The internal structure of natural numbers and one method for the definition of large prime numbers

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Abstract

It holds that every product of natural numbers can also be written as a sum. The inverse does not hold when 1 is excluded from the product. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article. We prove that primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa. The following theorem is proven: “Every natural number, except for 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2 with the coefficients of the linear combination being -1 or +1.” This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

Keywords: Prime numbers, Composite numbers.

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1 Introduction

It holds that every product of natural numbers can also be written as a sum. The inverse (i.e. each sum of natural numbers can be written as a product) does not hold when 1 is excluded from the product. This is due to prime numbers \( p \) which can be written as a product only in the form of \( p = 1 \cdot p \). For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article.

We prove that each natural number can be written as a sum of three or more consecutive natural numbers except of the powers of 2 and the prime numbers. Each power of 2 and each prime number cannot be written as a sum of three or more consecutive natural numbers. Primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa.
We prove a theorem which is analogous to the fundamental theorem of arithmetic, when we study the positive integers with respect to addition: ‘’Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.’’ This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

2 The sequence \( \mu(k,n) \)

We consider the sequence of natural numbers

\[
\mu(k,n) = k + (k + 1) + (k + 2) + \ldots + (k + n) = \frac{(n+1)(2k+n)}{2}
\]

\( k \in \mathbb{N}^* = \{1, 2, 3, \ldots\} \)

\( n \in A = \{2, 3, 4, \ldots\} \)

For the sequence \( \mu(k,n) \) the following theorem holds:

**Theorem 2.1.** For the sequence \( \mu(k,n) \) the following hold:

1. \( \mu(k,n) \in \mathbb{N}^* \).
2. No element of the sequence is a prime number.
3. No element of the sequence is a power of 2.
4. The range of the sequence is all natural numbers that are not primes and are not powers of 2.

**Proof.**

1. \( \mu(k,n) \in \mathbb{N}^* \) as a sum of natural numbers.

2. \( n \in A = \{2, 3, 4, \ldots\} \) and therefore it holds that

\[
\begin{align*}
  n &\geq 2 \\
  n + 1 &\geq 3
\end{align*}
\]

Also we have that

\[
2k + n \geq 4
\]

\[
\frac{2k + n}{2} \geq \frac{3}{2} > 1
\]
since \( k \in \mathbb{N}^+ \) and \( n \in A = \{2, 3, 4, \ldots\} \). Thus, the product

\[
\frac{(n+1)(2k+n)}{2} = \mu(k,n)
\]

is always a product of two natural numbers different than 1, thus the natural number \( \mu(k,n) \) cannot be prime.

3. Let that the natural number

\[
\mu(k,n) = \frac{(n+1)(2k+n)}{2}
\]

is a power of 2. Then, it exists \( \lambda \in \mathbb{N} \) such as

\[
\frac{(n+1)(2k+n)}{2} = 2^\lambda
\]

and equivalently

\[
(n+1)(2k+n) = 2^{\lambda+1}.
\]

Equation (2.2) can hold if and only if there exist \( \lambda_1, \lambda_2 \in \mathbb{N} \) such as

\[
n + 1 = 2^{\lambda_1} \land 2k + n = 2^{\lambda_2}
\]

and equivalently

\[
n = 2^{\lambda_1} - 1 \quad \text{and} \quad n = 2^{\lambda_2} - 2k.
\]

We eliminate \( n \) from equations (2.3) and we obtain

\[
2^\lambda - 1 = 2^{\lambda_2} - 2k
\]

\[
2k - 1 = 2^{\lambda_2} - 2^{\lambda_1},
\]

which is impossible since the first part of the equation is an odd number and the second part is an even number. Thus, the range of the sequence \( \mu(k,n) \) does not include the powers of 2.

4. We now prove that the range of the sequence \( \mu(k,n) \) includes all natural numbers that are not primes and are not powers of 2. Let a random natural number \( N \) which is not a prime nor a power of 2. Then, \( N \) can be written in the form

\[
N = \chi \psi
\]
where at least one of the $\chi,\psi$ is an odd number $\geq 3$. Let $\chi$ be an odd number $\geq 3$. We will prove that there are always exist $k \in \mathbb{N}$ and $n \in A = \{2,3,\ldots\}$ such as

$$ N = \chi \cdot \psi = \mu(k,n). $$

We consider the following two pairs of $k$ and $n$:

1. $\chi \leq 2\psi - 1, \chi, \psi \in \mathbb{N}$
   
   $$ k = k_1 = \frac{2\psi + 1 - \chi}{2}, \quad n = n_1 = \chi - 1 $$

2. $\chi \geq 2\psi + 1, \chi, \psi \in \mathbb{N}$
   
   $$ k = k_2 = \frac{2\chi + 1 - 2\psi}{2}, \quad n = n_2 = 2\psi - 1 $$

For every $\chi, \psi \in \mathbb{N}$ it holds either the inequality $\chi \leq 2\psi - 1$ or the inequality $\chi \geq 2\psi + 1$. Thus, for each pair of naturals $(\chi, \psi)$, where $\chi$ is odd, at least one of the pairs $(k_1, n_1), (k_2, n_2)$ of equations (2.4), (2.5) is defined. We now prove that when the natural number $k_1$ of equation (2.4) is $k_1 = 0$ then the natural number $k_2$ of equation (2.5) is $k_2 = 1$ and additionally it holds that $n_2 > 2$. For $k_1 = 0$ from equations (2.4) we take

$$ \chi = 2\psi + 1 $$

and from equations (2.5) we have that

$$ k_2 = \frac{(2\psi + 1) + 1 - 2\psi}{2} = 1 $$

$$ n_2 = 2\psi - 1 $$

and because $\psi \geq 2$ we obtain

$$ k_2 = 1 $$

$$ n_2 = 2\psi - 1 \geq 3 > 2 $$

We now prove that when $k_2 = 0$ in equations (2.5), then in equations (2.4) it is $k_1 = 1$ and $n_1 > 2$. For $k_2 = 0$, from equations (2.5) we obtain

$$ \chi = 2\psi - 1 $$

and from equations (2.4) we get
\[
\begin{align*}
k_1 &= \frac{2\nu +1-(2\nu -1)}{2} = 1, \\
n_1 &= \chi -1 = 2\nu - 2 \geq 2
\end{align*}
\]

We now prove that at least one of the \(k_1\) and \(k_2\) is positive. Let \(k_1 < 0 \land k_2 < 0\).

