0.2	0.3	0.6
2	0.2	0.3	0.6
3	0.7	0.1	0.7
4	0.4	0.4	0.9

Example 3.7. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.
Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by
Then

$$T_\in(A;\alpha) = \begin{cases} \{0,3\} & \text{if } \alpha \in (0.4, 0.5], \\ \{0,3,4\} & \text{if } \alpha \in (0.2, 0.4], \\ X & \text{if } \alpha \in (0, 0.2], \end{cases}$$

$$I_\in(A;\beta) = \begin{cases} \{0\} & \text{if } \beta \in (0.4, 0.5], \\ \{0,4\} & \text{if } \beta \in (0.3, 0.4], \\ \{0,1,2,4\} & \text{if } \beta \in (0.1, 0.3], \\ X & \text{if } \beta \in (0, 0.1], \end{cases}$$

$$F_\in(A;\gamma) = \begin{cases} X & \text{if } \gamma \in (0.9, 1), \\ \{0,1,2,3\} & \text{if } \gamma \in [0.7, 0.9), \\ \{0,1,2\} & \text{if } \gamma \in [0.6, 0.7), \\ \{0\} & \text{if } \gamma \in [0.5, 0.6), \end{cases}$$

which are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Using Theorem 3.3, $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. But it is not a $(q, \in\vee q)$-neutrosophic subalgebra of X since $2 \in T_q(A; 0.83)$ and $3 \in T_q(A; 0.4)$, but $2 * 3 = 2 \notin T_{\in\vee q}(A; 0.4)$.

We provide conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra.

Theorem 3.8. *Assume that any neutrosophic T_Φ-point and neutrosophic I_Φ-point has the value α and β in $(0, 0.5]$, respectively, and any neutrosophic F_Φ-point has the value γ in $[0.5, 1)$ for $\Phi \in \{\in, q, \in\vee q\}$. Then every $(\in, \in\vee q)$-neutrosophic subalgebra is a $(q, \in\vee q)$-neutrosophic subalgebra.*

Proof. Let X be a BCK/BCI-algebra and let $A = (A_T, A_I, A_F)$ be an $(\in, \in\vee q)$-neutrosophic subalgebra of X. For $x, y, a, b \in X$, let $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 0.5]$ be such that $x \in T_q(A;\alpha_x)$, $y \in T_q(A;\alpha_y)$, $a \in I_q(A;\beta_a)$ and $b \in T_q(A;\beta_b)$. Then $A_T(x)+\alpha_x > 1$, $A_T(y)+\alpha_y > 1$, $A_I(a)+\beta_a > 1$ and $A_I(b)+\beta_b > 1$. Since $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 0.5]$, it follows that $A_T(x) > 1-\alpha_x \geq \alpha_x$, $A_T(y) > 1-\alpha_y \geq \alpha_y$, $A_I(a) > 1-\beta_a \geq \beta_a$ and $A_I(b) > 1-\beta_b \geq \beta_b$, that is, $x \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a \in I_\in(A;\beta_a)$ and $b \in I_\in(A;\beta_b)$. Also, let $x \in F_q(A;\gamma_x)$ and $y \in F_q(A;\gamma_y)$ for $x, y \in X$ and $\gamma_x, \gamma_y \in [0.5, 1)$. Then $A_F(x)+\gamma_x < 1$ and $A_F(y)+\gamma_y < 1$, and so $A_F(x) < 1-\gamma_x \leq \gamma_x$ and $A_F(y) < 1-\gamma_y \leq \gamma_y$ since $\gamma_x, \gamma_y \in [0.5, 1)$. This shows that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. It follows from (3.4) that $x*y \in T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$, $a*b \in I_{\in\vee q}(A;\beta_a \wedge \beta_b)$, and $x*y \in F_{\in\vee q}(A;\gamma_x \vee \gamma_y)$. Consequently, $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic subalgebra of X. □

Theorem 3.9. *Both $(\in, \in)$-neutrosophic subalgebra and $(\in\vee q, \in\vee q)$-neutrosophic subalgebra are an $(\in, \in\vee q)$-neutrosophic subalgebra.*

Proof. It is clear that $(\in, \in)$-neutrosophic subalgebra is an $(\in, \in\vee q)$-neutrosophic subalgebra. Let $A = (A_T, A_I, A_F)$ be an $(\in\vee q, \in\vee q)$-neutrosophic subalgebra of X. For any $x, y, a, b \in X$, let α_x, α_y, β_a, $\beta_b \in (0, 1]$ be such that $x \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a \in I_\in(A;\beta_a)$ and $b \in I_\in(A;\beta_b)$. Then $x \in T_{\in\vee q}(A;\alpha_x)$, $y \in T_{\in\vee q}(A;\alpha_y)$, $a \in I_{\in\vee q}(A;\beta_a)$ and $b \in I_{\in\vee q}(A;\beta_b)$ by (3.1) and (3.2). It follows that $x*y \in T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$ and $a*b \in I_{\in\vee q}(A;\beta_a \wedge \beta_b)$. Now, let $x, y \in X$ and $\gamma_x, \gamma_y \in [0, 1)$ be such that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. Then $x \in F_{\in\vee q}(A;\gamma_x)$ and $y \in F_{\in\vee q}(A;\gamma_y)$ by (3.3). Hence $x*y \in F_{\in\vee q}(A;\gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

The converse of Theorem 3.9 is not true in general. In fact, the $(\in, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ in Example 3.7 is neither an $(\in, \in)$-neutrosophic subalgebra nor an $(\in\vee q, \in\vee q)$-neutrosophic subalgebra.

