# 30 Ideas about Prime Numbers

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**Abstract:** A Prime number (or a Prime) is a natural number greater than 1 that has no positive divisors other than 1 and itself.

The crucial importance of Prime numbers to number theory and mathematics in general stems from the fundamental theorem of arithmetic, which states that every integer larger than 1 can be written as a product of one or more Primes in a way that is unique except for the order of the Prime factors. Primes can thus be considered the "basic building blocks", the atoms, of the natural numbers.

In this paper we present 30 ideas about Primes. Some are based on the fact that all Primes greater than 3, are 1 unit away from a multiple of 1, 2, 3, 4, or 6, which is used to introduce new methods to factorize, to count Primes less than a given number, and to add some ideas to already famous Prime conjectures.

## 1. <u>Prime sequence</u>

A Prime number (or a Prime) is a natural number (a whole number) greater than 1 that has no positive divisors other than 1 and itself. The sequence of Prime numbers is infinite, is composed only by odd numbers (and the number 2) and there is no known close form formula to generate it.

The first 25 Prime numbers are given by:

$$\{2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,\ 53,59,61,67,71,\ 73,79,83,89,97\ \ldots\}$$

The sequence of Prime numbers is usually described as random. This statement does not reflect the fact that there are many known structures within the Prime sequence. There are infinitely many Primes, as demonstrated by Euclid around 300 BC. There is no known simple formula that separates Prime numbers from composite numbers.

However, the distribution of Primes, that is to say, the statistical behavior of Primes in the large, can be modelled. The first result in that direction is the Prime number theorem, proven at the end of the 19th century, which says that the probability that a given, randomly chosen number N is Prime is inversely proportional to its number of digits, which is equivalent to the logarithm of N.

The way to build the sequence of Prime numbers uses sieves, an algorithm yielding all Primes up to a given limit, or using trial division method which consists of dividing N by each integer M that is greater than 1 and less than or equal to the square root of N. If the result of any of these divisions is an integer, then <u>N</u> is not a Prime, otherwise it is a Prime.

Eratosthenes (276 BC – 194 BC) introduced a sieve to generate the Prime sequence: starting with 2, eliminate every 2 numbers, then find the next number, which is 3 and eliminate every 3 numbers, and repeat it sequentially with any P number that has not been eliminated previously and eliminate every P numbers. The remaining set are the Prime numbers.

Since Eratosthenes, there has been a continuous effort to find patterns, count the number of Primes, and efficiently factor very large integers.

New ideas regarding Primes are difficult to prove and some widely accepted conjectures are still unproven, such as Goldbach's and Grimm's, just to name famous ones.

**[IDEA #1]** In this paper we use the fact that all Primes can be expressed using one of the two following formulas for $a = 1,2,3,4,6$:

$p = a * k_{an} + 1 \qquad k_{an} \in N$ that we will call the $P^{a+}$series (1)

$p = a * k_{am} - 1 \qquad k_{am} \in N$ that we will call the $P^{a-}$series (2)

We will call $k_{an}$ and $k_{am}$ Prime Generators.

All Prime numbers, except in some cases 2 and 3, belong to either $P^{a+}$ or $P^{a-}$ series. We will call this kind of intertwined sequences the DNA-Prime Sequences as it resembles the intertwined DNA helix.

Counting Primes less than a number N is equivalent to counting how many Prime generators $k_{an}$ and $k_{am}$ are less than $floor(N/a)$ where $floor(x)$ is the function that gives the greatest integer that is less than or equal to $x$.

## 2. Prime Generator sequences $P^{a+} = \{k_{an}\}$ and $P^{a-} = \{K_{am}\}$

The only values of $\underline{a}$ that generate the complete sequence of primes are $a = 1, 2, 3, 4, 6$. For each one of them different conditions apply, although all those conditions have a similar structure.

2.1. Case a=6.

When a=6, it is known that all primes, except 2 and 3, are 1 unit away of a multiple of 6. For p prime, the following is true: $p = 6k \pm 1$

Not all values of $k \in N$ make $6k \pm 1$ prime. The following conditions apply:

If $k_{6n} \neq 6xy + x + y$
And $k_{6n} \neq 6xy - x - y$ $\quad then \quad p = 6 * k_{6n} + 1$ is prime
If $k_{6m} \neq 6xy + x - y$ $\quad then \quad p = 6 * k_{6m} - 1$ is prime

The set of Prime numbers using $\boldsymbol{k_{6n}}$ can be defined as **[IDEA #2]**:

**{Primes} = {2,3}** (3)

$$\cup \{\mathbf{6 * k_{6n} + 1 \mid k_{6n} \neq 6xy + x + y \text{ and } k_{6n} \neq 6xy - x - y \; for\; all\; x, y \in N}\}$$
$$\cup \{\mathbf{6 * k_{6m} - 1 \mid k_{6m} \neq 6xy - x + y \; for\; all\; x, y \in N}\}$$

Which is equivalent to say that any number that is not of the form 6*($6xy + x + y) + 1$, for any $x, y \in N$ is a Prime number because 6*($6xy + x + y) + 1 = (6x + 1)(6y + 1)$

First elements of $k_{6n} = \{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 16, 17, 18, 21, 23, 25, 26, 27, 30, 32, 33, 35, 37, 38, \ldots\}$
First elements of $k_{6m} = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 14, 15, 17, 18, 19, 22, 23, 25, 28, 29, 30, 32, 33, 38, 39, \ldots\}$

In theory, if we knew the sequences $k_n$ and $k_m$we would know all Primes as there is a bijective relationship between $k_n$ and $k_m$and the Primes that they generate, as we can see in the following table:

| P+ | P- | | |
|---|---|---|---|
| 6kn+1 | 6km-1 | kn | km |
| | | | |
| | 5 | 1 | |
| 7 | | | 1 |
| | 11 | 2 | |
| 13 | | | 2 |
| | 17 | 3 | |
| 19 | | | 3 |
| | 23 | 4 | |
| | 29 | | 5 |
| 31 | | 5 | |

Table 1

It can be easily observed that there are values of $k$ that do not generate Prime numbers:

| k | 6k+1 | 6k-1 |
|---|---|---|
| 1 | 7 | 5 |
| 2 | 13 | 11 |
| 3 | 19 | 17 |
| 4 | 25 | 23 |
| 5 | 31 | 29 |
| 6 | 37 | 35 |
| 7 | 43 | 41 |

Table 2

And there are other values of k that do not generate a Prime in either $P^{6+}$ and $P^{6-}$series: (OEIS A060461 Beedassy)

{20, 24, 31, 34, 36, 41, 48, 50, 54, 57, 69, 71, 79, 86, 88, 89, 92, 97, 104, 106, 111, 116, 119,…}

As commented by the researcher in that sequence: "All terms can be expressed as (6ab+a+b OR 6cd-c-d) AND (6xy+x-y) for a,b,c,d,x,y positive integers. Example: 20=6*2*2-2-2 AND 20=6*3*1+3-1))

2.2. Case a=1

When a=1 the prime sequence can be generated using condition:

$$\text{If } k_{1n} \neq xy + x + y \text{ then } p = k_{1n} + 1 \quad \text{is prime}$$

This is probably the simplest way to define a prime number. It is a powerful condition that can drive simple factorization and primality methods as we will describe later in the paper.

The set of Prime numbers using $\boldsymbol{k_{1n}}$ can be defined as **[IDEA #3]**:

$$\textbf{\{Primes\}} = \{\boldsymbol{k_{1n} + 1} \mid \boldsymbol{k_{1n} \neq xy + x + y}\} \quad (4)$$

Which is equivalent to say that any number that is not of the form $xy + x + y +$, for any $x, y \in N$ is a Prime number, which is obvious given that $xy + x + y + 1 = (x + 1) * (y + 1)$. The following matrix shows numbers of the form $xy + x + y + 1$ that are all composite. Each row x contains the multiples of $(x + 1)$:

First elements of $k_{1n} = \{1, 2, 4, 6, 10, 12, 16, 18, 22, 28, 30, 36, 40, 42, 46, 52, 58, 60, 66, 70, 72, 78, 82 \dots\}$

The matrix $\{xy + x + y + 1\}$ that we will call $c_{ij}$ or matrix of composite numbers:

| | 1 | 2 | 3 | 4 | 5 | 6 | Observation |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 6 | 8 | 10 | 12 | 14 | Multiples of 2 |
| 2 | 6 | 9 | 12 | 15 | 18 | 21 | Multiples of 3 |
| 3 | 8 | 12 | 16 | 20 | 24 | 28 | Multiples of 4 |
| 4 | 10 | 15 | 20 | 25 | 30 | 35 | Multiples of 5 |
| 5 | 12 | 18 | 24 | 30 | 36 | 42 | Multiples of 6 |
| 6 | 14 | 21 | 28 | 35 | 42 | 49 | Multiples of 7 |
| 7 | 16 | 24 | 32 | 40 | 48 | 56 | Multiples of 8 |

Table 3

2.3. Case a=2

When a=2 the prime sequence, except the number 2 can be generated using one condition:

$$\text{If } k_{2n} \neq 2xy + x + y \text{ then } p = 2k_{2n} + 1 \quad \text{is prime}$$

The set of Prime numbers using $\boldsymbol{k_{2n}}$ can be defined as **[IDEA #4]**:

$$\textbf{\{Primes\}} = \{2\} \cup \{\boldsymbol{2 * k_{2n} + 1 \mid k_{2n} \neq 2xy + x + y}\} \qquad (5)$$

First elements of $k_{2n} = \{1, 2, 3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 21, 23, 26, 29, 30, 33, 35, 36, 39, 41, 44, \dots\}$

Which is equivalent to say that any number that is not of the form 2*$(2xy + x + y) + 1$, for any $x, y \in N$ is a Prime number because 2*$(2xy + x + y) + 1 = (2x + 1)(2y + 1)$

2.4. Case a=3

When a=3 the prime sequence, except the number 3 can be generated using 3 conditions:

If $k_{3n} \neq 3xy + x + y$
And $k_{3n} \neq 3xy - x - y$ then $p = 3k_{3n} + 1$ is prime
If $k_{3m} \neq 3xy + x - y$ then $p = 3k_{3m} - 1$ is prime

The set of Prime numbers using $\boldsymbol{k_{3n}}$ can be defined as **[IDEA #5]**:

$$\textbf{\{Primes\} = \{3\}} \cup \{\boldsymbol{3 * k_{3n} + 1 \mid k_{3n} \neq 3xy + x + y \text{ and } k_{3n} \neq 3xy - x - y \text{ for all } x, y \in N}\} \cup \{\boldsymbol{3 * k_{3m} - 1 \mid k_{3m} \neq 3xy - x + y \text{ for all } x, y \in N}\} \qquad (6)$$