Then from equations (2.4) and (2.5) we have that 

\[
2\nu +1 - \chi < 0 \land \chi +1 - 2\nu < 0.
\]

(2.6)

Taking into account that \(\chi > 1\) is odd, that is \(\chi = 2\rho +1, \rho \in \mathbb{N}\), we obtain from inequalities (2.6)

\[
\begin{align*}
2\nu +1 -(2\rho -1) &< 0 \land (2\rho +1) +1 - 2\nu < 0 \\
2\nu - 2\rho &< 0 \land 2\rho - 2\nu + 2 > 0 \\
\nu &< \rho \land \nu > \rho + 1
\end{align*}
\]

which is absurd. Thus, at least one of \(k_1\) and \(k_2\) is positive.

For equations (2.4) we take

\[
\mu(k_1, n_1) = \left(\frac{n_1 +1}{2} \right) (2k_1 + n_1)
\]

\[
\mu(k_1, n_1) = \frac{(\chi -1 +1) \left(\frac{2\nu+1-\chi}{2} + \chi -1 \right)}{2} = \frac{\chi \cdot 2\cdot \nu}{2} = \chi \nu = N.
\]

For equations (2.5) we obtain

\[
\mu(k_2, n_2) = \left(\frac{n_2 +1}{2} \right) (2k_2 + n_2)
\]

\[
\mu(k_2, n_2) = \frac{(2\nu -1 +1) \left(\frac{2\nu +1 -2\nu}{2} + 2\nu -1 \right)}{2} = \frac{2\nu \chi}{2} = \chi \nu = N.
\]

Thus, there are always exist \(k \in \mathbb{N}^*\) and \(n \in A = \{2,3,4,...\}\) such as

\[
N = \chi \nu = \mu(k,n) \text{ for every } N \text{ which is not a prime number and is not a power of } 2. \quad \square
\]

Example 2.1. For the natural number \(N = 40\) we have
\[ N = 40 = 5 \cdot 8 \]
\[ \chi = 5 \]
\[ \psi = 8 \]

and from equations (2.4) we get
\[ k = k_1 = \frac{16 + 1 - 5}{2} = 6 \]
\[ n = n_1 = 5 - 1 = 4 \]

thus, we obtain
\[ 40 = \mu(6, 4). \]

**Example 2.2.** For the natural number \( N = 51, \)
\[ N = 51 = 3 \cdot 17 = 17 \cdot 3 \]

there are two cases. First case:
\[ N = 51 = 3 \cdot 17 \]
\[ \chi = 3 \]
\[ \psi = 17 \]

and from equations (2.4) we obtain
\[ k = k_1 = \frac{34 + 1 - 3}{2} = 16 \]
\[ n = n_1 = 3 - 1 = 2 \]

thus,
\[ 51 = \mu(16, 2). \]

Second case:
\[ N = 51 = 17 \cdot 3 \]
\[ \chi = 17 \]
\[ \psi = 3 \]

and from equations (2.5) we obtain
\[ k = k_2 = \frac{17 + 1 - 6}{2} = 6 \]
\[ n = n_2 = 6 - 1 = 5 \]

thus,
51 = \mu(6,5).

The second example expresses a general property of the sequence \( \mu(k, n) \). The more composite an odd number that is not prime (or an even number that is not a power of 2) is, the more are the \( \mu(k, n) \) combinations that generate it.

**Example 2.3.**

\[
135 = 15 \cdot 9 = 27 \cdot 5 = 9 \cdot 15 = 45 \cdot 3 = 5 \cdot 27 = 3 \cdot 45
\]

135 = \mu(2,14) = \mu(9,9) = \mu(11,8) = \mu(20,5) = \mu(25,4) = \mu(44,2)

a. 135 = 9 \cdot 15 = \mu(2,14) = \mu(11,8)

\[
135 = 2 + 3 + 4 + \ldots + 15 + 16 = 11 + 12 + 13 + \ldots + 18 + 19.
\]

b. 135 = 5 \cdot 27 = \mu(9,9) = \mu(25,4)

\[
135 = 9 + 10 + 11 + \ldots + 17 + 18 = 25 + 26 + 27 + 28 + 29.
\]

c. 135 = 3 \cdot 45 = \mu(20,5) = \mu(44,2)

\[
135 = 20 + 21 + 22 + 23 + 24 + 25 = 44 + 45 + 46.
\]

In the transitive property of multiplication, when writing a composite odd number or an even number that is not a power of 2 as a product of two natural numbers, we use the same natural numbers \( \chi, \psi \in \mathbb{N} \):

\[
\Phi = \chi \cdot \psi = \psi \cdot \chi.
\]

On the contrary, the natural number \( \Phi \) can be written in the form \( \Phi = \mu(k, n) \) using different natural numbers \( k \in \mathbb{N}^+ \) and \( n \in A = \{2, 3, 4, \ldots\} \), through equations (2.4), (2.5). This difference between the product and the sum can also become evident in example 2.3:

\[
135 = 3 \cdot 45 = 45 \cdot 3
\]

\[
135 = 44 + 45 + 46 = 20 + 21 + 22 + 23 + 24 + 25.
\]

From Theorem 2.1 the following corollary is derived:

**Corollary 2.1.** 1. Every natural number which is not a power of 2 and is not a prime can be written as the sum of three or more consecutive natural numbers.

2. Every power of 2 and every prime number cannot be written as the sum of three or more consecutive natural numbers.

*Proof. Corollary 2.1 is a direct consequence of Theorem 2.1.*
3 The concept of rearrangement

In this paragraph, we present the concept of rearrangement of the composite odd numbers and even numbers that are not power of 2. Moreover, we prove some of the consequences of the rearrangement in the Diophantine analysis. The concept of rearrangement is given from the following definition:

**Definition 3.1.** We say that the sequence \( \mu(k,n), k \in \mathbb{N}^+, n \in A = \{2,3,4,...\} \) is rearranged if there exist natural numbers \( k_1 \in \mathbb{N}^+, n_1 \in A, (k_1, n_1) \neq (k, n) \) such as

\[
\mu(k,n) = \mu(k_1, n_1).
\]

From equation (2.1) written in the form of

\[
\mu(k,n) = k + (k + 1) + (k + 2) + ... + (k + n)
\]

two different types of rearrangement are derived: The “compression”, during which \( n \) decreases with a simultaneous increase of \( k \). The «decompression», during which \( n \) increases with a simultaneous decrease of \( k \). The following theorem provides the criterion for the rearrangement of the sequence \( \mu(k,n) \).