Theorem 3.10. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in (0, 0.5)$, then*

$$\begin{aligned} &x \in T_\in(A;\alpha_x),\ y \in T_\in(A;\alpha_y) \ \Rightarrow\ x*y \in T_q(A;\alpha_x \vee \alpha_y), \\ &x \in I_\in(A;\beta_x),\ y \in I_\in(A;\beta_y) \ \Rightarrow\ x*y \in I_q(A;\beta_x \vee \beta_y), \\ &x \in F_\in(A;\gamma_x),\ y \in F_\in(A;\gamma_y) \ \Rightarrow\ x*y \in F_q(A;\gamma_x \wedge \gamma_y) \end{aligned} \tag{3.15}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0.5, 1]$ and $\gamma_x, \gamma_y \in (0, 0.5)$.

Proof. Let $x, y, a, b, u, v \in X$ and α_x, α_y, β_a, $\beta_b \in (0.5, 1]$ and $\gamma_u, \gamma_v \in (0, 0.5)$ be such that $x \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a \in I_\in(A;\beta_a)$, $b \in I_\in(A;\beta_b)$, $u \in F_\in(A;\gamma_u)$ and $v \in F_\in(A;\gamma_v)$. Then $A_T(x) \geq \alpha_x > 1-\alpha_x$, $A_T(y) \geq \alpha_y > 1-\alpha_y$, $A_I(a) \geq \beta_a > 1-\beta_a$, $A_I(b) \geq \beta_b > 1-\beta_b$, $A_F(u) \leq \gamma_u < 1-\gamma_u$ and $A_F(v) \leq \gamma_v < 1-\gamma_v$. It follows that $x, y \in T_q(A;\alpha_x \vee \alpha_y)$, $a, b \in I_q(A;\beta_a \vee \beta_b)$ and $u, v \in F_q(A;\gamma_u \wedge \gamma_v)$. Since $\alpha_x \vee \alpha_y, \beta_a \vee \beta_b \in (0.5, 1]$ and $\gamma_u \wedge \gamma_v \in (0, 0.5)$, we have $x*y \in T_q(A;\alpha_x \vee \alpha_y)$, $a*b \in I_q(A;\beta_a \vee \beta_b)$ and $u*v \in F_q(A;\gamma_u \wedge \gamma_v)$ by hypothesis. This completes the proof. □

The following corollary is by Theorem 3.10 and [7, Theorem 3.7].

Corollary 3.11. *Every $(\in, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X satisfies the condition (3.15).*

Corollary 3.12. *Every $(q, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X satisfies the condition (3.15).*

Proof. It is by Theorem 3.6 and Corollary 3.11. □

Theorem 3.13. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in (0.5, 1)$, then*

$$
\begin{aligned}
&x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y)\ \Rightarrow\ x * y \in T_\in(A;\alpha_x \vee \alpha_y),\\
&x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y)\ \Rightarrow\ x * y \in I_\in(A;\beta_x \vee \beta_y),\\
&x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y)\ \Rightarrow\ x * y \in F_\in(A;\gamma_x \wedge \gamma_y)
\end{aligned}
\tag{3.16}
$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in (0.5, 1)$.

Proof. Let $x, y, a, b, u, v \in X$ and α_x, α_y, β_a, $\beta_b \in (0, 0.5]$ and $\gamma_u, \gamma_v \in (0.5, 1)$ be such that $x \in T_q(A;\alpha_x)$, $y \in T_q(A;\alpha_y)$, $a \in I_q(A;\beta_a)$, $b \in I_q(A;\beta_b)$, $u \in F_q(A;\gamma_u)$ and $v \in F_q(A;\gamma_v)$. Then $x, y \in T_q(A;\alpha_x \vee \alpha_y)$, $a, b \in I_q(A;\beta_a \vee \beta_b)$ and $u, v \in F_q(A;\gamma_u \wedge \gamma_v)$. Since $\alpha_x \vee \alpha_y, \beta_a \vee \beta_b \in (0, 0.5]$ and $\gamma_u \wedge \gamma_v \in (0.5, 1)$, it follows from the hypothesis that $x * y \in T_q(A;\alpha_x \vee \alpha_y)$, $a * b \in I_q(A;\beta_a \vee \beta_b)$ and $u * v \in F_q(A;\gamma_u \wedge \gamma_v)$. Hence

$$
\begin{aligned}
&A_T(x * y) > 1 - (\alpha_x \vee \alpha_y) \geq \alpha_x \vee \alpha_y, \text{ that is, } x * y \in T_\in(A;\alpha_x \vee \alpha_y),\\
&A_I(a * b) > 1 - (\beta_a \vee \beta_b) \geq \beta_a \vee \beta_b, \text{ that is, } a * b \in I_\in(A;\beta_a \vee \beta_b),\\
&A_F(u * v) < 1 - (\gamma_u \wedge \gamma_v) \leq \gamma_u \wedge \gamma_v, \text{ that is, } u * v \in F_\in(A;\gamma_u \wedge \gamma_v).
\end{aligned}
$$