First elements of $k_{3n} = \{2, 4, 6, 10, 12, 14, 20, 22, 24, 26, 32, 34, 36, 42, 46, 50, 52, 54, 60, 64, 66, 70, 74, , \dots\}$
First elements of $k_{3m} = \{1, 2, 4, 6, 8, 10, 14, 16, 18, 20, 24, 28, 30, 34, 36, 38, 44, 46, 50, 56, 58, 60, 64, 66, \dots\}$

Which is equivalent to say that any number that is not of the form 3*$(3xy + x + y) + 1$, for any $x, y \in N$ is a Prime number because 3*$(3xy + x + y) + 1 = (3x + 1)(3y + 1)$

2.5. Case a=4

When a=4 the prime sequence, except numbers 2 and 3, can be generated using 3 conditions:

| | | | |
|---|---|---|---|
| If | $k_{4n} \neq 4xy + x + y$ | | |
| And | $k_{4n} \neq 4xy - x - y$ | $then \;\; p = 4k_{4n} + 1$ is prime | |
| If | $k_{4m} \neq 4xy + x - y$ | $then \;\; p = 4k_{4m} - 1$ is prime | |

The set of Prime numbers using $\boldsymbol{k_{4n}}$ can be defined as **[IDEA #6]**:

$$\textbf{\{Primes\} = \{2\}} \cup \{\mathbf{4 * k_{4n} + 1} \mid \mathbf{k_{4n} \neq 4xy + x + y} \textbf{ and } \mathbf{k_{4n} \neq 4xy - x - y} \; \boldsymbol{for\ all\ x, y \in N}\} \cup \{\mathbf{4 * k_{4m} - 1} \mid \mathbf{k_{4m} \neq 4xy - x + y} \; \boldsymbol{for\ all\ x, y \in N}\} \quad (7)$$

First elements of $k_{4n} = \{1, 3, 4, 7, 9, 10, 13, 15, 18, 22, 24, 25, 27, 28, 34, 37, 39, 43, 45, 48, 49, 57, 58, 60, \dots\}$
First elements of $k_{4m} = \{1, 2, 3, 5, 6, 8, 11, 12, 15, 17, 18, 20, 21, 26, 27, 32, 33, 35, 38, 41, 42, 45, 48, 50, \; \dots\}$

Which is equivalent to say that any number that is not of the form 4*$(4xy + x + y) + 1$, for any $x, y \in N$ is a Prime number because 4*$(4xy + x + y) + 1 = (4x + 1)(4y + 1)$

## 2. Characteristics of the DNA-Prime Sequences $P^{6+}$ and $P^{6-}$. Understanding $k_{6n}$ and $k_{6m}$ conditions.

The difference between two Primes in either sequence $P^{6+}$ and $P^{6-}$ is a multiple of 6. In the following table we show the two DNA-Prime series, the difference between two consecutive elements of the series, the difference divided by 6 and the cumulative difference divided by 6.

One can observe that the cumulative difference from any element of the series and the first element, that we will call $R_n$ for the $P^{6+}$ series, and $R_m$ for the $P^{6-}$ series, is equal to the $(k - 1)$, where k is any generator $k_{6n}$ $or$ $k_{6m}$. This key fact will help us formulate a way to generate the Prime sequence. For simplification we will denominate $P^{+} = P^{6+}$ and $P^{-} = P^{6-}$ in this section.

| P+ | | | | | P- | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Pn+ | Kn | P(n)-P(n-1) | (P(n)-P(n-1))/6 | Rn | Pm- | Km | P(m)-P(m-1) | (P(m)-P(m-1))/6 | Rm |
| 7 | 1 | | | | 5 | 1 | | | |
| 13 | 2 | 6 | 1 | 1 | 11 | 2 | 6 | 1 | 1 |
| 19 | 3 | 6 | 1 | 2 | 17 | 3 | 6 | 1 | 2 |
| 31 | 5 | 12 | 2 | 4 | 23 | 4 | 6 | 1 | 3 |
| 37 | 6 | 6 | 1 | 5 | 29 | 5 | 6 | 1 | 4 |
| 43 | 7 | 6 | 1 | 6 | 41 | 7 | 12 | 2 | 6 |
| 61 | 10 | 18 | 3 | 9 | 47 | 8 | 6 | 1 | 7 |
| 67 | 11 | 6 | 1 | 10 | 53 | 9 | 6 | 1 | 8 |
| 73 | 12 | 6 | 1 | 11 | 59 | 10 | 6 | 1 | 9 |
| 79 | 13 | 6 | 1 | 12 | 71 | 12 | 12 | 2 | 11 |

Table 4

We can see in the chart that:

$$P^{+}\ series \qquad P(n) - P(n-1)\ mod\ 6\ =\ 0$$
$$P^{-}\ series \qquad P(m) - P(m-1)\ mod\ 6\ =\ 0$$

a. The difference between any two Primes in either series is given by:

$$P^{+}\ series \qquad if\ p1 = 6 * k_1 + 1\ and\ \ p2 = 6 * k_2 + 1$$
$$P^{+}\ series \qquad if\ p1 = 6 * k_1 - 1\ and\ \ p2 = 6 * k_2 - 1$$
$$Then\ p_2 - p_1\ =\ 6 *\ (k_2 - k_1)$$

b. The difference between any Prime in either sequence P⁺ and P and the first one in the series is a multiple of the generators $R_n$ and $R_m$:

$$P^{+}\ series\ \ P^{+} = 7 + 6 * R_n$$
$$P^{-}\ series\ \ P^{-} = 5 + 6 * R_m$$
$$\text{Where}$$
$$R_n = k_n - 1 \quad \text{and} \quad R_m = k_m - 1$$

Let's take a look at the $R_n$ and $R_m$ sequences eliminating (in color) all those that don't generate a Prime using previous formulas:

**$R_n$**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 |
| 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 |
| 121 | 122 | 123 | 124 | 125 | 126 | 127 | 128 | 129 | 130 |
| 131 | 132 | 133 | 134 | 135 | 136 | 137 | 138 | 139 | 140 |
| 141 | 142 | 143 | 144 | 145 | 146 | 147 | 148 | 149 | 150 |
| 151 | 152 | 153 | 154 | 155 | 156 | 157 | 158 | 159 | 160 |
| 161 | 162 | 163 | 164 | 165 | 166 | 167 | 168 | 169 | 170 |
| 171 | 172 | 173 | 174 | 175 | 176 | 177 | 178 | 179 | 180 |
| 181 | 182 | 183 | 184 | 185 | 186 | 187 | 188 | 189 | 190 |

Table 5

**$R_m$**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 |
| 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 |
| 121 | 122 | 123 | 124 | 125 | 126 | 127 | 128 | 129 | 130 |
| 131 | 132 | 133 | 134 | 135 | 136 | 137 | 138 | 139 | 140 |
| 141 | 142 | 143 | 144 | 145 | 146 | 147 | 148 | 149 | 150 |
| 151 | 152 | 153 | 154 | 155 | 156 | 157 | 158 | 159 | 160 |
| 161 | 162 | 163 | 164 | 165 | 166 | 167 | 168 | 169 | 170 |
| 171 | 172 | 173 | 174 | 175 | 176 | 177 | 178 | 179 | 180 |
| 181 | 182 | 183 | 184 | 185 | 186 | 187 | 188 | 189 | 190 |

Table 6

## 3. Reasoning for the formulation of the Prime sequence using $P^{+}$ and $P^{-}$

3.1. A definition of Primes using the sequences $P^{+}$ and $P^{-}$

From the tables above and testing many potential combinations, we conclude that the sequence of DNA-Primes and their generators $R_n$ and $R_m$ can be formulated algebraically as follows:

**$P^{+}$ series** $\quad P^{+} = 7 + 6*R_n$

$$R_n \neq x + (6x + 1) * y - 1 \qquad x > 0, y > 1 \in N$$
$$R_n \neq -x + (6x - 1) * y - 1 \qquad x > 1, y > 1 \in N$$

**$P^{-}$ series** $\quad P^{-} = 5 + 6*R_m$

$$R_m \neq x + (6x - 1) * y - 1 \qquad x > 0, y > 1 \in N$$
$$R_m \neq -(x + 1) + (6x + 1) * y \qquad x > 0, y > 1 \in N$$

We defined $Kn = Rn + 1$ and $Km = Rm + 1$, so, these conditions can be simplified as follows:

**$P^{+}$ series** $\quad P^{+} = 6 * \mathrm{Kn} + 1$

Where
$$K_n \neq A_n = 6xy + x + y \qquad x > 0, y > 0 \in N$$
$$K_n \neq B_n = 6xy - x - y \qquad x > 0, y > 0 \in N$$

**$P^{-}$ series** $\quad P^{-} = 6 * \mathrm{Km} - 1$

Where
$$K_m \neq C_n = 6xy - x + y \qquad x > 0, y > 0 \in N$$

The tables showing the values of $A_n$ , $B_n$ , and $C_n$ are the following:

| An | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 8 | 15 | 22 | 29 | 36 | 43 | 50 | 57 | 64 | 71 | 78 | 85 | 92 | 99 | 106 |
| 2 | | 28 | 41 | 54 | 67 | 80 | 93 | 106 | 119 | 132 | 145 | 158 | 171 | 184 | 197 |
| 3 | | | 60 | 79 | 98 | 117 | 136 | 155 | 174 | 193 | 212 | 231 | 250 | 269 | 288 |
| 4 | | | | 104 | 129 | 154 | 179 | 204 | 229 | 254 | 279 | 304 | 329 | 354 | 379 |
| 5 | | | | | 160 | 191 | 222 | 253 | 284 | 315 | 346 | 377 | 408 | 439 | 470 |
| 6 | | | | | | 228 | 265 | 302 | 339 | 376 | 413 | 450 | 487 | 524 | 561 |
| 7 | | | | | | | 308 | 351 | 394 | 437 | 480 | 523 | 566 | 609 | 652 |
| 8 | | | | | | | | 400 | 449 | 498 | 547 | 596 | 645 | 694 | 743 |
| 9 | | | | | | | | | 504 | 559 | 614 | 669 | 724 | 779 | 834 |
| 10 | | | | | | | | | | 620 | 681 | 742 | 803 | 864 | 925 |
| 11 | | | | | | | | | | | 748 | 815 | 882 | 949 | 1016 |
| 12 | | | | | | | | | | | | 888 | 961 | 1034 | 1107 |
| 13 | | | | | | | | | | | | | 1040 | 1119 | 1198 |
| 14 | | | | | | | | | | | | | | 1204 | 1289 |
| 15 | | | | | | | | | | | | | | | 1380 |

