The internal structure of natural numbers and one method for the definition of large prime numbers

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Abstract

It holds that every product of natural numbers can also be written as a sum. The inverse does not hold when 1 is excluded from the product. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article. We prove that primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa. The following theorem is proven: “Every natural number, except for 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2 with the coefficients of the linear combination being -1 or +1.” This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

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1. INTRODUCTION

It holds that every product of natural numbers can also be written as a sum. The inverse (i.e. each sum of natural numbers can be written as a product) does not hold when 1 is excluded from the product. This is due to prime numbers $p$ which can be written as a product only in the form of $p = 1 \cdot p$. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article.

We prove that each natural number can be written as a sum of three or more consecutive natural numbers except of the powers of 2 and the prime numbers. Each power of 2 and each prime number cannot be written as a sum of three or more consecutive natural numbers. Primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa.

We prove a theorem which is analogous to the fundamental theorem of arithmetic, when we study the positive integers with respect to addition: “Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.” This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.
2. THE SEQUENCE $\mu(k, n)$

We consider the sequence of natural numbers

$$\mu(k, n) = k + (k + 1) + (k + 2) + \ldots + (k + n) = \frac{(n+1)(2k+n)}{2}$$

$$k \in \mathbb{N}^+ = \{1, 2, 3, \ldots\}$$

$$n \in A = \{2, 3, 4, \ldots\}$$

(2.1)

For the sequence $\mu(k, n)$ the following theorem holds:

Theorem 2.1.

For the sequence $\mu(k, n)$ the following hold:

1. $\mu(k, n) \in \mathbb{N}^+$.
2. No element of the sequence is a prime number.
3. No element of the sequence is a power of 2.
4. The range of the sequence is all natural numbers that are not primes and are not powers of 2.

Proof.

1. $\mu(k, n) \in \mathbb{N}^+$ as a sum of natural numbers.

2. $n \in A = \{2, 3, 4, \ldots\}$ and therefore it holds that

$$n \geq 2$$

$$n + 1 \geq 3$$

Also we have that

$$2k + n \geq 4$$

$$\frac{2k + n}{2} \geq \frac{3}{2} > 1$$

since $k \in \mathbb{N}^+$ and $n \in A = \{2, 3, 4, \ldots\}$. Thus, the product

$$\frac{(n+1)(2k+n)}{2} = \mu(k, n)$$

is always a product of two natural numbers different than 1, thus the natural number $\mu(k, n)$ cannot be prime.
3. Let that the natural number \( \mu(k,n) = \frac{(n+1)(2k+n)}{2} \) is a power of \( 2 \). Then, it exists \( \lambda \in \mathbb{N} \) such as
\[
\frac{(n+1)(2k+n)}{2} = 2^\lambda.
\]
Equation (2.2) can hold if and only if there exist \( \lambda_1, \lambda_2 \in \mathbb{N} \) such as
\[
n + 1 = 2^{\lambda_1} \wedge 2k + n = 2^{\lambda_2}
\]
and equivalently
\[
\begin{aligned}
n &= 2^{\lambda_1} - 1 \\
n &= 2^{\lambda_2} - 2k
\end{aligned}
\]
We eliminate \( n \) from equations (2.3) and we obtain
\[
2^{\lambda_1} - 1 = 2^{\lambda_2} - 2k \quad \text{and equivalently}
\]
\[
2k - 1 = 2^{\lambda_2} - 2^{\lambda_1}
\]
which is impossible since the first part of the equation is an odd number and the second part is an even number. Thus, the range of the sequence \( \mu(k,n) \) does not include the powers of \( 2 \).

4. We now prove that the range of the sequence \( \mu(k,n) \) includes all natural numbers that are not primes and are not powers of \( 2 \). Let a random natural number \( N \) which is not a prime nor a power of \( 2 \). Then, \( N \) can be written in the form
\[
N = \chi \cdot \psi
\]
where at least one of the \( \chi, \psi \) is an odd number \( \geq 3 \). Let \( \chi \) be an odd number \( \geq 3 \). We will prove that there are always exist \( k \in \mathbb{N} \) and \( n \in A = \{2, 3, \ldots\} \) such as
\[
N = \chi \cdot \psi = \mu(k,n).
\]
We consider the following two pairs of \( k \) and \( n \):

\[
\begin{aligned}
\chi &\leq 2\psi - 1, \chi, \psi \in \mathbb{N} \\
k &= k_1 = \frac{2\psi + 1 - \chi}{2} \\
n &= n_1 = \chi - 1
\end{aligned}
\]  \quad (2.4)

\[
\begin{aligned}
\chi &\geq 2\psi + 1, \chi, \psi \in \mathbb{N} \\
k &= k_2 = \frac{\chi + 1 - 2\psi}{2} \\
n &= n_2 = 2\psi - 1
\end{aligned}
\]  \quad (2.5)
For every \( \chi, \psi \in \mathbb{N} \) it holds either the inequality \( \chi \leq 2\psi - 1 \) or the inequality \( \chi \geq 2\psi + 1 \). Thus, for each pair of naturals \((\chi, \psi)\), where \( \chi \) is odd, at least one of the pairs \((k_1, n_1)\), \((k_2, n_2)\) of equations (2.4), (2.5) is defined. We now prove that “when the natural number \( k_1 \) of equation (2.4) is \( k_1 = 0 \) then the natural number \( k_2 \) of equation (2.5) is \( k_2 = 1 \) and additionally it holds that \( n_2 > 2 \).” For \( k_1 = 0 \) from equations (2.4) we take
\[
\chi = 2\psi + 1
\]
and from equations (2.5) we have that
\[
k_2 = \frac{(2\psi + 1) + 1 - 2\psi}{2} = 1
\]
\[
n_2 = 2\psi - 1
\]
and because \( \psi \geq 2 \) we obtain
\[
k_2 = 1
\]
\[
n_2 = 2\psi - 1 \geq 3 > 2
\].