**Theorem 3.1.** 1. The sequence \( \mu(k_1,n_1), (k_1,n_1) \in \mathbb{N}^+ \times A \) can be compressed

\[
\mu(k_1,n_1) = \mu(k_1 + \varphi, n_1 - \omega)
\]

if and only if there exist \( \varphi, \omega \in \mathbb{N}^+, \omega \leq n_1 - 2 \) which satisfies the equation

\[
\omega^2 - (2k_1 + 2n_1 + 1 + 2\varphi) \omega + 2(n_1 + 1) \varphi = 0
\]

\( \varphi, \omega \in \mathbb{N}^+ \) \quad (3.3)

\[
\omega \leq n_1 - 2
\]

2. The sequence \( \mu(k_2,n_2), (k_2,n_2) \in \mathbb{N}^+ \times A \) can be decompressed

\[
\mu(k_2,n_2) = \mu(k_2 - \varphi, n_2 + \omega)
\]

if and only if there exist \( \varphi, \omega \in \mathbb{N}^+, \varphi \leq k_2 - 1 \) which satisfies the equation

\[
\omega^2 + (2k_2 + 2n_2 + 1 - 2\varphi) \omega - 2(n_2 + 1) \varphi = 0
\]

\( \varphi, \omega \in \mathbb{N}^+ \) \quad (3.5)

\[
\varphi \leq k_2 - 1
\]
3. The odd number $\Pi \neq 1$ is prime if and only if the sequence
\[
\mu(k, n) = \Pi \cdot 2^l
\]
\[l, k \in \mathbb{N}^*, n \in A\] cannot be rearranged.

4. The odd $\Pi$ is prime if and only if the sequence
\[
\mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \Pi^2
\]
cannot be rearranged.

**Proof.** 1, 2. We prove part 1 of the corollary and similarly number 2 can also be proven. From equation (4.1) we conclude that the sequence $\mu(k, n)$ can be compressed if and only if there exist $\varphi, \omega \in \mathbb{N}^*$ such as
\[
\mu(k, n) = \mu(k + \varphi, n - \omega).
\]
In this equation the natural number $n - \omega$ belongs to the set $A = \{2, 3, 4, \ldots\}$ and thus
\[n - \omega \geq 2 \iff \omega \leq n - 2.\]
Next, from equations (2.1) we obtain
\[
\mu(k, n) = \mu(k + \varphi, n - \omega)
\]
\[
\frac{(n+1)(2k+n)}{2} = \frac{(n+1)(2k+\varphi+n-\omega)}{2}
\]
and after the calculations we get equation (3.3).

3. The sequence (3.6) is derived from equations (2.4) or (2.5) for $\chi = \Pi$ and $\psi = 2^l$. Thus, in the product $\chi \psi$, the only odd number is $\Pi$. If the sequence $\mu(k, n)$ in equation (3.6) cannot be rearranged then the odd number $\Pi$ has no divisors. Thus, $\Pi$ is prime. Obviously, the inverse also holds.

4. First, we prove equations (3.7). From equation (2.1) we obtain:
\[
\mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \frac{(\Pi-1+1)\left(\frac{2\Pi+1}{2} + \Pi-1\right)}{2} = \Pi^2.
\]
In case that the odd number $\Pi$ is prime in equations (2.4), (2.5) the natural numbers $\chi, \psi$ are unique $\chi = \Pi \wedge \psi = \Pi$, and from equation (2.5) we get
Thus, the sequence
\[ \mu(k, n) = \mu\left(\frac{\Pi+1}{2}, \Pi-1\right) \]
cannot be rearranged. Conversely, if the sequence
\[ \mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \Pi^2 = \Pi \cdot \Pi \]
cannot be rearranged the odd number \( \Pi \) cannot be composite and thus \( \Pi \) is prime. □

We now prove the following corollary:

**Corollary 3.1.**

1. The odd number \( \Phi \),

\[ \Phi = \Pi^2 = \mu\left(\frac{\Pi+1}{2}, \Pi-1\right) \]

\( \Pi = \text{odd} \)
\( \Pi \neq 1 \)

is decompressed and compressed if and only if the odd number \( \Pi \) is composite.

2. The even number \( \alpha_1 \),

\[ \alpha_1 = 2^{l'} \Pi = \mu\left(2^{l'} - \frac{\Pi-1}{2}, \Pi-1\right) \]

\( \Pi = \text{odd} \)
\( 3 \leq \Pi \leq 2^{l'} - 1 \)
\( l \in \mathbb{N}, l \geq 2 \)

cannot be decompressed, while it compresses if and only if the odd number \( \Pi \) is composite.

3. The even number \( \alpha_2 \),

\[ \alpha_2 = 2^{l'} \Pi = \mu\left(\frac{\Pi+1}{2} - 2^{l'-1} 2^{l'+1}, 2^{l'-1} - 1\right) \]

\( \Pi = \text{odd} \)
\( \Pi \geq 2^{l'+1} + 1 \)
\( l \in \mathbb{N}^* \)

cannot be compressed, while it decompresses if and only if the odd number \( \Pi \) is composite.
4. Every even number that is not a power of can be written either in the form of equation (3.9) or in the form of equation (3.10).

Proof. 1. It is derived directly through number (4) of Theorem 3.1. A second proof can be derived through equations (2.4), (2.5) since every composite odd $\Pi$ can be written in the form of $\Pi = \chi \psi$, $\chi, \psi \in \mathbb{N}$, $\chi, \psi$ odds.

2, 3. Let the even number $\alpha$, 

$$\alpha = 2^l \Pi$$

$$\Pi = odd.$$ (3.11)

$$l \in \mathbb{N}^*$$

From equation (2.4) we obtain

$$k = \frac{2 \cdot 2^l + \Pi - 1}{2} = 2^l - \frac{\Pi - 1}{2}$$ (3.12)

$$n = \Pi - 1$$

and since $k, n \in \mathbb{N}, k \geq 1 \wedge n \geq 2$ we get

$$\frac{2 \cdot 2^l + \Pi - 1}{2} \geq 1$$

$$\Pi - 1 \geq 2$$

and equivalently

$$3 \leq \Pi \leq 2^{l+1} - 1.$$ 

In the second of equations (3.12) the natural number $n$ obtains the maximum possible value of $n = \Pi - 1$, and thus the natural number $k$ takes the minimum possible value in the first of equations (3.12). Thus, the even number

$$\alpha = \mu \left( 2^l - \frac{\Pi - 1}{2}, \Pi - 1 \right)$$

cannot decompress. If the odd number $\Pi$ is composite then it can be written in the form of $\Pi = \chi \psi$, $\chi, \psi \in \mathbb{N}^*$, $\chi, \psi$ odds, $\chi, \psi < \Pi$, $\alpha = 2^l \chi \psi$. Therefore, the natural number $\alpha = 2^l \chi \psi$ decompresses since from equations (3.11) it can be written in the form of $\alpha = \mu (k, n)$ with $n = \chi - 1 < \Pi - 1$. Similarly, the proof of 3 is derived from equations (2.5).