| Bn | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 9 | 14 | 19 | 24 | 29 | 34 | 39 | 44 | 49 | 54 | 59 | 64 | 69 | 74 |
| 2 | | 20 | 31 | 42 | 53 | 64 | 75 | 86 | 97 | 108 | 119 | 130 | 141 | 152 | 163 |
| 3 | | | 48 | 65 | 82 | 99 | 116 | 133 | 150 | 167 | 184 | 201 | 218 | 235 | 252 |
| 4 | | | | 88 | 111 | 134 | 157 | 180 | 203 | 226 | 249 | 272 | 295 | 318 | 341 |
| 5 | | | | | 140 | 169 | 198 | 227 | 256 | 285 | 314 | 343 | 372 | 401 | 430 |
| 6 | | | | | | 204 | 239 | 274 | 309 | 344 | 379 | 414 | 449 | 484 | 519 |
| 7 | | | | | | | 280 | 321 | 362 | 403 | 444 | 485 | 526 | 567 | 608 |
| 8 | | | | | | | | 368 | 415 | 462 | 509 | 556 | 603 | 650 | 697 |
| 9 | | | | | | | | | 468 | 521 | 574 | 627 | 680 | 733 | 786 |
| 10 | | | | | | | | | | 580 | 639 | 698 | 757 | 816 | 875 |
| 11 | | | | | | | | | | | 704 | 769 | 834 | 899 | 964 |
| 12 | | | | | | | | | | | | 840 | 911 | 982 | 1053 |
| 13 | | | | | | | | | | | | | 988 | 1065 | 1142 |
| 14 | | | | | | | | | | | | | | 1148 | 1231 |
| 15 | | | | | | | | | | | | | | | 1320 |

| Cn | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | 11 | 16 | 21 | 26 | 31 | 36 | 41 | 46 | 51 | 56 | 61 | 66 | 71 | 76 |
| 2 | 13 | 24 | 35 | 46 | 57 | 68 | 79 | 90 | 101 | 112 | 123 | 134 | 145 | 156 | 167 |
| 3 | 20 | 37 | 54 | 71 | 88 | 105 | 122 | 139 | 156 | 173 | 190 | 207 | 224 | 241 | 258 |
| 4 | 27 | 50 | 73 | 96 | 119 | 142 | 165 | 188 | 211 | 234 | 257 | 280 | 303 | 326 | 349 |
| 5 | 34 | 63 | 92 | 121 | 150 | 179 | 208 | 237 | 266 | 295 | 324 | 353 | 382 | 411 | 440 |
| 6 | 41 | 76 | 111 | 146 | 181 | 216 | 251 | 286 | 321 | 356 | 391 | 426 | 461 | 496 | 531 |
| 7 | 48 | 89 | 130 | 171 | 212 | 253 | 294 | 335 | 376 | 417 | 458 | 499 | 540 | 581 | 622 |
| 8 | 55 | 102 | 149 | 196 | 243 | 290 | 337 | 384 | 431 | 478 | 525 | 572 | 619 | 666 | 713 |
| 9 | 62 | 115 | 168 | 221 | 274 | 327 | 380 | 433 | 486 | 539 | 592 | 645 | 698 | 751 | 804 |
| 10 | 69 | 128 | 187 | 246 | 305 | 364 | 423 | 482 | 541 | 600 | 659 | 718 | 777 | 836 | 895 |
| 11 | 76 | 141 | 206 | 271 | 336 | 401 | 466 | 531 | 596 | 661 | 726 | 791 | 856 | 921 | 986 |
| 12 | 83 | 154 | 225 | 296 | 367 | 438 | 509 | 580 | 651 | 722 | 793 | 864 | 935 | 1006 | 1077 |
| 13 | 90 | 167 | 244 | 321 | 398 | 475 | 552 | 629 | 706 | 783 | 860 | 937 | 1014 | 1091 | 1168 |
| 14 | 97 | 180 | 263 | 346 | 429 | 512 | 595 | 678 | 761 | 844 | 927 | 1010 | 1093 | 1176 | 1259 |
| 15 | 104 | 193 | 282 | 371 | 460 | 549 | 638 | 727 | 816 | 905 | 994 | 1083 | 1172 | 1261 | 1350 |

Table 7

Some observations regarding matrices $A_n$, $B_n$, and $C_n$:

- $A_n$ and $B_n$ have a symmetry over the main diagonal
- There are duplicates within $A_n$, $B_n$, and $C_n$ and between $A_n$ and $B_n$

From tables [5],[6],[7], we can list the elements of $K_n$ and $K_m$, as:

$$\{K_n\}=\{N\} - \{A_n \cup B_n\}$$
$$\{K_m\}=\{N\} - \{C_n\}$$

Where {N} is the set of Natural numbers.

And the set of Prime numbers can be expressed as:

$$\{Primes\} = \{2,3\}$$
$$\cup \{6k_n + 1 \mid k_n \neq 6xy + x + y \; and \; k_n \neq 6xy - x - y \; for \; all \; x, y \in N\}$$
$$\cup \{6k_m - 1 \mid k_m \neq 6xy - x + y \; for \; all \; x, y \in N\}$$

The generation of Primes using this algorithm is complete based on the following observation:

1) With $k = 6xy + x + y$, we have:
   $$6k + 1 = 36xy + 6x + 6y + 1 = (6x + 1)(6y + 1),$$
   i.e. all products of two factors both equivalent to $+1 \ (mod\ 6)$

2) With $k = 6xy - x - y$, we have:

$$6k + 1 = 36xy - 6x - 6y + 1 = (6x - 1)(6y - 1),$$

i.e. all products of two factors both equivalent to $-1\ (mod\ 6)$

2) With $k = 6xy - x + y$, we have:

$$6k - 1 = 36xy - 6x + 6y - 1 = (6x + 1)(6y - 1),$$

i.e. all products of two factors, one equivalent to $+1\ (mod\ 6)$ and the other equivalent to $-1\ (mod\ 6)$.

Starting with the integers equivalent to $\pm 1\ (mod\ 6)$ and excluding these three sets leaves those integers equivalent to $\pm 1\ (mod\ 6)$ which cannot be represented as a product of two factors equivalent to $\pm 1\ (mod\ 6)$, i.e. the Primes $p \geq 5$.

The first numbers in the generator series $k_{6n}$:

$$k_{6n} = 1, 2, 3, 5, 7, 10, 11, 12, 13, 16, 17, 18, 21, \ldots$$

Generating Primes $P^{6+} = 6 * k_n + 1$

$$P^{6+} = 7, 13, 19, 31, 43, 61, 67, 73, 79, 97, 103, 109, 127, \ldots$$

The first numbers in the generator series $k_m$:

$$k_{6m} = 1, 2, 3, 5, 7, 8, 9, 10, 12, 14, 15, 17, 18, 19, 22\ \ldots$$

Generating Primes $P^{6-} = 6 * k_{6m} - 1$:

$$P^{6-} = 5, 11, 17, 29, 41, 47, 53, 59, 71, 83, 89, 101, 107, 113, 131 \ldots$$

For any given number N, the number of unique values in $P^{-}$and $P^{-}$are almost the same as can be seen in the following chart:

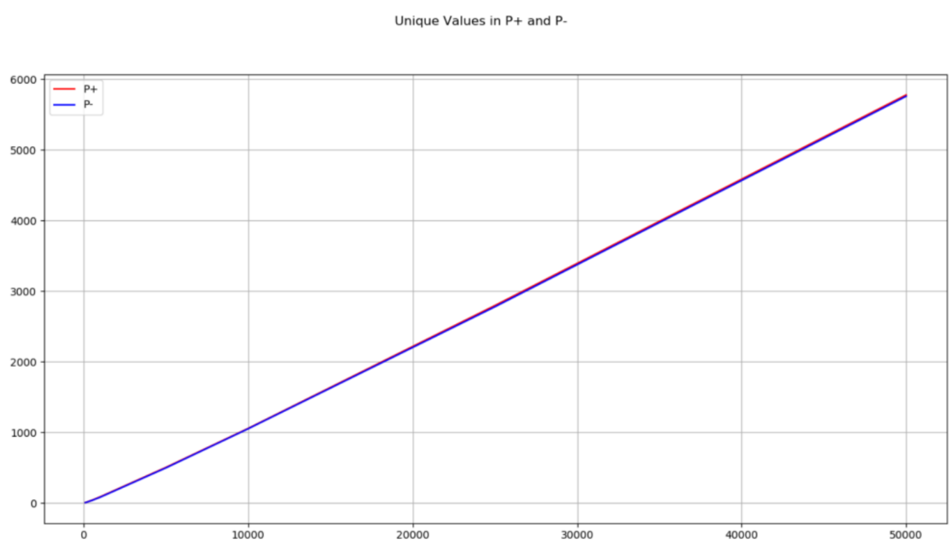


Fig 1

The difference in counts between $\pi(P^{-})$ and $\pi(P^{+})$ for Primes less than N is plotted in the next chart for values Prime values $\leq 10^7$:

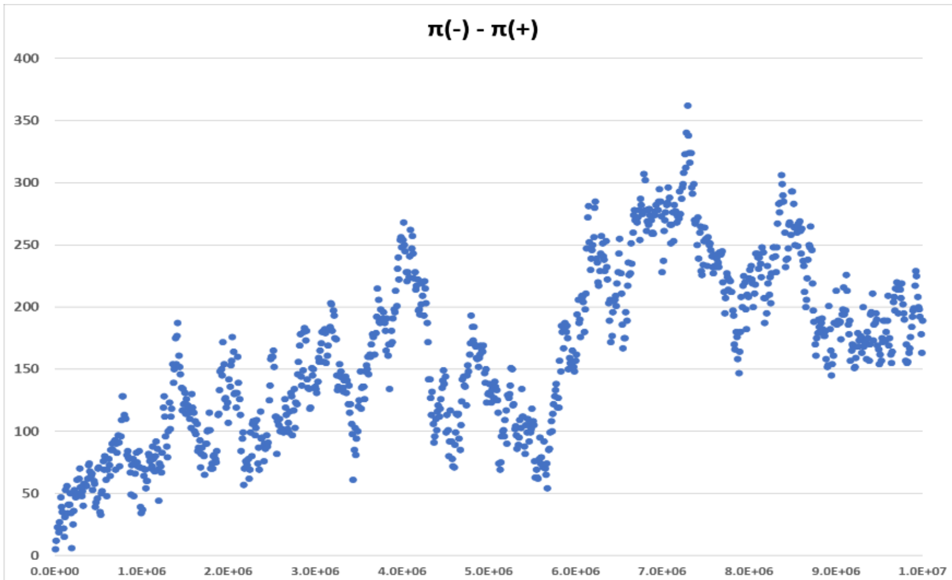


Fig 2

The following chart shows $\pi(P^{-})$ and $\pi(P^{+})$ as a percentage of N:

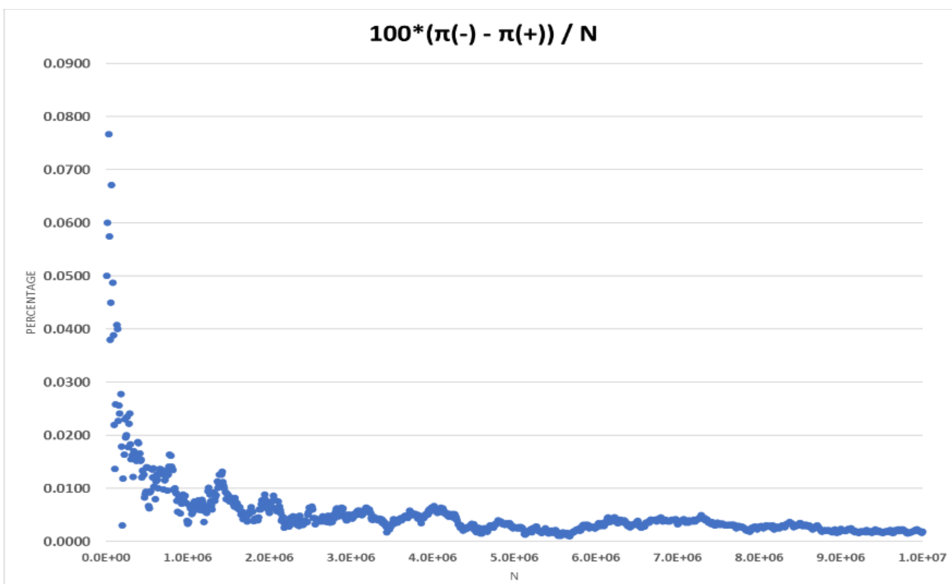


Fig 3

**[IDEA #7]** With $\lim_{N\to\infty} \frac{\pi(P^{-})-\pi(P^{+})}{N} = 0$

There are countless sequences that can be built based on the sequences of Prime Generators $P^{6+}$ and $P^{6-}$. These are a few examples:

a. Numbers n such that (6k-1) for k=n, n+1, n+2, n+3 are all primes with no primes of the form (6k+1) in between. This sequence of numbers is formed by positive integers k that make 6k-1, 6k+5, 6k+11 and 6k+17 prime numbers with no primes of the form 6k+1 in between. (OEIS A296011 Caceres):

$$\{42, 897, 1052, 2107, 2242, 2457, 2632, 2912, 3887, 4362, 9347, 10367, 12587, 13132, 13797, \ldots\}$$

b. Numbers n such that 6k+1 is prime for k=n, n+1, n+2, n+3 with no primes of the form 6k-1 in between. This sequence of numbers is formed by positive integers k that make 6k+1, 6k+7, 6k+13 and 6k+19 prime numbers with no primes of the form 6k-1 in between. (OEIS A296055 Caceres):

$$\{ 290, 550, 850, 1060, 2650, 3035, 3245, 5015, 5105, 8935, 10615, 11890, 12925, 13485, 13905, \ldots\}$$

## 4. Differences between consecutive Primes

The differences between two of these consecutive Primes is calculated to be:

| **Prime** | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 | 53 | 59 | 61 | 67 | 71 | 73 | 79 | 83 | 89 | 97 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Gap** | | 1 | 2 | 2 | 4 | 2 | 4 | 2 | 4 | 6 | 2 | 6 | 4 | 2 | 4 | 6 | 6 | 2 | 6 | 4 | 2 | 6 | 4 | 6 | 8 |

Table 8

The next figure shows the Prime gaps for Primes up to 10000 [5][6]:

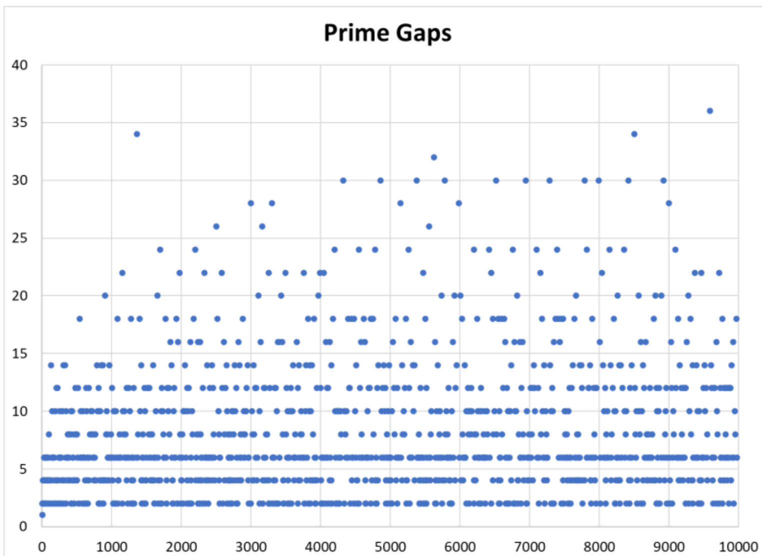


Fig 4

The Prime gap function is defined as [2]:

$$g_n = p_n - p_{n-1}$$

Verifying that the gap can get infinitely large with:

$$\lim_{n\to\infty} g_n = \infty$$

But it grows slower than the sequence of Primes, therefore:

$$\lim_{n\to\infty} \frac{g_n}{p_n} = 0$$

The differences between Primes are increasing and the Prime number theorem proves that these gaps grow with log(n). The function is neither multiplicative nor additive. The Merit of a gap is defined by:

$$\text{Merit(g(n))} = \frac{g_n}{\ln{(p_{n)}}}$$

The race to find larger Prime gaps as well as Prime numbers never stops. The maximal prime gap $G(N)$ is the length of the largest prime gap that begins with a prime $p_k$ less than some maximum value $N$.

The following chart represents the gaps between elements of $P^+$and $P^-$ for Primes less than 1,000,000. It shows again the similar behavior of both Prime sequences $P^+$and $P^-$:

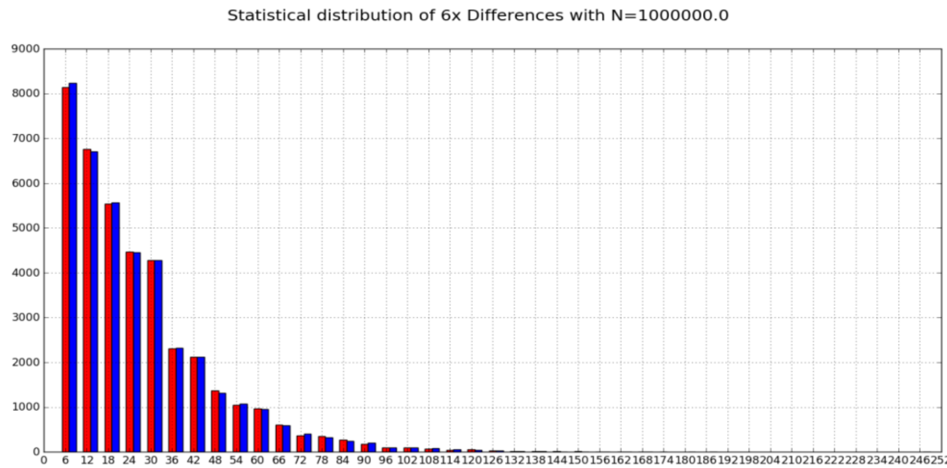


Fig 5

### 5. Ratios between consecutive Primes

The ratios between two consecutive Primes is given by:

| **Prime** | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Jump** | | 1.500 | 1.667 | 1.400 | 1.571 | 1.182 | 1.308 | 1.118 | 1.211 | 1.261 | 1.069 | 1.194 | 1.108 | 1.049 | 1.093 |

Table 9

The next figure plots the function $\frac{p_n}{p_{n-1}}$ for the Primes ≤1000:

These ratios are decreasing with: $\lim_{n\to\infty} \frac{p_n}{p_{n-1}} = 1$

The gaps are not consistently decreasing, and important research has been done on the limits of those gaps. This research is related to the counting of the number of Primes less than a given number.

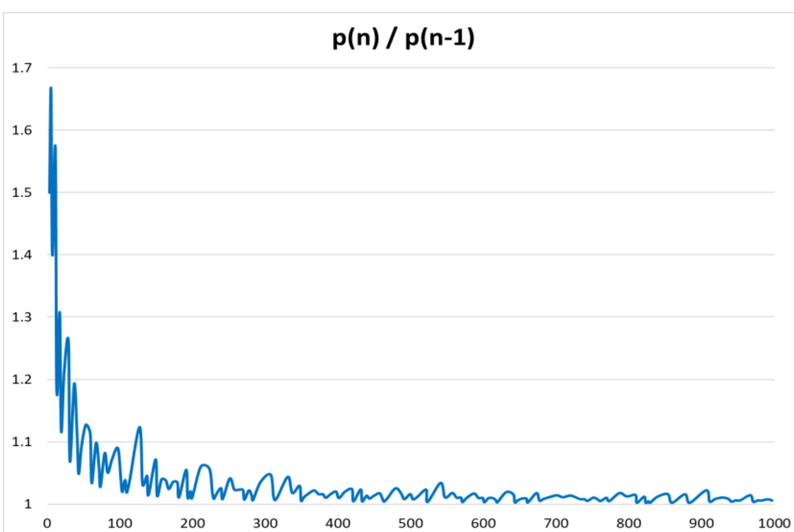


Fig 6

## 6. Twin Primes

Twin Primes are Primes that are two units apart. We will use $S_{2n}$ to refer to the set of twin Primes. The first few twin Prime pairs are:

$$S_{2n} = \{(3,5),(5,7),(11,13),(17,19),(29,31),(41,43),(59,61),(71,73),(101,103),\ldots\}$$

It is easily observable that every twin pair, other than $(3,5)$ is of the form $(6k-1, 6k+1)$ for some value of k.