Consequently, the condition (3.16) is valid for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in (0.5, 1)$. □

Theorem 3.14. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee q$-subsets $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then the following assertions are valid.*

$$
\begin{aligned}
&x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y)\ \Rightarrow\ x * y \in T_{\in\vee q}(A;\alpha_x \vee \alpha_y),\\
&x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y)\ \Rightarrow\ x * y \in I_{\in\vee q}(A;\beta_x \vee \beta_y),\\
&x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y)\ \Rightarrow\ x * y \in F_{\in\vee q}(A;\gamma_x \wedge \gamma_y)
\end{aligned}
\tag{3.17}
$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in [0.5, 1)$.

Proof. Let $x, y, a, b, u, v \in X$ and α_x, α_y, β_a, $\beta_b \in (0, 0.5]$ and $\gamma_u, \gamma_v \in [0.5, 1)$ be such that $x \in T_q(A;\alpha_x)$, $y \in T_q(A;\alpha_y)$, $a \in I_q(A;\beta_a)$, $b \in I_q(A;\beta_b)$, $u \in F_q(A;\gamma_u)$ and $v \in F_q(A;\gamma_v)$. Then $x \in T_{\in\vee q}(A;\alpha_x)$, $y \in T_{\in\vee q}(A;\alpha_y)$, $a \in I_{\in\vee q}(A;\beta_a)$, $b \in I_{\in\vee q}(A;\beta_b)$, $u \in F_{\in\vee q}(A;\gamma_u)$ and $v \in F_{\in\vee q}(A;\gamma_v)$. It follows that $x, y \in T_{\in\vee q}(A;\alpha_x \vee \alpha_y)$, $a, b \in I_{\in\vee q}(A;\beta_a \vee \beta_b)$ and $u, v \in F_{\in\vee q}(A;\gamma_u \wedge \gamma_v)$ which imply from the hypothesis that $x * y \in T_{\in\vee q}(A;\alpha_x \vee \alpha_y)$, $a * b \in I_{\in\vee q}(A;\beta_a \vee \beta_b)$ and $u * v \in F_{\in\vee q}(A;\gamma_u \wedge \gamma_v)$. This completes the proof. □

Corollary 3.15. *Every $(\in, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the condition (3.17).*

Proof. It is by Theorem 3.14 and [7, Theorem 3.9]. □

Theorem 3.16. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$, then the following assertions are valid.*

$$
\begin{aligned}
&x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y) \ \Rightarrow\ x * y \in T_{\in\vee\, q}(A;\alpha_x \vee \alpha_y),\\
&x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y) \ \Rightarrow\ x * y \in I_{\in\vee\, q}(A;\beta_x \vee \beta_y),\\
&x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y) \ \Rightarrow\ x * y \in F_{\in\vee\, q}(A;\gamma_x \wedge \gamma_y)
\end{aligned}
\tag{3.18}
$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0.5, 1]$ and $\gamma_x, \gamma_y \in [0, 0.5)$.

Proof. It is similar to the proof Theorem 3.14. □

Corollary 3.17. *Every $(q, \in\vee\, q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the condition (3.18).*

Proof. It is by Theorem 3.16 and [7, Theorem 3.10]. □

Combining Theorems 3.14 and 3.16, we have the following corollary.

Corollary 3.18. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then the following assertions are valid.*

$$
\begin{aligned}
&x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y) \ \Rightarrow\ x * y \in T_{\in\vee\, q}(A;\alpha_x \vee \alpha_y),\\
&x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y) \ \Rightarrow\ x * y \in I_{\in\vee\, q}(A;\beta_x \vee \beta_y),\\
&x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y) \ \Rightarrow\ x * y \in F_{\in\vee\, q}(A;\gamma_x \wedge \gamma_y)
\end{aligned}
$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Conclusions

We have considered relations between an $(\in, \in\vee\, q)$-neutrosophic subalgebra and a $(q, \in\vee\, q)$-neutrosophic subalgebra. We have discussed characterization of an $(\in, \in\vee\, q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets, and have provided conditions for an $(\in, \in\vee\, q)$-neutrosophic subalgebra to be a $(q, \in\vee\, q)$-neutrosophic subalgebra. We have investigated properties on neutrosophic q-subsets and neutrosophic $\in\vee\, q$-subsets. Our future research will be focused on the study of generalization of this paper.

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