4. From the above proof process it follows that every even number that is not a power of 2 can be written either in the form of equation (3.9) or in the form of equation (3.10).
By substituting $\Pi = P = \text{prime}$ in equations of Theorem 3.1 and of corollary 3.1 four sets of equations are derived, each including infinite impossible diophantine equations.

**Example 3.1.** The odd number $P = 999961$ is prime. Thus, combining (1) of Theorem 3.1 with (1) of corollary 3.1 we conclude that there is no pair $(\omega,\varphi) \in \mathbb{N}^2$ with $\omega \leq 999958$ which satisfies the diophantine equation

$$\omega^2 - (2999883 + 2\varphi)\omega + 1999922\varphi = 0.$$

We now prove the following corollary:

**Corollary 3.2.** The square of every prime number can be uniquely written as the sum of consecutive natural numbers.

**Proof.** For $\Pi = P = \text{prime}$ in equation (3.5) we obtain

$$P^2 = \mu\left(\frac{P+1}{2}, P-1\right).$$

(3.13)

According with 4 of Theorem 3.1 the odd $P^2$ cannot be rearranged. Thus, the odd can be uniquely written as the sum of consecutive natural numbers, as given from equation (3.13).

**Example 3.2.** The odd $P = 17$ is prime. From equation (3.13) for $P = 17$ we obtain

$$289 = \mu(9,16)$$

and from equation (2.1) we get

$$289 = 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 + 24 + 25$$

which is the only way in which the odd number 289 can be written as a sum of consecutive natural numbers.

## 4 Natural numbers as linear combination of consecutive powers of 2

According to the fundamental theorem of arithmetic, every natural number can be uniquely written as a product of powers of prime numbers. The previously presented study reveals a correspondence between odd prime numbers and the powers of 2. Thus, the question arises whether there exists a theorem for the powers of 2 corresponding to the fundamental theorem of arithmetic. The answer is given by the following theorem:

**Theorem 4.1.** Every natural number, with the exception of 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.
Proof. Let the odd number $\Pi$ as given from equation

$$\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^\nu \pm 2^{\nu-1} \pm 2^{\nu-2} \pm \ldots \pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^\nu + \sum_{i=0}^{\nu-1} \beta_i 2^i$$

$$\beta_i = \pm 1, i = 0, 1, 2, \ldots, \nu - 1$$

$$\nu \in \mathbb{N}$$

From equation (4.1) for $\nu = 0$ we obtain

$$\Pi = 2^1 + 2^0 = 2 + 1 = 3.$$ 

We now examine the case where $\nu \in \mathbb{N}^*$. The lowest value that the odd number $\Pi$ of equation (4.1) can obtain is

$$\Pi_{\text{min}} = \Pi(\nu) = 2^{\nu+1} + 2^\nu - 2^{\nu-1} - 2^{\nu-2} - \ldots - 2^1 - 1$$

$$\Pi_{\text{min}} = \Pi(\nu) = 2^{\nu+1} + 1.$$ (4.2)

The largest value that the odd number $\Pi$ of equation (4.1) can obtain is

$$\Pi_{\text{max}} = \Pi(\nu) = 2^{\nu+2} - 1.$$ (4.3)

Thus, for the odd numbers $\Pi = \Pi(\nu, \beta_i)$ of equation (4.1) the following inequality holds

$$\Pi_{\text{min}} = 2^{\nu+1} + 1 \leq \Pi(\nu, \beta_i) \leq 2^{\nu+2} - 1 = \Pi_{\text{max}}.$$ (4.4)

The number $N(\Pi(\nu, \beta_i))$ of odd numbers in the closed interval $[2^{\nu+1} + 1, 2^{\nu+2} - 1]$ is

$$N(\Pi(\nu, \beta_i)) = \frac{\Pi_{\text{max}} - \Pi_{\text{min}}}{2} + 1 = \frac{(2^{\nu+2} - 1) - (2^{\nu+1} + 1)}{2} + 1$$

$$N(\Pi(\nu, \beta_i)) = 2^\nu.$$ (4.5)

The integers $\beta_i, i = 0, 1, 2, \ldots, \nu - 1$ in equation (4.1) can take only two values, $\beta_i = -1 \vee \beta_i = +1$, thus equation (4.1) gives exactly $2^\nu = N(\Pi(\nu, \beta_i))$ odd numbers. Therefore, for every $\nu \in \mathbb{N}^*$ equation (4.1) gives all odd numbers in the interval $[2^{\nu+1} + 1, 2^{\nu+2} - 1]$.

We now prove the theorem for the even numbers. Every even number $\alpha$ which is a power of 2 can be uniquely written in the form of $\alpha = 2^\nu, \nu \in \mathbb{N}^*$. We now consider the case where the even number $\alpha$ is not a power of 2. In that case, according to corollary 3.1 the even number $\alpha$ is written in the form of
\[ \alpha = 2^l \Pi \Pi = \text{odd}, \Pi \neq 1, l \in \mathbb{N}^+. \] (4.6)