The sequence of Twin Primes generators is: (OEIS A002822 Sloane):

$$\{1,2,3,5,7,10,12,17,18,23,25,30,32,33,38,40,45,47,52,58,70,72,77,87\ldots\}$$

**[IDEA #8]** We have previously formulated that if $k$ is a twin pair generator, $k$ cannot be represented by any of these three equations with $x, y$ positive integers:

$$k = 6xy + x + y$$

$$k = 6xy - x - y$$

$$k = 6xy + x - y$$

### 6.1. Brun's Theorem

It is conjectured that there are an infinite number of twin Primes (this is one form of the twin Prime conjecture) but proving this remains one of the most elusive open problems in number theory. An important result for twin Primes is Brun's theorem, which states that the number obtained by adding the reciprocals of the odd twin Primes,

$$B_2 = (\frac{1}{3}+\frac{1}{5}) + (\frac{1}{5}+\frac{1}{7}) + (\frac{1}{11}+\frac{1}{13}) + (\frac{1}{17}+\frac{1}{19}) + \ldots$$

converges to a definite number ("Brun's constant" $B$_2), a value that gets updated based on the larger number of twin Primes available for the calculation. The number of terms has since been calculated using twin Primes up to $10^{16}$ [11], giving the result

$$B_2 = 1.902160583104$$

**[IDEA #9]** Let (L) be the lesser of a twin Prime pair and (G) the greater. We know that every L is of the form $6k-1$ and every G is of the form $6k+1$, so

$$\left(\frac{1}{L}\right) - \left(\frac{1}{G}\right) = \frac{1}{6k-1} - \frac{1}{6k+1} = \frac{2}{(36k^2-1)}$$

And:

$$\sum_{k=1}^{\infty} \frac{2}{(36k^2-1)} = 1 - \frac{\pi}{2\sqrt{3}}$$

Therefore:

$$\sum \frac{1}{L} - \sum \frac{1}{G} <= 1 - \frac{\pi}{2\sqrt{3}} = 0.09310032\ldots = D_2$$

We know that the Brun's constant is:

$$\sum \frac{1}{L} + \sum \frac{1}{G} = B_2$$

so:

$$\sum \frac{1}{L} = 0.99763$$

$$\sum \frac{1}{G} = 0.90453$$

and:

$$\sum \frac{1}{L} / \sum \frac{1}{G} <= 1.10296$$

Also, the theorem can be expressed:

$$\sum \frac{1}{L} \leq \frac{B_2 + D_2}{2}$$

$$\sum \frac{1}{G} >= \frac{B_2 - D_2}{2}$$

6.2. Twin Pair Centers

If we call Twin Prime Center the composite number in the middle of a Twin pair. The sequence of Twin Centers is (OEIS A014574 Guy, Sloane, Weisstein):

$$\{4, 6, 12, 18, 30, 42, 60, 72, 102, 108, 138, 150, 180, 192, 198, 228, 240, 270 \ldots\}$$

Twin Centers greater than 18 can be written as the sum of two smaller twin centers. The result is the sequence (OEIS A305825 Caceres):

{0, 0, 0, 1, 1, 1, 1, 2, 2, 1, 1, 2, 3, 2, 3, 1, 4, 3, 3, 3, 2, 6, 3, 5, 3, 3…}

plotted in the following figure:

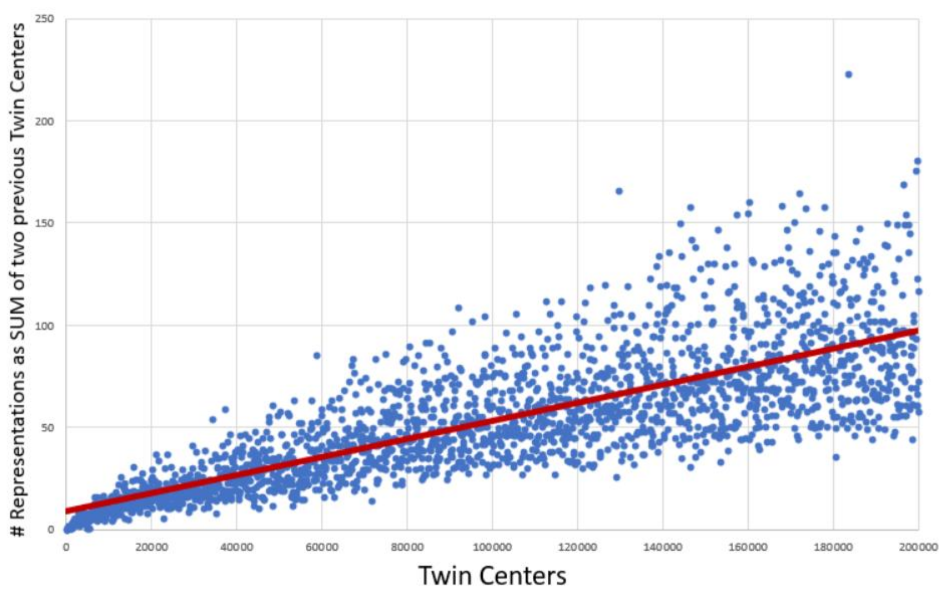


Fig 7

6.3. **[IDEA #10]** An equation with solutions that are twin Primes:

If k is a Twin Prime generator, then:

$$36k^2 - 12kx + x^2 - 1 = 0$$

for any $k$, has two solutions that are twin Primes.

Examples:

| k (Twin Prime Generator) | solution 1 | solution 2 |
|---|---|---|
| 1 | 5 | 7 |
| 2 | 11 | 13 |
| 3 | 17 | 19 |
| 5 | 29 | 31 |
| 7 | 41 | 43 |
| 10 | 59 | 61 |
| 12 | 71 | 73 |
| 17 | 101 | 103 |
| 18 | 107 | 109 |

Table 10

6.4. **[IDEA #11]** Infinite roots with twin Primes

Let's define a seed (s) and a recurrence $z = r(n, m, x)$ to build infinite roots of the form:

$$z = \sqrt{s} + \sqrt{z}$$

for instance, if:

$$seed = n$$

$$z = r(n, m, x) = n + (m + j) * \sqrt{z}$$

then we have a Ramanujan infinite root. For example, for n=1, m=2:

$$z = \sqrt{(1 + 2\sqrt{(1 + 3\sqrt{(1 + 4\sqrt{\ldots}}}}$$

We can build an infinite root using $n$ and $m$ Prime numbers:

$$\text{Example: for } n = 3, m = 7 \ \rightarrow \ z = \sqrt{(3 + 7\sqrt{(3 + 8\sqrt{(3 + 9\sqrt{(3 + 10\sqrt{\ldots}}}}}$$

And calculate the infinite roots that provide an integer solution:

$$n = \ 107 \ \& m = \ 101 \ \ {-}{-}{-}{>} \ z = \ 103$$
$$n = \ 113 \ \& m = \ 107 \ \ {-}{-}{-}{>} \ z = \ 109$$
$$n = \ 197 \ \& m = \ 191 \ \ {-}{-}{-}{>} \ z = \ 193$$
$$n = \ 233 \ \& m = \ 227 \ \ {-}{-}{-}{>} \ z = \ 229$$
$$n = \ 317 \ \& m = \ 311 \ \ {-}{-}{-}{>} \ z = \ 313$$
$$n = \ 353 \ \& m = \ 347 \ \ {-}{-}{-}{>} \ z = \ 349$$

It can be observed that $(m, z)$ are twin Primes and $(m, z, n)$ are three consecutive Primes.

## 7. Prime factorization

In number theory, the Prime factors of a positive integer are the Prime numbers that divide that integer with no remainder. The crucial importance of Prime numbers to number theory and mathematics in general stems from the Fundamental Theorem of Arithmetic, which states that every integer larger than 1 can be written as a product of one or more Primes in a way that is unique except for the order of the Prime factors. Primes can thus be considered the “basic building blocks”, the atoms, of the natural numbers.

If n is divided by p, there is a $k, r \in Z$ such that:

$$n = k*p + r$$

p is a Prime factor of n, if and only if r = 0, which can also be expressed using the mod(ulo) function by:

$$n \ mod \ p = 0$$

Where the function mod (modulo) is defined as follows:

$$r = p - n * trunc(\frac{p}{n})$$

The Prime factorization of a positive integer is a list of the integer's Prime factors, together with their multiplicities; the process of determining these factors is called integer factorization. The fundamental theorem of arithmetic says that every positive integer has a single unique Prime factorization.[7]

One useful fact is that any composite number has at least one factor that is less or equal than the square root of the number.

The test to verify if a number is prime is called primality test. According to [13]: “A primality test is a test to determine whether or not a given number is prime, as opposed to actually decomposing the number into its constituent prime factors (which is known as prime factorization). Primality tests come in two varieties: deterministic and probabilistic. Deterministic tests determine with absolute certainty whether a number is prime. Examples of deterministic tests include the Lucas-Lehmer test and elliptic curve primality proving. Probabilistic tests can potentially (although with very small probability) falsely identify a composite number as prime (although not vice versa). However, they are in general much faster than deterministic tests. Numbers that have passed a probabilistic prime test are therefore properly referred to as probable primes until their primality can be demonstrated deterministically.”

Among other fields of mathematics, prime factorization is used extensively in asymmetric public key cryptography. Our inability to factorize large numbers with current methods and computing power is the basis of internet security and most security protocols in networks and information systems in general. One of the methods used in cryptography are the RSA codes which consist of very large composite numbers that have exactly two known Prime factors. These numbers are called Semiprimes. Finding those two factors require very complex algorithms as the numbers are composed by two Prime numbers of more than one hundred digits. As an example:

RSA-220 = {200 digits long}
2601385262034057849416540486101975135080389157197767183211977681094456418179666766085931213065825772506315628866769704480700018111497118630021124879281994874820660701310665866460833279828035603792053919801399464969552 61

Has the following two factors:

FACTOR 1 of RSA-220 =
6863656412267566274382371499288437800130842239979164844621244993321541061 44

146426679382136442084201920549996877

FACTOR 2 of RSA-220 =
32929074394863498120493015492129352919164551965362339524626860511692903493094652463337824866390738191765712603

The simple factorization method is the trial division method which consists in dividing sequentially by all known Primes until we find a factor. Then we reduce the number by the factor and start again. This method is unpractical for large Primes.