We now prove that when \( k_2 = 0 \) in equations (2.5), then in equations (2.4) it is \( k_1 = 1 \) and \( n_1 > 2 \). For \( k_2 = 0 \), from equations (2.5) we obtain
\[
\chi = 2\psi - 1
\]
and from equations (2.4) we get
\[
k_1 = \frac{2\psi + 1 - (2\psi - 1)}{2} = 1
\]
\[
n_1 = \chi - 1 = 2\psi - 2 \geq 2
\].

We now prove that at least one of the \( k_1 \) and \( k_2 \) is positive. Let \( k_1 < 0 \land k_2 < 0 \).
Then from equations (2.4) and (2.5) we have that
\[
2\psi + 1 - \chi < 0 \land \chi + 1 - 2\psi < 0 .
\] (2.6)

Taking into account that \( \chi > 1 \) is odd, that is \( \chi = 2\rho + 1, \rho \in \mathbb{N} \), we obtain from inequalities (2.6)
\[
2\psi + 1 - (2\rho - 1) < 0 \land (2\rho + 1) + 1 - 2\psi < 0 \\
2\psi - 2\rho < 0 \land 2\rho - 2\psi + 2 > 0 \\
\psi < \rho \land \psi > \rho + 1
\]
which is absurd. Thus, at least one of \( k_1 \) and \( k_2 \) is positive.

For equations (2.4) we take
\[
\mu(k_i, n_i) = \frac{(n_i + 1)(2k_i + n_i)}{2} \\
= \frac{(\chi - 1 + 1)(2 \psi^2 \chi + 1 - \chi - 1)}{2} = \frac{\chi(2) \psi}{2} = \chi \psi = N
\]

For equations (2.5) we obtain
\[
\mu(k_2, n_2) = \frac{(n_2 + 1)(2k_2 + n_2)}{2} \\
= \frac{(2\psi - 1 + 1)(2 \chi^2 + 1 - 2\psi + 2\psi - 1)}{2} = \frac{2\psi \chi}{2} = \chi \psi = N
\]

Thus, there are always exist \( k \in \mathbb{N}^* \) and \( n \in A = \{2, 3, 4,...\} \) such as
\[N = \chi \psi = \mu(k, n)\] for every \( N \) which is not a prime number and is not a power of 2. \( \square \)

**Example 2.1.** For the natural number \( N = 40 \) we have
\[N = 40 = 5 \cdot 8\] \( \chi = 5 \)
\( \psi = 8 \)
and from equations (2.4) we get
\[k = k_i = \frac{16 + 1 - 5}{2} = 6 \]
\[n = n_i = 5 - 1 = 4 \]
thus, we obtain
\[40 = \mu(6, 4).\]

**Example 2.2.** For the natural number \( N = 51, \)
\[N = 51 = 3 \cdot 17 = 17 \cdot 3 \]
there are two cases. First case:
\[N = 51 = 3 \cdot 17\] \( \chi = 3 \)
\( \psi = 17 \)
and from equations (2.4) we obtain
\[k = k_i = \frac{34 + 1 - 3}{2} = 16 \]
\[n = n_i = 3 - 1 = 2 \]
thus,
\[51 = \mu(16, 2).\]
Second case:
\[ N = 51 = 17 \cdot 3 \]
\[ \chi = 17 \]
\[ \psi = 3 \]
and from equations (2.5) we obtain
\[ k = k_2 = \frac{17 + 1 - 6}{2} = 6 \]
\[ n = n_2 = 6 - 1 = 5 \]
thus,
51 = \mu(6, 5).

The second example expresses a general property of the sequence \( \mu(k, n) \). The more composite an odd number that is not prime (or an even number that is not a power of 2) is, the more are the \( \mu(k, n) \) combinations that generate it.

**Example 2.3.**

\[ 135 = 15 \cdot 9 = 27 \cdot 5 = 9 \cdot 15 = 45 \cdot 3 = 5 \cdot 27 = 3 \cdot 45 \]
\[ 135 = \mu(2, 14) = \mu(9, 9) = \mu(11, 8) = \mu(20, 5) = \mu(25, 4) = \mu(44, 2) \]

a. 135 = 9 \cdot 15 = \mu(2, 14) = \mu(11, 8)
135 = 2 + 3 + 4 + \ldots + 15 + 16 = 11 + 12 + 13 + \ldots + 18 + 19.

b. 135 = 5 \cdot 27 = \mu(9, 9) = \mu(25, 4)
135 = 9 + 10 + 11 + \ldots + 17 + 18 = 25 + 26 + 27 + 28 + 29.

c. 135 = 3 \cdot 45 = \mu(20, 5) = \mu(44, 2)
135 = 20 + 21 + 22 + 23 + 24 + 25 = 44 + 45 + 46.

In the transitive property of multiplication, when writing a composite odd number or an even number that is not a power of 2 as a product of two natural numbers, we use the same natural numbers \( \chi, \psi \in \mathbb{N} \):

\[ \Phi = \chi \cdot \psi = \psi \cdot \chi. \]

On the contrary, the natural number \( \Phi \) can be written in the form \( \Phi = \mu(k, n) \) using different natural numbers \( k \in \mathbb{N}^* \) and \( n \in A = \{2, 3, 4, \ldots\} \), through equations (2.4), (2.5). This difference between the product and the sum can also become evident in example 2.3:

\[ 135 = 3 \cdot 45 = 45 \cdot 3 \]
\[ 135 = 44 + 45 + 46 = 20 + 21 + 22 + 23 + 24 + 25. \]
From Theorem 2.1 the following corollary is derived:

**Corollary 2.1.** “1. Every natural number which is not a power of 2 and is not a prime can be written as the sum of three or more consecutive natural numbers.