We now prove that the even number \( \alpha \) can be uniquely written in the form of equation (4.6). If we assume that the even number \( \alpha \) can be written in the form of

\[ \alpha = 2^l \Pi = 2^l \Pi' \]
\[ l \neq l' (l > l') \]
\[ \Pi \neq \Pi' \] (4.7)

\[ l, l' \in \mathbb{N}^+ \]
\[ \Pi, \Pi' = \text{odd} \]
the we obtain

\[ 2^l \Pi = 2^{l'} \Pi' \]
\[ 2^{l-l'} \Pi = \Pi' \]
which is impossible, since the first part of this equation is even and the second odd. Thus, it is \( l = l' \) and we take that \( \Pi = \Pi' \) from equation (4.7). Therefore, every even number \( \alpha \) that is not a power of 2 can be uniquely written in the form of equation (4.6). The odd number \( \Pi \) of equation (4.6) can be uniquely written in the form of equation (4.1), thus from equation (4.6) it is derived that every even number \( \alpha \) that is not a power of 2 can be uniquely written in the form of equation

\[ \alpha = \alpha (l, v, \beta_i) = 2^l \left( 2^{v+1} + 2^v + \sum_{i=0}^{v-1} \beta_i 2^i \right) \]
\[ l \in \mathbb{N}^+, v \in \mathbb{N} \]
\[ \beta_i = \pm 1, i = 0, 1, 2, \ldots, v - 1 \] (4.8)

and equivalently

\[ \alpha = \alpha (l, v, \beta_i) = 2^{l+v+1} + 2^{l+v} + \sum_{i=0}^{v-1} \beta_i 2^{l+i} \]
\[ l \in \mathbb{N}^+, v \in \mathbb{N} \]
\[ \beta_i = \pm 1, i = 0, 1, 2, \ldots, v - 1 \] (4.9)

For \( l = 0 \) we take

\[ 1 = 2^0 \]
\[ 1 = 2^1 - 2^0 \]
thus, it can be written in two ways in the form of equation (4.1). Both the odds of equation (4.1) and the evens of the equation (4.8) are positive. Thus, 0 cannot be written either in the form of equation (4.1) or in the form of equation (4.8). \( \square \)
In order to write an odd number $\Pi \neq 1,3$ in the form of equation (4.1) we initially define the $\nu \in \mathbb{N}^*$ from inequality (4.4). Then, we calculate the sum $2^{\nu+1} + 2^{\nu}$.

If it holds that $2^{\nu+1} + 2^{\nu} < \Pi$ we add the $2^{\nu-1}$, whereas if it holds that $2^{\nu+1} + 2^{\nu} > \Pi$ then we subtract it. By repeating the process exactly $\nu$ times we write the odd number $\Pi$ in the form of equation (4.1). The number of $\nu$ steps needed in order to write the odd number $\Pi$ in the form of equation (4.1) is extremely low compared to the magnitude of the odd number $\Pi$, as derived from inequality (4.4).

**Example 4.1.** For the odd number $\Pi = 23$ we obtain from inequality (4.4)

$2^{\nu+1} + 1 < 23 < 2^{\nu+2} - 1$

$2^{\nu+1} + 2 < 24 < 2^{\nu+2}$

$2^{\nu} < 12 < 2^{\nu+1}$

thus $\nu = 3$. Then, we have

$2^{\nu+1} + 2^{\nu} = 2^4 + 2^3 = 24 > 23$ (thus $2^2$ is subtracted)

$2^4 + 2^3 - 2^2 = 20 < 23$ (thus $2^1$ is added)

$2^4 + 2^3 - 2^2 + 2^1 = 22 < 23$ (thus $2^0 = 1$ is added)

$2^4 + 2^3 - 2^2 + 2^1 + 1 = 23$.

Fermat numbers $F_s$ can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\text{min}}$,

$$F_s = 2^{2^s} + 1 = \Pi_{\text{min}} \left( 2^s - 1 \right) = 2^{2^s} + 2^{2^s-1} - 2^{2^s-2} - 2^{2^s-3} - \ldots - 2^1 - 1.$$  \hspace{1cm} (4.10)

$s \in \mathbb{N}$

Mersenne numbers $M_p$ can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\text{max}}$,

$$M_p = 2^p - 1 = \Pi_{\text{max}} \left( p - 2 \right) = 2^{p-1} + 2^{p-2} + 2^{p-3} + \ldots + 2^1 + 1.$$ \hspace{1cm} (4.11)

$p = \text{prime}$

In order to write an even number $\alpha$ that is not a power of 2 in the form of equation (4.1), initially it is consecutively divided by 2 and it takes of the form of equation (4.6). Then, we write the odd number $\Pi$ in the form of equation (4.1).

**Example 4.2.** By consecutively dividing the even number $\alpha = 368$ by 2 we obtain

$\alpha = 368 = 2^4 \cdot 23$. 

15
Then, we write the odd number $\Pi = 23$ in the form of equation (4.1),

$$23 = 2^4 + 2^3 - 2^2 + 2^1 + 1,$$

and we get

$$368 = 2^4 \left( 2^4 + 2^3 - 2^2 + 2^1 + 1 \right)$$

$$368 = 2^8 + 2^7 - 2^6 + 2^5 + 2^4.$$  

This equation gives the unique way in which the even number $\alpha = 368$ can be written in the form of equation (4.9).

From inequality (4.4) we obtain

$$2^{\nu+1} + 1 \leq \Pi \leq 2^{\nu+2} - 1$$

$$2^{\nu+1} < 2^{\nu+1} + 1 \leq \Pi \leq 2^{\nu+2} - 1 < 2^{\nu+2}$$

$$2^{\nu+1} < \Pi < 2^{\nu+2}$$

$$(\nu + 1) \log 2 < \log \Pi < (\nu + 2) \log 2$$

from which we get

$$\frac{\log \Pi}{\log 2} - 1 < \nu + 1 < \frac{\log \Pi}{\log 2}$$

and finally

$$\nu + 1 = \left\lfloor \frac{\log \Pi}{\log 2} \right\rfloor$$

(4.12)

where $\left\lfloor \frac{\log \Pi}{\log 2} \right\rfloor$ the integer part of $\frac{\log \Pi}{\log 2} \in \mathbb{R}$.

We now give the following definition:

**Definition 4.1.** We define as the conjugate of the odd

$$\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^\nu + \sum_{i=0}^{\nu-1} \beta_i 2^i$$

$$\beta_i = \pm 1, i = 0, 1, 2, 
\nu \in \mathbb{N}^*$$

the odd $\Pi^*$,
\[ \Pi^{*} = \Pi^{*}(v, \gamma_{j}) = 2^{v+1} + 2^{v} + \sum_{j=0}^{v-1} \gamma_{j} 2^{j} \]

\[ \gamma_{i} = \pm 1, j = 0, 1, 2, \ldots, v-1 \]

\[ v \in \mathbb{N}^{*} \]

for which it holds

\[ \gamma_{k} = -\beta_{k} \forall k = 0, 1, 2, \ldots, v-1 . \]

For conjugate odds, the following corollary holds:

**Corollary 4.1.** For the conjugate odds \( \Pi = \Pi(v, \beta_{i}) \) and \( \Pi^{*} = \Pi^{*}(v, \gamma_{i}) \) the following hold:

1. \( (\Pi^{*})^{*} = \Pi . \) \hspace{1cm} (4.16)
2. \( \Pi + \Pi^{*} = 3 \cdot 2^{v+1} . \) \hspace{1cm} (4.17)
3. \( \Pi \) is divisible by 3 if and only if \( \Pi^{*} \) is divisible by 3 .