The fastest-known fully proven deterministic algorithm is the Pollard-Strassen method (Pomerance 1982; Hardy et al. 1990). [8]

Wolfram Math World mentions the following list of factorization methods: [7][10]:

- Brent's Factorization Method,
- Class Group Factorization Method,
- Continued Fraction Factorization Algorithm,
- Direct Search Factorization,
- Dixon's Factorization Method,
- Elliptic Curve Factorization Method,
- Euler's Factorization Method,
- Excludent Factorization Method,
- Fermat's Factorization Method,
- Legendre's Factorization Method,
- Number Field Sieve,
- Pollard p-1 Factorization Method,
- Pollard rho Factorization Algorithm,
- Quadratic Sieve,
- Trial Division,
- Williams p+1 Factorization Method

## 7.1. **[IDEA #12]** Primality test using DNA-Prime sequences $P^{6+}$ and $P^{6-}$

We are going to formulate a new factorization method base on the $P^{+}$ and $P^{-}$seres. We know that for a number N to be Prime, the following conditions must be met:

a) If (N-1) mod 6≠0 and (N+1) mod 6≠0 the number is not Prime

b) If (N-1) mod 6=0 then $N = 6 * k_{6n} + 1$

CONDITION C1 $(K_{6n} - s)\ mod\ (6s + 1) \neq 0$ for $s \in N$<$k_n$

CONDITION C2 $(K_{6n} + s)\ mod\ (6s - 1) \neq 0$ for $s \in N$<$k_n$

If s=1 or s=$k_n$ then N is Prime.

c) If (N+1) mod 6=0 then $N = 6 * k_{6m} - 1$

CONDITION C3 $(K_{6m} + \mathrm{s})\ mod\ (6s + 1) \neq 0$ for $s \in N$<$k_m$

CONDITION C4 $(K_{6m} - s)\ mod\ (6s - 1) \neq 0$ for $s \in N$<$k_m$

If s=1 or s=$k_m$ then N is Prime.

| N | kn=(N+1)/6 | kn=(N-1)/6 | s | Primality? | Factors |
|---|---|---|---|---|---|
| 4489 | 748.33 | 748.00 | 11 | No | 67x67 |
| 6839 | 1140.00 | 1139.67 | 1 | No | 7x977 |
| 9973 | 1662.33 | 1662.00 | - | Yes | - |
| 100001 | 16667.00 | 16666.67 | 2 | No | 11x9091 |

Table 11

7.2. A practical factorization method using DNA-Prime sequences

The algorithm to factorize N using DNA-Primes logic would be as follows:

```
While N>1
    If (N-1)=0 mod 6 then
            N∈ P+
            K=(N-1)/6
            S=2
            While S<K:
                    If (K-S)=0 mod (6S+1) then add factor (6S+1)        [Condition C1]
                    If (K+S)=0 mod (6S-1) then add factor (6S-1)        [Condition C2]
            If S=K then N is Prime
            N=N/factor
    Else if (N+1)=0 mod 6 then
            N∈ P−
            K=(N+1)/6
            While S<K:
                    If (K-S)=0 mod (6S-1) then add factor (6S-1)        [Condition C3]
                    If (K+S)=0 mod (6S+1) then add factor (6S+1)        [Condition C4]
            If S=K then N is Prime
            N=N/factor
    Else    N is Prime
```

Some examples of factorization with code FACTORIZA7.SIX (using Python 3.7.):

| N | Factors (FACTORIZA7.SIX) | Time Elapsed (sec) |
|---|---|---|
| 10^10+1 | 101 * 3541 * 27961 | 0.0123 |
| 10^20+1 | 73 * 137 * 1676321 * 5964848081 | 0.0469 |
| 10^30+1 | 61 * 101 * 3541 * 9901 * 27961 * 4188901 * 39526741 | 0.1406 |
| 10^40+1 | 17 * 5070721 * 5882353 * 19721061166646717498359681 | 0.0625 |
| 10^50+1 | 101 * 3541 * 27961 * 60101 * 7019801 * 14103673319201 * 1680588011350901 | 339.8750 |
| 10^60+1 | 73 * 137 * 1676321 * 99990001 * 5964848081 * 1000099999999899989999000000010001 | 1.2031 |
| 10^70+1 | 29 * 101 * 281 * 421 * 3541 * 27961 * 3471301 * 13489841 * 121499449 * 60368344121 * 848654483879497562821 | 88.8906 |
| 10^80+1 | 353 * 449 * 641 * 1409 * 69857 * 1634881 * 18453761 * 947147262401 * 349954396040122577928041596214187605761 | 8.4375 |

Table 12

The algorithm checks the remainders of $(k \pm s)/(6s \pm 1)$. When these remainders hit zero, a factor is found. The algorithm performs a sequential search. One important component of any optimized strategies has to do with the remainders of $(k \pm s)/(6s \pm 1)$. These remainders have very interesting behaviors. The following chart plots the value of the remainders obtain for the search of the first factor of a composite number 693949:

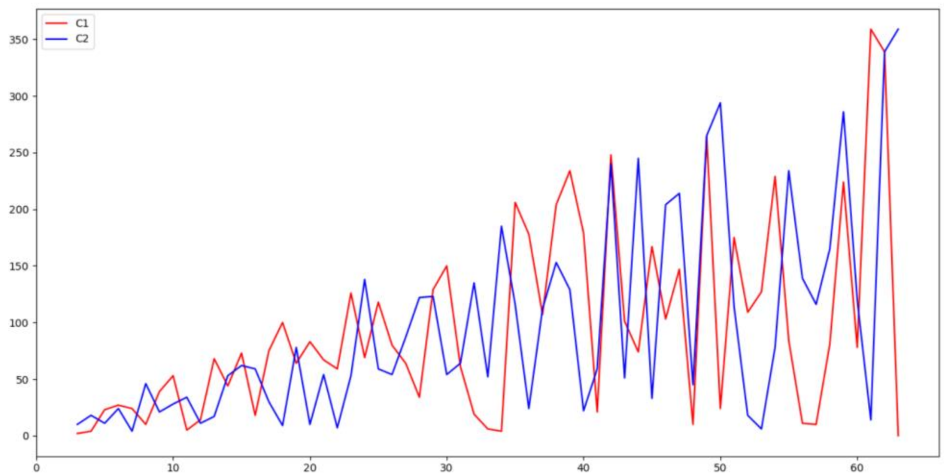


Fig 8

And the following chart plots the remainder when the number to factor is prime:

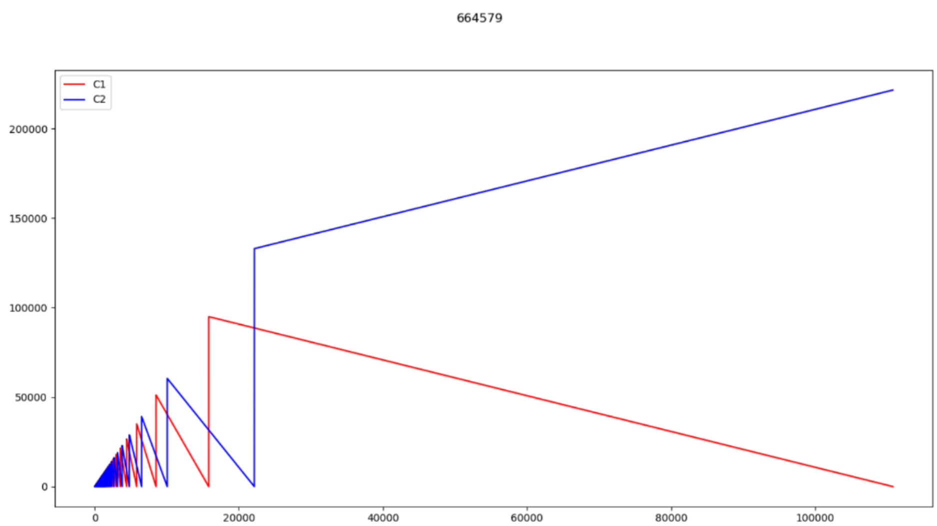


Fig 9

**[IDEA #13]** One can observe that the plot of remainders for a Prime number when verifying conditions C1, C2, C3, C4 has a linear structure for $s$ greater than a certain $s_0$.

Using the fact that, at some time in the sequence, the remainders of prime numbers are part of straight lines with slopes that are one away from a multiple of six, one can build a code to perform primality tests.

Running the code created based on the shape of the remainder curves (FACTORIZA10 in Python 3.7) we can check the primality of the integer plotted in the previous chart N=664579 with the following result:

```
--> N   = 664579 ( 6 ) is in Series: P+
--> k   = 110763 ( 6 )
---------------------------------------------------------------------
C1:  110764  -> New s= 200  slope -553
C2:  110764  -> New s= 199  slope -557
N= 664579 --> PRIME
---------------------------------------------------------------------
Time Elapsed =  0.001 seconds
```

It can be observed that:

a. 664579 is in the series $P^{+}$, which means that (N-1) = 0 mod 6 so we will check conditions C1 and C2.
b. k = (N-1)/6 = 110763
c. Remainder when testing for Condition 1 (C1) gives at some point (s=200) a straight line
    a. The slope of the line verifies (slope+1) = 0 mod 6 (-553+1 = 0 mod 6)
d. So, there are no divisors of the form C1 (6s+1)
e. Remainder when testing for Condition 2 (C2) gives at some point (s=199) a straight line
    a. The slope of the line verifies (slope-1) = 0 mod 6 (-557-1 = 0 mod 6)
f. So, there are no divisors of the form C2 (6s-1)
g. Therefore 664579 is prime

7.3. **[IDEA #14]** Factorization and Primality test using DNA-Prime sequences $P^{1+}$ and $P^{1-}$

We are going to formulate a new factorization method base on the $P^{1+}$ and $P^{1-}$series. We know that for a number N to be Prime, the following conditions must be met:

d) If (N) mod 2≠0 the number is not Prime

e) If (N-1) mod 2=0 then $N = kn + 1$

CONDITION C1 $(K_n - s)\ mod\ (s+1) \neq 0$ for $s \in N$<$k_n$
If s=1 or s=$k_n$ then N is Prime.