2. Every power of 2 and every prime number cannot be written as the sum of three or more consecutive natural numbers.”

**Proof.** Corollary 2.1 is a direct consequence of Theorem 2.1. □

### 3. THE CONCEPT OF REARRANGEMENT

In this paragraph, we present the concept of rearrangement of the composite odd numbers and even numbers that are not power of 2. Moreover, we prove some of the consequences of the rearrangement in the Diophantine analysis. The concept of rearrangement is given from the following definition:

**Definition.** “We say that the sequence \( \mu(k,n) \), \( k \in \mathbb{N}^+, n \in A = \{2,3,4,...\} \) is rearranged if there exist natural numbers \( k_1 \in \mathbb{N}^+, n_1 \in A \), \( (k_1,n_1) \neq (k,n) \) such as

\[
\mu(k,n) = \mu(k_1,n_1).
\]  

From equation (2.1) written in the form of

\[
\mu(k,n) = k + (k+1) + (k+2) + ..... + (k+n)
\]

two different types of rearrangement are derived: The “compression”, during which \( n \) decreases with a simultaneous increase of \( k \). The «decompression», during which \( n \) increases with a simultaneous decrease of \( k \). The following theorem provides the criterion for the rearrangement of the sequence \( \mu(k,n) \).

**Theorem 3.1.** ”’1. The sequence \( \mu(k_1,n_1),(k_1,n_1) \in \mathbb{N}^+ \times A \) can be compressed

\[
\mu(k_1,n_1) = \mu(k_1 + \varphi,n_1 - \omega)
\]  

if and only if there exist \( \varphi, \omega \in \mathbb{N}^+, \omega \leq n_1 - 2 \) which satisfies the equation

\[
\omega^2 - (2k_1 + 2n_1 +1 + 2\varphi)\omega + 2(n_1+1)\varphi = 0
\]

\( \varphi, \omega \in \mathbb{N}^+ \)

\( \omega \leq n_1 - 2 \)  

2. The sequence \( \mu(k_2,n_2),(k_2,n_2) \in \mathbb{N}^+ \times A \) can be decompressed

\[
\mu(k_2,n_2) = \mu(k_2 - \varphi,n_2 + \omega)
\]  

(3.4)
if and only if there exist $\varphi, \omega \in \mathbb{N}^*, \varphi \leq k_2 - 1$ which satisfies the equation

$$\omega^2 + (2k_2 + 2n_2 + 1 - 2\varphi)\omega - 2(n_2 + 1)\varphi = 0$$

$$\varphi, \omega \in \mathbb{N}^*$$

$$\varphi \leq k_2 - 1$$

(3.5)

3. The odd number $\Pi \neq 1$ is prime if and only if the sequence

$$\mu(k, n) = \Pi \cdot 2^i$$

$$l, k \in \mathbb{N}^*, n \in A$$

cannot be rearranged.

4. The odd $\Pi$ is prime if and only if the sequence

$$\mu \left( \frac{\Pi + 1}{2}, \Pi - 1 \right) = \Pi^2$$

(3.7)

cannot be rearranged."

**Proof.** 1,2. We prove part 1 of the corollary and similarly number 2 can also be proven. From equation (4.1) we conclude that the sequence $\mu(k, n_i)$ can be compressed if and only if there exist $\varphi, \omega \in \mathbb{N}^*$ such as

$$\mu(k, n_i) = \mu(k + \varphi, n_i - \omega).$$

In this equation the natural number $n_i - \omega$ belongs to the set $A = \{2, 3, 4, \ldots\}$ and thus $n_i - \omega \geq 2 \iff \omega \leq n_i - 2$. Next, from equations (2.1) we obtain

$$\mu(k, n_i) = \mu(k + \varphi, n_i - \omega)$$

$$\frac{(n_i + 1)(2k_i + n_i)}{2} = \frac{(n_i - \omega + 1)[2(k_i + \varphi) + n_i - \omega]}{2}$$

and after the calculations we get equation (3.3).

3. The sequence (3.6) is derived from equations (2.4) or (2.5) for $\chi = \Pi$ and $\psi = 2^i$. Thus, in the product $\chi \psi$ the only odd number is $\Pi$. If the sequence $\mu(k, n)$ in equation (3.6) cannot be rearranged then the odd number $\Pi$ has no divisors. Thus, $\Pi$ is prime. Obviously, the inverse also holds.

4. First, we prove equations (3.7). From equation (2.1) we obtain:
\[
\mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \frac{(\Pi-1+1)(2 \cdot \frac{\Pi+1}{2} + \Pi-1)}{2} = \Pi^2.
\]

In case that the odd number \( \Pi \) is prime in equations (2.4), (2.5) the natural numbers \( \chi, \psi \) are unique \( \chi = \Pi \land \psi = \Pi \), and from equation (2.5) we get \( k = \frac{\Pi+1}{2} \land n = \Pi-1 \). Thus, the sequence \( \mu(k,n) = \mu\left(\frac{\Pi+1}{2}, \Pi-1\right) \) cannot be rearranged. Conversely, if the sequence \( \mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \Pi^2 = \Pi \cdot \Pi \) cannot be rearranged the odd number \( \Pi \) cannot be composite and thus \( \Pi \) is prime. □

We now prove the following corollary:

**Corollary 3.1.** 1. The odd number \( \Phi \),

\[
\Phi = \Pi^2 = \mu\left(\frac{\Pi+1}{2}, \Pi-1\right)
\]

\( \Pi = \text{odd} \) \hspace{0.5cm} (3.8)

\( \Pi \neq 1 \)

is decompressed and compressed if and only if the odd number \( \Pi \) is composite.