**Proof.** 1. The 1 of the corollary is an immediate consequence of definition 4.1.

2. From equations (4.13), (4.14) and (4.15) we get

\[ \Pi + \Pi^{*} = \left(2^{v+1} + 2^{v}\right) + \left(2^{v+1} + 2^{v}\right) \]

and equivalently

\[ \Pi + \Pi^{*} = 3 \cdot 2^{v+1} . \]

3. If the odd \( \Pi \) is divisible by 3 then it is written in the form \( \Pi = 3x, x = odd \) and from equation (4.17) we get \( 3x + \Pi^{*} = 3 \cdot 2^{v+1} \) and equivalently \( \Pi^{*} = 3\left(2^{v+1} - x \right) \). Similarly we can prove the inverse. \( \square \)

5 **The T symmetry and a method for defining large prime numbers**

We now give the following definition:

**Definition 5.1.** Define as “symmetry” every specific algorithm which determines the signs of \( \beta_{i} = \pm 1, i = 0, 1, 2, \ldots, v-1 \) in equation (4.1):

\[ \Pi = \Pi(v, \beta_{i}) = 2^{v+1} + 2^{v} \pm 2^{v-1} \pm 2^{v-2} \pm \ldots \pm 2^{1} \pm 2^{0} = 2^{v+1} + 2^{v} + \sum_{i=0}^{v-1} \beta_{i} 2^{i} \]

\[ \beta_{i} = \pm 1, i = 0, 1, 2, \ldots, v-1 \]

\[ v \in \mathbb{N} \]
Next, we develop a specific symmetry, the $T$ symmetry.

If the natural number $\nu$, in the equation (4.1), is not a prime and is not a power of 2, the equation (2.1) gives

$$\nu = \mu(k,n) = \frac{(n+1)(2k+n)}{2} = k + (k+1) + (k+2) + \ldots + (k+n).$$  \hspace{1cm} (5.1)

$k \in \mathbb{N}^*, n \in A = \{2,3,4,\ldots\}$

We define the odd number $T_i = \prod(\nu = \mu(k,n)) = T_i(k,n)$ as follows: In the right side of equation (4.1), from left to right, we take $k$ signs -1, and then $(k+1)$ signs +1, $(k+2)$ signs -1, $(k+3)$ signs +1 etc., according to the right side of equation (5.1). After making some calculations we have

$$T_i = T_i(k,n) = 2^{\frac{(n+1)(2k+n)}{2}} + \left( \sum_{j=0}^{n} (-1)^j \times 2^{\frac{(n+1)(2k+n)}{2} + \frac{1}{2} \sum_{i=j}^{j}(k+i)} \right) - (-1)^n.$$  \hspace{1cm} (5.2)

$k \in \mathbb{N}^*, n \in A$

and

$$T_i^* = T_i^*(k,n) = 3 \times 2^{\mu(k,n)+1} - T_i(k,n)$$

and equivalently

$$T_i^* = T_i^*(k,n) = 2^{\frac{(n+1)(2k+n)}{2}} - \left( \sum_{j=1}^{n} (-1)^j \times 2^{\frac{(n+1)(2k+n)}{2} + \frac{1}{2} \sum_{i=j}^{j}(k+i)} \right) + (-1)^n.$$  \hspace{1cm} (5.3)

$k \in \mathbb{N}^*, n \in A$

We write the equation (5.1) in the form

$$\nu = \mu(k,n) = \frac{(n+1)(2k+n)}{2} = (k+n) + (k+n-1) + (k+n-2) + \ldots + k.$$  \hspace{1cm} (5.4)

$k \in \mathbb{N}^*, n \in A$

We define the odd number $T_2 = \prod(\nu = \mu(k,n)) = T_2(k,n)$ by the same way as we defined $T_i = \prod(\nu = \mu(k,n)) = T_i(k,n)$ but the signs in equation (4.1) are now determined according to the right side of equation (5.4), $(k+n)$ signs -1, $(k+n-1)$ signs +1, $(k+n-2)$ signs -1, $(k+n-3)$ signs +1 etc. After making some calculations we have
\[
T_2 = T_2(k, n) = 2^{\frac{(n+1)(2k+n)}{2}} + \left( \sum_{j=0}^{n} (-1)^{j} \times 2^{\frac{(n+1)(2k+n)}{2} - \sum_{i=0}^{j}(k+n-i)} \right) - (-1)^n
\]  
(5.5)

for \( k \in \mathbb{N}^+, n \in A \)

and

\[
T_2^{*} = T_2^{*}(k, n) = 3 \times 2^{\mu(k, n)+1} - T_2(k, n)
\]

and equivalently

\[
T_2^{*} = T_2^{*}(k, n) = 2^{\frac{(n+1)(2k+n)}{2}} - \left( \sum_{j=0}^{n} (-1)^{j} \times 2^{\frac{(n+1)(2k+n)}{2} - \sum_{i=0}^{j}(k+n-i)} \right) + (-1)^n.
\]  
(5.6)

for \( k \in \mathbb{N}^+, n \in A \)

Equations (5.2), (5.3), (5.5) and (5.6) define the T symmetry.

A method for the determination of large prime numbers emerges from the study we presented. This method is completely different from previous methods [1-5]. For the T symmetry holds:

“There are pairs \((k, n) \in \mathbb{N}^+ \times A\),

\[n \neq 3 + 4l, l \in \mathbb{N}\]

for which one or more of \(T_1(k, n), T_1^{*}(k, n), T_2(k, n), T_2^{*}(k, n)\) are prime numbers.”

We will present three examples:

1. The number
\[T_2(11, 5) = 2^{82} + 2^{66} - 2^{51} + 2^{37} - 2^{24} + 2^{12} - 2^1 + 1 = 4835777063183149145526271\] is a prime.

The number
\[T_1^{*}(11, 5) = 2^{83} - 2^{71} + 2^{59} - 2^{46} + 2^{32} - 2^{17} + 2^1 - 1 = 9669045950065986429124609\] is a prime.