Examples:

| N | kn=(N-1) | s | Primality? | Factors |
|---|---|---|---|---|
| 4489 | 4488.00 | 66 | No | 67x67 |
| 6839 | 6838.00 | 6 | No | 7x977 |
| 9973 | 9972.00 | - | Yes | - |
| 100001 | 100000.00 | 10 | No | 11x9091 |

Table 13

We can compare the speed of FACTORIZA7.UNO code using $P^{1+}$ and $P^{1-}$with the previous results of FACTORIZA7.SIX in table 12:

| N | Factors (FACTORIZA7.UNO) | Time Elapsed (sec) |
|---|---|---|
| 10^10+1 | 101 * 3541 * 27961 | 0.1719 |
| 10^20+1 | 73 * 137 * 1676321 * 5964848081 | 0.1719 |
| 10^30+1 | 61 * 101 * 3541 * 9901 * 27961 * 4188901 * 39526741 | 0.2656 |
| 10^40+1 | 17 * 5070721 * 5882353 * 19721061166646717498359681 | 0.1875 |
| 10^50+1 | 101 * 3541 * 27961 * 60101 * 7019801 * 14103673319201 * 1680588011350901 | 209.4844 |
| 10^60+1 | 73 * 137 * 1676321 * 99990001 * 5964848081 * 100009999998999989999000000010001 | 1.2188 |
| 10^70+1 | 29 * 101 * 281 * 421 * 3541 * 27961 * 3471301 * 13489841 * 121499449 * 60368344121 * 848654483879497562821 | 68.0938 |
| 10^80+1 | 353 * 449 * 641 * 1409 * 69857 * 1634881 * 18453761 * 947147262401 * 349954396040122577928041596214187605761 | 5.3594 |

Table 14

Comparing Table 14 to Table 12, one can see that FACTORIZA7.UNO is faster that FACTORIZA7 for larger N.

The algorithm checks the remainders of $(k-s)/(s+1)$. When the remainder hits zero, a factor is found. The algorithm performs a sequential search. The following chart plots the value of the remainders obtain for the search of the first factor of a composite number 877193 = 2 * 3 * 19 * 739 * 1187, until the code finds factor 739:

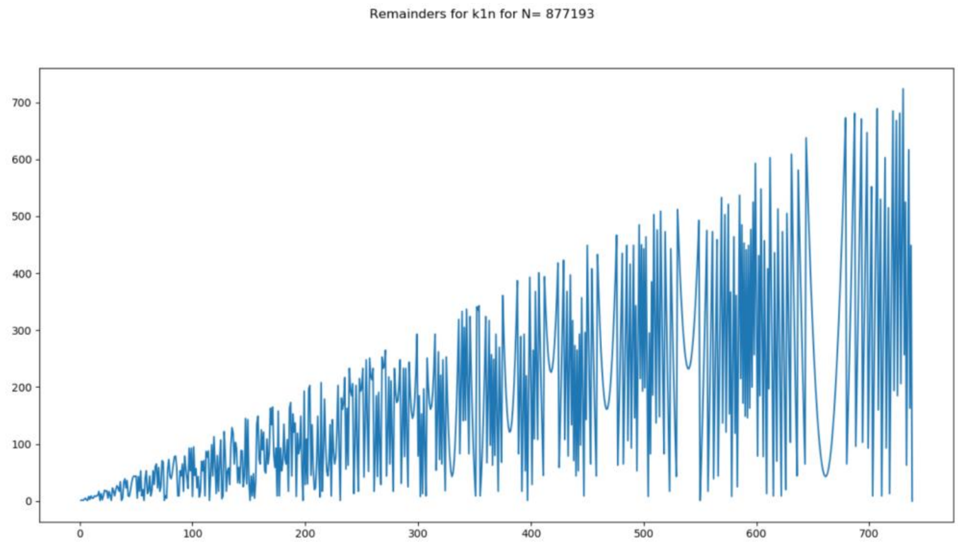


Fig 10

Again, we can see the different structure of the remainder plot when N is prime:

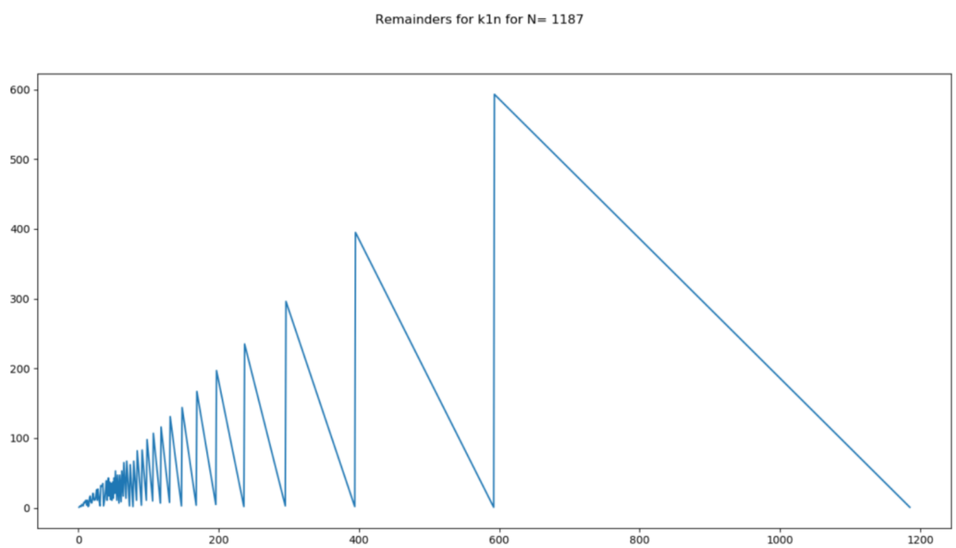


Fig 11

As done before, we will test primality and processing time for larger primes optimizing the code using the characteristics of the straight lines in the plot of remainders of Primes using $(K-s)\ mod\ (s+1)$:

Code: Factoriza10.UNO v. 04/26/2020 PJC

------------------------------------------------------------

--> N = 100000003319

------------------------------------------------------------

JUMP C1:  2  -> New s= 175  -> resto1[-3]= 131   -> resto11[-1]= -62
JUMP C1:  4  -> New s= 1000  -> resto1[-3]= 422   -> resto11[-1]= -103
JUMP C1:  3  -> New s= 2924  -> resto1[-3]= 2198   -> resto11[-1]= -627
JUMP C1:  100000003319  -> New s= 447809  -> resto1[-3]= 446647   -> resto11[-1]= -223309

N= 100000003319 --> PRIME
--------------------------------------------------------------------
Time Elapsed =  1.3125 seconds

The code found at s=4447809 a straight that verifies the conditions for primality.

7.4. Some other interesting (and impractical) factorization methods or primality tests

a. **[IDEA #15]** Recurrence to check Primality

The following recurrence that will generate a positive integer only if m is Prime:

$$b(1) = 1 \tag{8}$$

$$b(n) = ((n-1)^2 / n) * (b(n-1) + (n-3)/(n-1))$$

Also, the integer values for b(m) when m is Prime are the elements of the OEIS A091330 sequence [13]. This has been added as a comment to that sequence.

The values of b(n) are:

$$b(n) = [1, 0, 0, \frac{3}{4}, 4, \frac{115}{6}, 102, \frac{5033}{8}, \frac{40312}{9}, \frac{362871}{10}, 329890, \dots]$$

b. Wilson’s theorem.

First stated by Ibn al-Haytham (c. 1000 AD),and, in the 18th century, by John Wilson. Edward Waring announced the theorem in 1770, although neither he nor his student Wilson could prove it. Lagrange gave the first proof in 1771. There is evidence that Leibniz was also aware of the result a century earlier, but he never published it.

In essence, it says that: A natural number n > 1 is a Prime number if and only if the product of all the positive integers less than n is one less than a multiple of n. That is (using the notations of modular arithmetic), one has that the factorial  (n−1)!=1×2×3×⋯×(n−1) satisfies (n−1)! ≡ −1 (mod n) exactly when n is a Prime number.

We can rewrite Wilson’s theorem saying that if:

$$K(n) = \frac{1}{n} + \frac{\Gamma(n)}{n} - 1 \tag{9}$$

is integer, then n is Prime. ($\Gamma(n) = (n-1)!$ Is the gamma function. It is not a very efficient algorithm for primality as gamma(n) can get very large very quickly. The values of K(n) are:

$$K(n) = \{1, 0, 0, \frac{3}{4}, 4, \frac{115}{6}, 102, \frac{5033}{8}, \frac{40312}{9}, \frac{362871}{10}, 329890, \dots\}$$

One can observe the equivalence of K(n) with b(n) from 7.3.a.

c. **[IDEA #16]** Using combinations to check primality

p is Prime if $C(p,n) = 0 \; mod \; p$ for all positive integer n such that n<p.

Example:

7 is Prime because C(7,1)/7=1, C(7,2)/7=3, C(7,3)/7=5, are integers, and obviously C(7,4)/7, C(7,5)/7, and C(7,6)/7 as well.

8 is not Prime because: C(8,2)/8=3.5 is not integer.

as factorials grow fast, the best way to code this is by simplifying:

$$C(p,n) = \; p * (p-1) * \dots * \frac{p-n+1}{n!} = \; p \; * \frac{p-1}{n} * \dots * \; (p-n+1)$$

And:

$$\frac{C(p,n)}{p} = \frac{p-1}{n} * \dots * \; (p-n+1) = \prod_{k=1}^{n-1} \frac{p-k}{n+1-k} \qquad (10)$$

d. **[IDEA #17]** Primality test using powers of 2 and 3:

For n>3:
$gcd(2\text{^}n \; + \; n \,, 3\text{^}n \; + n \,, n+1) \; = \; n+1$ if and only if $(n+1)$ is Prime

Using WolframAlpha for n=1..100

[1, 1, 1, 5, 1, 7, 1, 1, 1, 11, 1, 13, 1, 1, 1, 17, 1, 19, 1, 1, 1, 23, 1, 1, 1, 1, 1, 29, 1, 31, 1, 1, 1, 1, 1, 37, 1, 1, 1, 41, 1, 43, 1, 1, 1, 47, 1, 1, 1, 1, 1, 53, 1, 1, 1, 1, 1, 59, 1, 61, 1, 1, 1, 1, 1, 67, 1, 1, 1, 71, 1, 73, 1, 1, 1, 1, 1, 79, 1, 1, 1, 83, 1, 1, 1, 1, 1, 89, 1, 1, 1, 1, 1, 1, 1, 97, 1, 1, 1]
(Checked to n=1,000,000)