2. The even number \( \alpha_1 \),

\[
\alpha_1 = 2^l \Pi = \mu\left(2^l - \frac{\Pi-1}{2}, \Pi-1\right)
\]

\( \Pi = \text{odd} \) \hspace{0.5cm} (3.9)

\( 3 \leq \Pi \leq 2^l - 1 \)

\( l \in \mathbb{N}, l \geq 2 \)

cannot be decompressed, while it compresses if and only if the odd number \( \Pi \) is composite.

3. The even number \( \alpha_2 \),

\[
\alpha_2 = 2^{l+1} \Pi = \mu\left(\frac{\Pi+1}{2} - 2^l, 2^{l+1} - 1\right)
\]

\( \Pi = \text{odd} \) \hspace{0.5cm} (3.10)

\( \Pi \geq 2^{l+1} + 1 \)

\( l \in \mathbb{N}^* \)

cannot be compressed, while it decompresses if and only if the odd number \( \Pi \) is composite.
4. Every even number that is not a power of can be written either in the form of equation (3.9) or in the form of equation (3.10)."

Proof.

1. It is derived directly through number (4) of Theorem 3.1. A second proof can be derived through equations (2.4), (2,5) since every composite odd \( \Pi \) can be written in the form of \( \Pi = \chi \psi \), \( \chi, \psi \in \mathbb{N} \), \( \chi, \psi \) odds.

2,3.

Let the even number \( \alpha \),

\[ \alpha = 2^l \Pi \]
\( \Pi = \text{odd} \).
\( l \in \mathbb{N}^* \)

From equation (2.4) we obtain

\[ k = \frac{2 \cdot 2^l + 1 - \Pi}{2} = 2^l - \frac{\Pi - 1}{2} \]
\( n = \Pi - 1 \)

and since \( k, n \in \mathbb{N}, k \geq 1 \land n \geq 2 \) we get

\[ \frac{2 \cdot 2^l + 1 - \Pi}{2} \geq 1 \]
\( \Pi - 1 \geq 2 \)

and equivalently

\[ 3 \leq \Pi \leq 2^{l+1} - 1. \]

In the second of equations (3.12) the natural number \( n \) obtains the maximum possible value of \( n = \Pi - 1 \), and thus the natural number \( k \) takes the minimum possible value in the first of equations (3.12). Thus, the even number

\[ \alpha_i = \mu \left( 2^l - \frac{\Pi - 1}{2}, \Pi - 1 \right) \]

cannot decompress. If the odd number \( \Pi \) is composite then it can be written in the form of \( \Pi = \chi \psi \), \( \chi, \psi \in \mathbb{N}^* \), \( \chi, \psi \) odds, \( \chi, \psi < \Pi \), \( \alpha_i = 2^l \chi \psi \). Therefore, the natural number \( \alpha_i = 2^l \chi \psi \) decompresses since from equations (3.11) it can be written in the form of \( \alpha_i = \mu(k, n) \) with \( n = \chi - 1 < \Pi - 1 \). Similarly, the proof of 3 is derived from equations (2.5).

4. From the above proof process it follows that every even number that is not a power of 2 can be written either in the form of equation (3.9) or in the form of equation (3.10). □
By substituting \( \prod = P = \text{prime} \) in equations of Theorem 3.1 and of corollary 3.1 four sets of equations are derived, each including infinite impossible diophantine equations.

**Example 3.1.** The odd number \( P = 999961 \) is prime. Thus, combining (1) of Theorem 3.1 with (1) of corollary 3.1 we conclude that there is no pair \((\omega, \varphi) \in \mathbb{N}^2 \) with \( \omega \leq 999958 \) which satisfies the diophantine equation

\[
\omega^2 - (2999883 + 2\varphi) \omega + 1999922\varphi = 0.
\]

We now prove the following corollary:

**Corollary 3.2** “The square of every prime number can be uniquely written as the sum of consecutive natural numbers.”

**Proof.** For \( \prod = P = \text{prime} \) in equation (3.5) we obtain

\[
P^2 = \mu \left( \frac{P+1}{2}, P-1 \right).
\]

(3.13)

According with 4 of Theorem 3.1 the odd \( P^2 \) cannot be rearranged. Thus, the odd can be uniquely written as the sum of consecutive natural numbers, as given from equation (3.13).

**Example 3.2.** The odd \( P = 17 \) is prime. From equation (3.13) for \( P = 17 \) we obtain

\[
289 = \mu(9,16)
\]

and from equation (2.1) we get

\[
289 = 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 + 24 + 25
\]

which is the only way in which the odd number 289 can be written as a sum of consecutive natural numbers.

**4. NATURAL NUMBERS AS LINEAR COMBINATION OF CONSECUTIVE POWERS OF 2**

According to the fundamental theorem of arithmetic, every natural number can be uniquely written as a product of powers of prime numbers. The previously presented study reveals a correspondence between odd prime numbers and the powers of 2. Thus, the question arises whether there exists a theorem for the powers of 2 corresponding to the fundamental theorem of arithmetic. The answer is given by the following theorem:

**Theorem 4.1.** ‘Every natural number, with the exception of 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.’

**Proof.** Let the odd number \( \prod \) as given from equation
\[ \Pi = \prod (\nu, \beta_i) = 2^{\nu+1} + 2^\nu \pm 2^{\nu+2} \pm \ldots \pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^\nu + \sum_{i=0}^{\nu-1} \beta_i 2^i \]

\[ \beta_i = \pm 1, i = 0, 1, 2, \ldots, \nu - 1 \]
\[ \nu \in \mathbb{N} \]

From equation (4.1) for \( \nu = 0 \) we obtain
\[ \Pi = 2^1 + 2^0 = 2 + 1 = 3. \]