2. The number
\[T_1(23, 4) = 2^{126} + 2^{103} - 2^{79} + 2^{54} - 2^{28} + 2^1 - 1 = 85070601871438813228787070915221389313\] is a prime.

3. The number \(T_1^{*}(80, 2) = 2^{245} - 2^{164} + 2^{83} - 2^1 + 1 = 56539106072908298546665496639747195212032793441072154605920000979840794623\) (74 digits) is a prime.

The number of digits of the primes calculated by the method is of order
The smallest prime number given by the method is \( T_1(2, 2) = 2^{10} + 2^5 - 2^5 + 2^1 - 1 = 1249 \). Also, it doesn't give prime numbers Fermat and Mersenne. The method may be further investigated for the form of the pairs \((k, n) \in \mathbb{N}^* \times A\) in equations (5.2), (5.3), (5.5) and (5.6).

We now cite three remarkable properties of the \( T \) symmetry. When the numbers of the \( T \) symmetry are not primes, with high probability, one or more of them are the product of a set of small primes with a large prime (with ratio of the number of digits at least 3:1 in the decimal system). We give an example for \( n=4 \) and \( k=1, 2, 3, \ldots, 23 \).

**Example 5.1.**

1. \( T_1^*(1, 4) = 5 \times 21107. \)
2. \( T_1(2, 4) = 3 \times 853291 \)
   \[ T_2(2, 4) = 3 \times 709651 \]
   \[ T_2^*(2, 4) = 3 \times 1387501. \]
3. \( T_1^*(3, 4) = 126337279 \) (9 digits) is a prime
   \[ T_2^*(3, 4) = 133701391 \) (9 digits) is a prime.
4. \( T_2^*(4, 4) = 3 \times 13 \times 109913929. \)
5. \( T_2(5, 4) = 68853 \ 174209 \) (11 digits) is a prime
   \[ T_2^*(5, 4) = 19 \times 7226592421. \]
6. \( T_2(6, 4) = 3 \times 7 \times 104817 \ 455293. \)
7. \( T_2(7, 4) = 37 \times 1 \ 902785 \ 687213 \)
   \[ T_2^*(7, 4) = 11 \times 12791196555101. \]
8. \( T_1(8, 4) = 3 \times 47 \times 16032473 \ 358917. \)
9. \( T_1^*(11, 4) = 1301 \times 113403 \ 483925 \ 962179. \)
10. \( T_1^*(12, 4) = 3 \times 13 \times 121 \times 071540 \times 832866 \times 439273 \)
\( T_2(12, 4) = 3 \times 7 \times 112 \times 439012 \times 815828 \times 430653 \)
\( T_2^*(12, 4) = 3 \times 89 \times 17 \times 686630 \times 918247 \times 456093. \)

11. \( T_1^*(13, 4) = 5 \times 30221 \times 300928 \times 544913 \times 175347 \)
\( T_2^*(13, 4) = 2239 \times 67 \times 492251 \times 451483 \times 773121. \)

12. \( T_1^*(14, 4) = 3 \times 19 \times 107 \times 792 \times 844025 \times 087630 \times 419877 \)

13. \( T_1^*(15, 4) = 23 \times 6 \times 727832 \times 337541 \times 722681 \times 821273. \)

14. \( T_2(16, 4) = 3 \times 825 \times 294146 \times 583166 \times 134057 \times 740971. \)

15. \( T_2(17, 4) = 3541 \times 22 \times 374527 \times 052572 \times 768094 \times 438269. \)

16. \( T_1^*(18, 4) = 3 \times 73 \times 2903 \times 7 \times 975677 \times 388569 \times 543733 \times 588379. \)

17. \( T_2^*(19, 4) = 11 \times 641 \times 23012 \times 234740 \times 860744 \times 903766 \times 035421. \)

18. \( T_1^*(20, 4) = 3 \times 6 \times 643069 \times 260 \times 536928 \times 672371 \times 642740 \times 686521. \)

19. \( T_2(21, 4) = 7 \times 79 \times 150 \times 229208 \times 340754 \times 381651 \times 561471 \times 195673 \) (33 digits).

20. \( T_1^*(22, 4) = 3 \times 29 \times 1259 \times 24 \times 270828 \times 201501 \times 431550 \times 885053 \times 400181 \) (32 digits)
\( T_1^*(22, 4) = 3 \times 1933 \times 916 \times 866933 \times 835909 \times 456002 \times 715952 \times 336617 \) (33 digits).

21. \( T_1^*(23, 4) = 85 \times 070601 \times 871438 \times 813228 \times 787070 \times 915221 \times 389313 \) (38 digits) is a prime
\( T_1^*(23, 4) = 18269 \times 9313 \times 108178 \times 842029 \times 359502 \times 101081 \times 537291 \) (34 digits)
\( T_2^*(23, 4) = 19 \times 89 \times 100615 \times 720181 \times 338817 \times 896100 \times 110722 \times 568301 \) (36 digits).

For
\[ n = 3 + 4l, l \in \mathbb{N} \] 
(5.9)

the numbers of the T symmetry have 3 as a factor. In these cases, we factorize the numbers of the T symmetry in order to identify the ones which are the product of a set of small primes with a
large prime (with ratio of the number of digits at least 3:1 in the decimal system). We give an example for l=0 and k=1, 2, 3,........, 33.

Example 5.2.

1. $T_2^*(1,3) = 3 \times 1327$.

2. $T_2^*(2,3) = 3 \times 21523$.

3. $T_2^*(4,3) = 3 \times 5 \times 570891$.

4. $T_1(5,3) = 3 \times 46 \times 115669$.

5. $T_2^*(7,3) = 3 \times 5 \times 4579 \times 065839$.

6. $T_2^*(9,3) = 3^2 \times 1 \times 954448 \times 845369$.

7. $T_1(10,3) = 3 \times 73 \times 643264 \times 201901$.

8. $T_1(11,3) = 3 \times 5 \times 23 \times 653042 \times 193943$.

9. $T_1(12,3) = 3^2 \times 4004 \times 176893 \times 145543$.

10. $T_2(13,3) = 3 \times 7 \times 31 \times 107 \times 151 \times 54806 \times 826689$.

11. $T_1(14,3) = 3 \times 11 \times 279513 \times 180897 \times 836063$.

12. $T_1^*(16,3) = 3^3 \times 7 \times 19 \times 73 \times 18014 \times 329790 \times 791679$.