We now examine the case where \( \nu \in \mathbb{N}^* \). The lowest value that the odd number \( \Pi \) of equation (4.1) can obtain is
\[ \Pi_{\text{min}} = \Pi (\nu) = 2^{\nu+1} + 2^\nu - 2^{\nu+1} - 2^{\nu-1} - \ldots - 2^1 - 1 \]
\[ \Pi_{\text{min}} = \Pi (\nu) = 2^{\nu+1} + 1. \]  \( \quad \) (4.2)

The largest value that the odd number \( \Pi \) of equation (4.1) can obtain is
\[ \Pi_{\text{max}} = \Pi (\nu) = 2^{\nu+1} + 2^\nu + 2^{\nu+1} + \ldots + 2^1 + 1 \]
\[ \Pi_{\text{max}} = \Pi (\nu) = 2^{\nu+2} - 1. \]  \( \quad \) (4.3)

Thus, for the odd numbers \( \Pi = \Pi (\nu, \beta_i) \) of equation (4.1) the following inequality holds
\[ \Pi_{\text{min}} = 2^{\nu+1} + 1 \leq \Pi (\nu, \beta_i) \leq 2^{\nu+2} - 1 = \Pi_{\text{max}}. \]  \( \quad \) (4.4)

The number \( N(\Pi (\nu, \beta_i)) \) of odd numbers in the closed interval \([2^{\nu+1} + 1, 2^{\nu+2} - 1]\) is
\[ N(\Pi (\nu, \beta_i)) = \frac{\Pi_{\text{max}} - \Pi_{\text{min}}}{2} + 1 = \frac{(2^{\nu+2} - 1) - (2^{\nu+1} + 1)}{2} + 1 \]
\[ N(\Pi (\nu, \beta_i)) = 2^{\nu}. \]  \( \quad \) (4.5)

The integers \( \beta_i, i = 0, 1, 2, \ldots, \nu - 1 \) in equation (4.1) can take only two values, \( \beta_i = -1 \vee \beta_i = +1 \), thus equation (4.1) gives exactly \( 2^\nu = N(\Pi (\nu, \beta_i)) \) odd numbers. Therefore, for every \( \nu \in \mathbb{N}^* \) equation (4.1) gives all odd numbers in the interval \([2^{\nu+1} + 1, 2^{\nu+2} - 1]\).

We now prove the theorem for the even numbers. Every even number \( \alpha \) which is a power of 2 can be uniquely written in the form of \( \alpha = 2^\nu, \nu \in \mathbb{N}^* \). We now consider the case where the even number \( \alpha \) is not a power of 2. In that case, according to corollary 3.1 the even number \( \alpha \) is written in the form of
\[ \alpha = 2^l \Pi, \Pi = \text{odd}, \Pi \neq 1, l \in \mathbb{N}^*. \]  \( \quad \) (4.6)
We now prove that the even number $\alpha$ can be uniquely written in the form of equation (4.6). If we assume that the even number $\alpha$ can be written in the form of

$$\alpha = 2^l \Pi = 2^{l'} \Pi'$$

$l \neq l' (l > l')$

$$\Pi \neq \Pi'$$

$$l, l' \in \mathbb{N}^*$$

$$\Pi, \Pi' = \text{odd}$$

the we obtain

$$2^l \Pi = 2^{l'} \Pi'$$

$$2^{l-i} \Pi = \Pi'$$

which is impossible, since the first part of this equation is even and the second odd. Thus, it is $l = l'$ and we take that $\Pi = \Pi'$ from equation (4.7). Therefore, every even number $\alpha$ that is not a power of 2 can be uniquely written in the form of equation (4.6). The odd number $\Pi$ of equation (4.6) can be uniquely written in the form of equation (4.1), thus from equation (4.6) it is derived that every even number $\alpha$ that is not a power of 2 can be uniquely written in the form of equation

$$\alpha = \alpha (l, v, \beta_i) = 2^l \left( 2^{v+1} + 2^v + \sum_{i=0}^{v-1} \beta_i 2^i \right)$$

$$l \in \mathbb{N}^*, v \in \mathbb{N}$$

$$\beta_i = \pm 1, i = 0, 1, 2, \ldots, v - 1$$

(4.8)

and equivalently

$$\alpha = \alpha (l, v, \beta_i) = 2^{l+v+1} + 2^{l+v} + \sum_{i=0}^{v-1} \beta_i 2^{l+i}$$

$$l \in \mathbb{N}^*, v \in \mathbb{N}$$

$$\beta_i = \pm 1, i = 0, 1, 2, \ldots, v - 1$$

(4.9)

For 1 we take

$$1 = 2^0$$

$$1 = 2^1 - 2^0$$

thus, it can be written in two ways in the form of equation (4.1). Both the odds of equation (4.1) and the evens of the equation (4.8) are positive. Thus, 0 cannot be written either in the form of equation (4.1) or in the form of equation (4.8). □

In order to write an odd number $\Pi \neq 1, 3$ in the form of equation (4.1) we initially define the $\nu \in \mathbb{N}^*$ from inequality (4.4). Then, we calculate the sum
$2^{r+1} + 2^r$.

If it holds that $2^{r+1} + 2^r < \Pi$ we add the $2^{r-1}$, whereas if it holds that $2^{r+1} + 2^r > \Pi$ then we subtract it. By repeating the process exactly $\nu$ times we write the odd number $\Pi$ in the form of equation (4.1). The number of $\nu$ steps needed in order to write the odd number $\Pi$ in the form of equation (4.1) is extremely low compared to the magnitude of the odd number $\Pi$, as derived from inequality (4.4).

Example 4.1. For the odd number $\Pi = 23$ we obtain from inequality (4.4)

\[
\begin{align*}
2^{r+1} + 1 &< 23 < 2^{r+2} - 1 \\
2^{r+1} + 2 &< 24 < 2^{r+2} \\
2^r &< 12 < 2^{r+1}
\end{align*}
\]

thus $\nu = 3$. Then, we have

\[
\begin{align*}
2^{r+1} + 2^r &= 2^4 + 2^3 = 24 > 23 \text{ (thus } 2^2 \text{ is subtracted)} \\
2^4 + 2^3 - 2^2 &= 20 < 23 \text{ (thus } 2^1 \text{ is added)} \\
2^4 + 2^3 - 2^2 + 2^1 &= 22 < 23 \text{ (thus } 2^0 = 1 \text{ is added)} \\
2^4 + 2^3 - 2^2 + 2^1 + 1 &= 23.
\end{align*}
\]

Fermat numbers $F_s$ can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\text{min}}$,

\[
F_s = 2^s + 1 = \Pi_{\text{min}} \left(2^s - 1\right) = 2^{2^s} + 2^{2^{s-1}} - 2^{2^{s-2}} - 2^{2^{s-3}} \ldots - 2^1 - 1, \quad s \in \mathbb{N}^*.
\]  

Mersenne numbers $M_p$ can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\text{max}}$,

\[
M_p = 2^p - 1 = \Pi_{\text{max}} \left(p - 2\right) = 2^{p-1} + 2^{p-2} + 2^{p-3} \ldots + 2^1 + 1, \quad p = \text{prime}
\]

In order to write an even number $\alpha$ that is not a power of 2 in the form of equation (4.1), initially it is consecutively divided by 2 and it takes of the form of equation (4.6). Then, we write the odd number $\Pi$ in the form of equation (4.1).

Example 4.2. By consecutively dividing the even number $\alpha = 368$ by 2 we obtain $\alpha = 368 = 2^4 \cdot 23$. Then, we write the odd number $\Pi = 23$ in the form of equation (4.1), $23 = 2^4 + 2^3 - 2^2 + 2^1 + 1$, and we get

\[
\begin{align*}
368 &= 2^4 \left(2^4 + 2^3 - 2^2 + 2^1 + 1\right) \\
368 &= 2^8 + 2^7 - 2^6 + 2^5 + 2^4.
\end{align*}
\]
This equation gives the unique way in which the even number $\alpha = 368$ can be written in the form of equation (4.9).

From inequality (4.4) we obtain

\[
\begin{align*}
2^{\nu+1} + 1 & \leq \Pi \leq 2^{\nu+2} - 1 \\
2^{\nu+1} & < 2^{\nu+1} + 1 \leq \Pi \leq 2^{\nu+2} - 1 < 2^{\nu+2} \\
2^{\nu+1} & < \Pi < 2^{\nu+2} \\
(\nu + 1) \log 2 & < \log \Pi < (\nu + 2) \log 2
\end{align*}
\]

from which we get

\[
\frac{\log \Pi}{\log 2} - 1 < \nu + 1 < \frac{\log \Pi}{\log 2}
\]

and finally

\[
\nu + 1 = \left\lfloor \frac{\log \Pi}{\log 2} \right\rfloor
\]

(4.12)

‘where \( \left\lfloor \frac{\log \Pi}{\log 2} \right\rfloor \) the integer part of \( \frac{\log \Pi}{\log 2} \in \mathbb{R} \).

We now give the following definition:

**Definition 4.1.** We define as the conjugate of the odd

\[
\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^\nu + \sum_{i=0}^{i=\nu-1} \beta_i 2^i
\]

\[
\beta_i = \pm 1, i = 0, 1, 2, \ldots, \nu - 1
\]

\( \nu \in \mathbb{N}^* \)

the odd \( \Pi^* \),

\[
\Pi^* = \Pi^*(\nu, \gamma_j) = 2^{\nu+1} + 2^\nu + \sum_{j=0}^{j=\nu-1} \gamma_j 2^j
\]

\[
\gamma_i = \pm 1, j = 0, 1, 2, \ldots, \nu - 1
\]

\( \nu \in \mathbb{N}^* \)

for which it holds

\[
\gamma_k = -\beta_k \forall k = 0, 1, 2, \ldots, \nu - 1.
\]

(4.15)

For conjugate odds, the following corollary holds:

**Corollary 4.1.** " For the conjugate odds \( \Pi = \Pi(\nu, \beta_i) \) and \( \Pi^* = \Pi^*(\nu, \gamma_j) \) the following hold:
1. $(\Pi^*)^2 = \Pi$. 

(4.16)

2. $\Pi + \Pi^* = 3 \cdot 2^{\nu+1}$. 

(4.17)

3. $\Pi$ is divisible by 3 if and only if $\Pi^*$ is divisible by 3.”

**Proof.** 1. The 1 of the corollary is an immediate consequence of definition 4.1.

2. From equations (4.13), (4.14) and (4.15) we get

$$\Pi + \Pi^* = (2^{\nu+1} + 2^\nu) + (2^{\nu+1} + 2^\nu)$$

and, equivalently

$$\Pi + \Pi^* = 3 \cdot 2^{\nu+1}.$$ 

3. If the odd $\Pi$ is divisible by 3 then it is written in the form $\Pi = 3x$, $x = odd$ and from equation (4.17) we get $3x + \Pi^* = 3 \cdot 2^{\nu+1}$ and equivalently $\Pi^* = 3 (2^{\nu+1} - x)$. Similarly we can prove the inverse. □