13. $T_1(17,3) = 3 \times 12593 \times 178481 \times 196 \times 7683 \times 97934 \times 630229$.

14. $T_1(18,3) = 3^2 \times 9239 \times 7 \times 269488 \times 227993 \times 959889$.

15. $T_1(19,3) = 3 \times 5 \times 73 \times 331 \times 26 \times 683841 \times 696377 \times 422587$

   $T_1^*(19,3) = 3 \times 7^2 \times 127 \times 269 \times 337 \times 11429 \times 204013 \times 400937$

   $T_2^*(19,3) = 3 \times 5 \times 557 \times 2315 \times 117990 \times 184578 \times 945803$.

16. $T_2(20,3) = 3 \times 23 \times 89 \times 683 \times 9041 \times 4080 \times 688125 \times 380017$.

17. $T_1^*(21,3) = 3 \times 47 \times 178481 \times 196 \times 765246 \times 663328 \times 879957$
\[ T_2^* (21,3) = 3^3 \times 83 \times 23473 \times 282403703 \times 315945 \times 507251. \]

18. \( T_2^* (22,5) = 3 \times 5273 \times 5008417 \times 809828 \times 231066 \times 746851. \)

19. \( T_2^* (23,3) = 3 \times 5^2 \times 137 \times 211 \times 379 \times 1542 \times 751819 \times 716389 \times 148523. \)

20. \( T_2^* (24,3) = 3 \times 2731 \times 7487 \times 8191 \times 20183749 \times 276015 \times 547071. \)

21. \( T_1^* (25,3) = 3 \times 13 \times 269 \times 636697 \times 323309 \times 475541. \)

22. \( T_1^* (26,3) = 3 \times 47 \times 829 \times 22210 \times 374525 \times 858205 \times 252016 \times 927831. \)

23. \( T_1^* (27,3) = 3 \times 233 \times 1103 \times 2089 \times 48091 \times 1072 \times 567317 \times 671651 \times 381903. \)

24. \( T_1^* (28,3) = 3^2 \times 7 \times 11 \times 31 \times 151 \times 331 \times 1237940038 \times 132458773513 \times 502719. \)

25. \( T_1^* (29,3) = 3 \times 5 \times 13 \times 29 \times 43 \times 83 \times 113 \times 127 \times 151 \times 59 \times 359638 \times 928368 \times 977041. \)

26. \( T_1^* (30,3) = 3^2 \times 47 \times 829 \times 22210 \times 374525 \times 858205 \times 252016 \times 927831. \)

27. \( T_1^* (31,3) = 3 \times 7 \times 23 \times 83 \times 89 \times 109 \times 599479 \times 23 \times 353056 \times 263230 \times 084539 \times 231539. \)

28. \( T_1^* (32,3) = 3 \times 37 \times 39237684 \times 468660 \times 613729 \times 344084 \times 167488 \times 872817 \times 39 \text{ digits}. \)

29. \( T_1^* (33,3) = 3^2 \times 23 \times 857 \times 3 \times 928422 \times 863348 \times 787826 \times 215906 \times 015441 \times 564473 \times 37 \text{ digits}. \)

\[ T_2^* (33,3) = 3^4 \times 20 \times 286419 \times 848 \times 220926 \times 630659 \times 241732 \times 391340 \times 317419 \times 33 \text{ digits}. \]

Fermat and Mersenne, for odds \( N \neq 3 \) of the form \( N = 2^n + 1 = 2^n + 1^n \), \( n \in \mathbb{N}^+ \) and \( N = 2^n - 1 = 2^n - 1^n \), \( n \in \mathbb{N}^+ \), respectively, chose the values of \( n \in \mathbb{N} \) for which the odd \( N \), firstly, does not have 3 as a factor (\( n \equiv 2^n, s \equiv \mathbb{N} \) and \( n = \text{prime} \) respectively). This has as a consequence that the Fermat and Mersenne numbers are not divisible by 3, that is, they are not divisible by \( \frac{1}{3} \) of the odd numbers (that are smaller than \( N \)). This non-divisibility by 3, is a property of the numbers of the T symmetry for \( n=5 \). Consequently, the odds \( T_1^*(k,5) \), \( T_2^*(k,5) \), \( T_2(k,5) \), \( k \in \mathbb{N}^+ \) are not divisible by \( \frac{1}{3} \) of the odd numbers (that are smaller than \( T_1^*(k,5) \), \( T_1^*(k,5) \), \( T_2(k,5) \), \( T_2^*(k,5) \), \( k \in \mathbb{N}^+ \)). Because of this, the method is particularly efficient for \( n=5 \). We give an example for \( n=5 \) and for small values of \( k \), \( k=1, 2, 3, \ldots, 18 \).
Example 5.3.

1. $T_2(2,5) = 270\,500807$ (9 digits) is a prime.

2. $T_2(3,5) = 17246\,461711$ (11 digits) is a prime

   $T_i^*(3,5) = 32342\,343169$ (11 digits) is a prime.

3. $T_i^*(4,5) = 2\,132417\,969153$ (13 digits) is a prime.

4. $T_1^*(8,5) = 36\,821571\,153497\,669633$ (20 digits) is a prime.

5. $T_2(9,5) = 1180\,663669\,517502\,645247$ (22 digits) is a prime.

6. $T_1(10,5) = 75631\,614682\,207162\,007551$ (23 digits) is a prime.

7. $T_2(11,5) = 4\,835777\,063183\,149145\,526271$ (25 digits) is a prime

   $T_i^*(11,5) = 9\,669045\,950065\,986429\,124609$ (25 digits) is a prime.

8. $T_1^*(12,5) = 618\,894471\,001773\,327207\,104513$ (27 digits) is a prime.

9. $T_2(16,5) = 5192\,299334\,412545\,020553\,193752\,494079$ (34 digits) is a prime.

10. $T_i^*(18,5) = 42\,535214\,735633\,635683\,576920\,453379\,260417$ (38 digits) is a prime.

From the above study it emerges that the method is applied in two ways:

a. We factorize the numbers of the T symmetry and identify the ones that are products of a set of prime numbers with a comparatively larger prime number.

b. We identify the prime numbers of the T symmetry, via a primality test, when the relation (5.7) holds.

We suggest, in both cases, that a specific $n \in \mathbb{A} = \{2,3,4,...\}$ should be chosen, and then the values $k=1,2,3,...$ can be given in equations (5.2), (5.3), (5.5) and (5.6).

References