**5. THE HARMONIC ODD NUMBERS AND A METHOD FOR DEFINING LARGE PRIME NUMBERS**

The harmonic symmetry: We define as harmonic the odd numbers of equation (4.1) for which the signs of $\beta_i = \pm 1$, $i = 0, 1, 2, 3 \ldots \ldots \nu - 1$ alternate:

$$\Pi_1 = 2^{\nu+1} + 2^\nu - 2^{\nu-1} + 2^{\nu-2} - \ldots \ldots - 2^1 + 1$$

$$\Pi_2 = 2^{\nu+1} + 2^\nu + 2^{\nu-1} - 2^{\nu-2} + \ldots \ldots + 2^1 - 1.$$ 

$$\nu = 2 \lambda, \lambda \in \mathbb{N}^*$$

(5.1)

$$\Pi_1 = 2^{\nu+1} + 2^\nu - 2^{\nu-1} + 2^{\nu-2} - \ldots \ldots - 2^1 + 1$$

$$\Pi_2 = 2^{\nu+1} + 2^\nu + 2^{\nu-1} - 2^{\nu-2} + \ldots \ldots + 2^1 - 1.$$ 

$$\nu = 2 \lambda + 1, \lambda \in \mathbb{N}^*$$

(5.2)

From equations (5.1), (5.2) and definition 4.1 we obtain

$$\Pi_2 = \Pi^*_1 = 3 \times 2^{\nu+1} - \Pi_1$$

(5.3)

for the pair of harmonic odd numbers.

A method for the determination of large prime numbers emerges from the study we presented. This method is completely different from previous methods [1-11]. When we consider the prime factorization of the odd integers

$$\Phi_1 = 2 + \Pi_1 = 2 + 2^{\nu+1} + 2^\nu - 2^{\nu-1} + 2^{\nu-2} - \ldots \ldots - 2^1 + 1$$

$$\nu = 2 \lambda, \lambda \in \mathbb{N}^*$$

(5.4)
\[
\Phi_2 = -2 + \Pi_2 = -2 + 2^{-1} + 2^v + 2^{v-1} - 2^{v-2} + \ldots + 2^1 - 1
\]
\[\nu = 2\lambda, \lambda \in \mathbb{N}
\]  
(5.5)

\[
\Phi_2 = \Phi_1^* = 3 \times 2^{v-1} - \Phi_1
\]  
(5.6)

we have the following statement:

The factors of either \(\Phi_1\) or \(\Phi_2 = \Phi_1^*\) consist of a set of small prime factors and one large factor. Hence from the factorization of \(\Phi_1\) and \(\Phi_2 = \Phi_1^*\) of equations (5.4), (5.5) we get a large prime number.

Following are 11 examples where we have chosen arbitrary even \(\nu\), \(600 \leq \nu \leq 1000\), in equations (5.4), (5.5).

1. \(\nu = 604\)
\[
\Phi_1 = 3 \times 5 \times 1423 \times 2677 \times 103967 \times 1465469 \times 2033624 \times 136455 \times 907062 \times 140355 \times 606581 \times 460617 \\
960329 \times 378244 \times 909713 \times 340374 \times 546035 \times 722007 \times 834481 \times 807880 \times 893223 \times 943637 \times 129816 \times 307143 \times 883666 \times 509589 \times 711687 \times 765791
\]

2. \(\nu = 626\)
\[
\Phi_1 = 13 \times 186653 \times 306 \times 32599 \times 340492 \times 581270 \times 323029 \times 570138 \times 222136 \times 733600 \times 420877 \times 92183 \times 139417 \\
574185 \times 782109 \times 955578 \times 496315 \times 765962 \times 131603 \times 14089 \times 519221 \times 871827 \times 181120 \times 845674 \times 859725 \times 387186 \\
219442 \times 305406 \times 755275 \times 821605 \times 426602 \times 403741 \times 599957
\]

3. \(\nu = 644\)
\[
\Phi_1 = 5 \times 79 \times 12 \times 671297 \times 38892 \times 671359 \times 559494 \times 324882 \times 180204 \times 888273 \times 078001 \times 950134 \times 412751 \\
881230 \times 225550 \times 378061 \times 442396 \times 379471 \times 711953 \times 850899 \times 474720 \times 409489 \times 565536 \times 036909 \times 109253 \times 945965 \\
590266 \times 361910 \times 559333 \times 142120 \times 493266 \times 182138 \times 997818 \times 136400 \times 630503
\]

4. \(\nu = 688\)
\[
\Phi_1 = 3^3 \times 5 \times 137 \times 2357 \times 84239 \times 14 \times 276659 \times 111598 \times 463167 \times 164995 \times 567141 \times 3547 \times 493034 \\
864246 \times 374604 \times 223939 \times 439254 \times 117526 \times 195183 \times 644765 \times 258504 \times 745395 \times 441461 \times 348003 \times 624541 \times 265182 \\
053620 \times 319595 \times 210678 \times 493117 \times 621150 \times 188802 \times 864705 \times 030169 \times 622562 \times 000148 \times 389984 \times 593085 \times 80457
\]

5. \(\nu = 732\)
\[
\Phi_1 = 5^5 \times 19 \times 4357 \times 10093 \times 2 \times 901193 \times 373 \times 058471 \times 21 \times 318693 \times 003272 \times 810610 \times 223875 \times 009176 \\
985967 \times 454655 \times 131235 \times 547239 \times 807330 \times 702392 \times 207730 \times 665072 \times 351378 \times 572215 \times 387223 \times 133953 \times 567092 \\
456647 \times 869354 \times 874941 \times 347502 \times 746701 \times 928543 \times 247909 \times 407783 \times 975122 \times 056127 \times 018272 \times 539991 \times 430427 \times 637981
6. $\nu = 818$

$$\Phi_2 = 5 \times 826599 \times 918309 \times 521729 \times 414628 \times 892756 \times 111346 \times 582385 \times 085483 \times 938095 \times 996388 \times 692690 \times 239258\text{901551 139189 714409 550909 093308 382061 608683 211278 156913 402724 889465 422572 029940 229036 211569 513811 is prime}

7. $\nu = 838$?

8. $\nu = 842$

$$\Phi_1 = 13 \times 811 \times 7789 \times 15271 \times 66809 \times 933 \times 419184 \times 297225 \times 688884 \times 84133 \times 741618 \times 091582 \times 561157\text{362135 750330 558036 085494 747230 138415 970602 017694 350758 458917 589235 971861 548843 635060 827053 633582 882443 092203 262135 552296 661334 709021 156021 405492 515100 671199 284761 072521 866782 927154 434480 887521}

9. $\nu = 914$?

10. $\nu = 986$?

11. $\nu = 998$

$$\Phi_2 = 23 \times 277 \times 4211 \times 1385899 \times 240154 \times 091459 \times 652243 \times 015812 \times 929515 \times 159070 \times 212159 \times 918817\text{425875 611004 712759 052716 135663 441910 181493 025014 669780 274245 881010 561780 858639 784499 969926 885693 756207 174479 909272 942309 784548 553831 369221 141895 942976 579419 394048 307219 568666 715750 728448 387606 183250 921312 430705 694057 415487 884739 523892 723969}

Equations (5.4), (5.5) and (5.6) are a special case of equations

$$\Phi_1 (v, \xi) = \Phi_1 (v, 2^{2\xi+1}) = 2^{2\xi+1} + \Pi_1 (v) = 2^{2\xi+1} + 2^{v+1} + 2^v - 2^{v-1} + 2^{v-2} - \ldots - 2^1 + 1$$

$$\nu = 2\lambda, \lambda \in \mathbb{N}^+ \quad (5.7)$$

$$\xi = 0, 1, 2, \ldots, \frac{v-2}{2} = \lambda - 1$$

$$\Phi_2 (v, \xi) = \Phi_2 (v, 2^{2\xi+1}) = -2^{2\xi+1} + \Pi_2 (v) = -2^{2\xi+1} + 2^{v+1} + 2^v + 2^{v-1} - 2^{v-2} + \ldots + 2^1 - 1$$

$$\nu = 2\lambda, \lambda \in \mathbb{N}^+ \quad (5.8)$$

$$\xi = 0, 1, 2, \ldots, \frac{v-2}{2} = \lambda - 1$$

$$\Phi_2 (v, \xi) = 3 \times 2^{v+1} - \Phi_1 (v, \xi) = \Phi_1 \ast (v, \xi) \quad (5.9)$$

for $\xi = 0$. The general equations (5.7), (5.8) and (5.9) give all possible variations of the method. For example, for $\nu = 838$, a value of $\nu$ that did not give a large prime number in the previous examples, from equation (5.7) for $\xi = 1$ we get
\[ \Phi_1(838,1) = 3 \times 251 \times 124 \times 958179 \times 125661 \times 642577 \times 51 \times 945201 \times 394308 \times 356447 \times 274374 \times 943957 
\]
\[ 749268 \times 889249 \times 128703 \times 205379 \times 933327 \times 597692 \times 534177 \times 000888 \times 147927 \times 160249 \times 734500 \times 867000 \times 722765 \]
\[ 431922 \times 957290 \times 626876 \times 299700 \times 840201 \times 468643 \times 187688 \times 745195 \times 339241 \times 792572 \times 155819 \times 582073 \times 320776 \times 475981 \times 870379 \times 650986 \times 830637 \times 696975 \times 455178 \times 897139 \]

For \( \nu = 66 \) and \( \xi = 1, 2, 3, \ldots, \frac{\nu - 2}{2} = 32 \) from equation (5.7) we get

\[ \Pi_1 = 196765270119568550571 \]
\[ \Pi_2 = 245956587649460688213 = 3^3 \times 27 \times 328509 \times 738828 \times 965357 \]
\[ \Phi_2(\nu, 2) = \Phi_1^*(\nu, 2) = 73 \times 3 \times 369268 \times 323965 \times 214907 \]
\[ \Phi_1(\nu, 2^3) = 1 \times 645337 \times 119 \times 589646 \times 449067 \]
\[ \Phi_2(\nu, 2^5) = 4140 \times 643009 \times 59400 \times 577909 \]
\[ \Phi_2(\nu, 2^7) = \Phi_1^*(\nu, 2^7) = 5 \times 13907 \times 3537 \times 162402 \times 379531 \]
\[ \Phi_2(\nu, 2^9) = \Phi_1^*(\nu, 2^9) = 13 \times 601 \times 25184 \times 342777 \times 367023 \]
\[ \Phi_2(\nu, 2^{11}) = \Phi_1^*(\nu, 2^{11}) = 5 \times 49 \times 191317 \times 529892 \times 137233 \]
\[ \Phi_1(\nu, 2^{13}) = 196765270119568550571 \text{ is prime} \]
\[ \Phi_2(\nu, 2^{15}) = \Phi_1^*(\nu, 2^{15}) = 5 \times 3259 \times 36269 \times 416167 \times 836559 \]
\[ \Phi_2(\nu, 2^{17}) = \Phi_1^*(\nu, 2^{17}) = 157 \times 12 \times 248491 \times 127901 \times 671243 \]
\[ \Phi_1(\nu, 2^{19}) = 13 \times 19 \times 4643 \times 32083 \times 5347 \times 833013 \]
\[ \Phi_1(\nu, 2^{21}) = 271 \times 5903 \times 123 \times 000357 \times 013771 \]
\[ \Phi_1(\nu, 2^{23}) = 7 \times 257879 \times 2 \times 350441 \times 46 \times 375123 \]
\[ \Phi_1(\nu, 2^{25}) = 634853 \times 309 \times 938316 \times 617551 \]
\[ \Phi_2(\nu, 2^{27}) = \Phi_1^*(\nu, 2^{27}) = 5 \times 49 \times 191317 \times 529865 \times 294097 \]
\[ \Phi_1(\nu, 2^{29}) = 7^2 \times 107 \times 37529 \times 137921 \times 057681 \]
Theorem 4.1 highlights additional symmetries of the internal structure of the natural numbers. We will not expand upon these symmetries in the current article.
References