The condition implies that if $(n+1)$ is Prime then:

$$(2^n \; + \; n) \; = \; 0 \; mod(n+1)$$

and

$$(3^n \; + \; n) \; = \; 0 \; mod(n+1)$$

which can be proved using Fermat's Little Theorem (p Prime and n=p-1):

$$(2^p \; - \; 2) \; = 0 \; mod \; (p)$$

and

$$\frac{\left[\frac{2^p - \; 2}{p} + 2\right]}{2} = \frac{2^n \; + n}{n+1}$$

$$\frac{\left[\frac{3^p - \; 3}{p} + 3\right]}{3} = \frac{3^n \; + n}{n+1}$$

which proves that:

$$(2^n + n) = \; 0 \; mod \; (n+1)$$

$$(3^n + n) = 0 \; mod \; (n+1)$$

this will be true for any $n = p - 1$ with $p$ Prime.

e. **[IDEA #18]** Primality test using the Lambert W-function: :

As a corollary, the following formula has integer solutions only if $(p+1)$ is Prime:

$$p = -\frac{k+1}{k} - \frac{LambertW\left(-1, -\frac{log(2)}{2k \, * \, 2^{\frac{1}{k}}}\right)}{log(2)} \tag{11}$$

where LambertW is the Lambert W function. The formula provides the following integer solutions:

$$\{k, p\} = \{(1,2), (3,4), (9,6), (93,10), (315,12), (3855,16), \dots\}$$

which makes $\{p+1\} = \{3,5,7,11,13,17, \dots\}$ Prime

7.5. Other interesting conjectures on Primes and divisions:

a. Grimm's Conjecture

In number theory, Grimm's conjecture (named after Carl Albert Grimm) states that to each element of a set of consecutive composite numbers one can assign a distinct Prime that divides it. [4]

Formal statement: if n + 1, n + 2, …, n + k are all composite numbers, then there are k distinct Primes $p_j$ such that $p_j$ divides n + j for 1 ≤ j ≤ k.

Weaker version: A weaker, though still unproven, version of this conjecture goes: If there is no Prime in the interval $[n+1, n+k]$, then:

$\prod_{x \le k}(n+x)$ has at least k distinct Prime divisors.

**[IDEA #19]** We propose the following proof for this conjecture:

i. Lemma 1: There is a Prime between n and 2n (Bertrand's Theorem)
ii. Corollary: There is a Prime in [n, 3n], [n,5n], … [n, p*n] for every Prime
iii. Lemma 2: There are no ODD composites with same Prime factors without a Prime number in between.
iv. If $c1 = \prod\{p \; primes\} p^{k1p}$ and $c2 = \prod\{p \; primes\} p^{k2p}$ with same Prime factors p, then, from Lemma 1, there is a Prime in the interval [c1,c2].
v. Theorem 1: For a sequence of ODD composite numbers $(2k+1), (2k+3), (2k+5), .. (2k+n)$, there are, from Lemma 2, at least (n+2) Prime factors to factor these composites.
vi. To prove Grimm's conjecture, we can separate the sequence of composites on evens and odds. The odds are proved through theorem 1, the evens can be reduced to odds dividing by 2 and using Theorem 1.

b. **[IDEA #20]** Maximum Common Divisors of multiple of Primes

If $P = 6p + 1$ is Prime and $Q = 6q - 1$ is Prime, then the following sequence a(n) gives the greatest common divisors for $(P^n - 1)$ and $(Q^n + 1)$:

$\{6, 24, 18, 240, 6, 72, 6, 480, 54, 264, 6, 720, 6, 24, 18, 16320, 6, 216, 6, 13200, 18, 552, 6, 1440, 6, \ldots$

Examples:

- All Primes squared are 1 away from a multiple of 24

- All Primes to the 8th power are 1 away from a multiple of 480

c. **[IDEA #21]** Divisibility criteria

We define a Digit Vector for any integer N from its expanded form using power of 10:

$15263 = 1 * 10^5 + 5 * 10^4 + 2 * 10^3 + 6 * 10^1 + 3 * 10^0$

Then the Digit Vector is :

D(15263)=[1,5,2,6,3]

We define the following Key vectors:

Key03= [1]
Key07= [2,3,1,5,4,6]
Key11= [1,10]
Key13= [1,4,3,12,9,10]
Key17= [1,12,8,11,13,3,2,7,16,5,9,6,4,14,15,10]
Key19= [1,2,4,8,16,13,7,14,9,18,17,15,11,3,6,12,5,10]
Key23= [1,7,3,21,9,17,4,5,12,15,13,22,16,20,2,14,6,19,18,11,8,10]
Key29= [1,3,9,27,23,11,4,12,7,21,5,15,16,19,28,26,20,2,6,18,25,17,22,8,24,14,13,10]

These keys can be extended repeating the sequence, for example:

Key07= [2,3,1,5,4,6,2,3,1,5,4,6,2,3,1,5,4,6,…]

The divisibility criteria we propose says that an integer N with Digit Ventor D of length L(D) is divisible by a factor X if the scalar product of the vectors D and KeyX (extended to length L(D)) is divisible by X, or formulated, D°KeyX = 0 mod X.

The proof is based on the fact that the Key vectors are residuals of mod(10^k, X) with k∈N.

For example:

2,900,519,955 is divisible by 19 because (extend digits in KeyX if necessary):

[2,9,0,0,5,1,9,9,5,5] ° [1,2,4,8,16,13,7,14,9,18,17(,15,11,3,6,12,5,10)] = 437

[4,3,7] ° [1,2,4] = 38

[3,8]*[1,2] = 19

The most astonishing property of the Key vectors is that if a number N is divisible by X, this will be true for any permutation of the elements of KeyX containing complete replications of the initial core sequence where order is maintained. For example:

[1,0,6,8,5,4,0,1,6,5] ° [2,3,1,5,4,6,2,3,1,5,4,6] = 126 = 18 * 7
[1,0,6,8,5,4,0,1,6,5] ° [6,2,3,1,5,4,6,2,3,1,5,4] =  98 = 14 * 7
[1,0,6,8,5,4,0,1,6,5] ° [4,6 2,3,1,5,4,6,2,3,1,5] =  98 = 14 * 7

As a larger example, the following number N with 79 digits is divisible by 17:

N=1068540165654659843210003216540065687987946512103468798415498798746513216873038

[N] ° key17 = 3060
[3,0,6,0] ° key17 = 51
[5,1] ° key17 = 17

d. PrimeFactorial and Primorial functions.

If p(n) is the n[th] prime, Primorial(p(n)) or n# is the multiplication of all primes up to p(n) :

$$n\# = \prod_{k=1}^{n} p(k)$$

**[IDEA #22]** We define PrimeFactorial or $n_¡$ as the product of all primes less than n:

$$n_¡ = \prod_{k=1}^{n} (p(k) < n)$$

Examples for n=5:

$$5! \ = \ 5 * 4 * 3 * 2 * 1 = \ 120 \text{ (Factorial function)}$$
$$5\# \ = \ 2 * 3 * 5 * 7 * 11 = 2310 \text{ (Primorial Function)}$$
$$5_¡ = 2 * 3 = 6 \text{ (PrimeFactorial function)}$$

From the definitions, one can also obtain:

$$\sum_{k=1}^{\infty} \frac{1}{n!} = e = \ 2.7181828184590455 ..$$

$$\sum_{k=1}^{\infty} \frac{1}{n^{\#}} = 1.70523017171801\ldots$$

$$\sum_{k=1}^{\infty} \frac{1}{n_¡} = \ 3.9200509773161327\ldots$$

Also:

$\mathrm{a(n)} \ = n^{\#}/n_¡$ is always integer (OEIS A301600 Caceres) with offset 0,2:

$$\{1, 2, 6, 15, 35, 385, 1001, 17017, 46189, 1062347, 30808063, 955049953, 3212440751 \ldots\}$$

$a(n) = n!/n\#$ is always integer (OEIS A300902 Caceres) with offset 0,3:

$\{1, 1, 2, 3, 4, 20, 24, 168, 192, 1728, 17280, 190080, 207360, 2695680, 2903040, \dots\}$

## 8. Number of Primes less than a given number. Function π(x)

Let's call $\pi(n)$ the number of Primes less than n. The Prime number theorem says that:

$$\lim_{n\to\infty} \pi(n)/(n/\ln n)) = 1 \qquad (12)$$

A better approximation given by Riemann is [3]:

$$\lim_{n\to\infty} \pi(n)/li(n) = 1 \qquad (13)$$

Where li(n) is the logarithmic integral function:

$$li(n) = \int_0^n \frac{dt}{\ln(t)} \qquad (14)$$

In 1899, de la Vallee Poussin proved that:

$$\pi(x) = li(x) + O(x\, e^{-\left(a\sqrt{\ln(x)}\right)})$$

For some positive constant '$a$' and $O()$ being the $big\ O\ notation$.

The following table shows the results of these approximations [6]:

| x | π(x) | π(x) − x / ln x | li(x) − π(x) | π(x) / li(x) |
|---|---|---|---|---|
| 1E+01 | 4 | -0.34 | 2.200 | 0.645161290 |
| 1E+02 | 25 | 3.29 | 5.100 | 0.830564784 |
| 1E+03 | 168 | 23.24 | 10.000 | 0.943820225 |
| 1E+04 | 1,229 | 143.26 | 17.000 | 0.986356340 |
| 1E+05 | 9,592 | 906.11 | 38.000 | 0.996053998 |
| 1E+06 | 78,498 | 6115.59 | 130.000 | 0.998346645 |
| 1E+07 | 664,579 | 44158.31 | 339.000 | 0.999490163 |
| 1E+08 | 5,761,455 | 332773.98 | 754.000 | 0.999869147 |
| 1E+09 | 50,847,534 | 2592591.57 | 1701.000 | 0.999966548 |
| 1E+10 | 455,052,511 | 20758029.10 | 3104.000 | 0.999993179 |
| 1E+11 | 4,118,054,813 | 169923159.33 | 11588.000 | 0.999997186 |

Table 15

The effort in this direction is to find more accurate approximations to $\pi(n)$. All these expressions involve complex algebraic expressions of ln(n), or the Riemann Zeta function, and li(x).

As an example, the Riemann hypothesis is equivalent to a much tighter bound on the error in the estimate for $\pi(n)$. and hence to a more regular distribution of Prime numbers, Specifically, [9]

$$|\pi(n) - li(x)| < \frac{1}{8\pi}\sqrt{x}\ln x \text{ for all x>2657}$$

## 9. [IDEA #23] Counting Primes less than a given number N using Prime generators $K_{6n}$ and $K_{6m}$

9.1. Based on:

**{Primes} = {2,3}**

$$\cup \{6k_n + 1 \mid k_n \neq 6xy + x + y \text{ and } k_n \neq 6xy - x - y \text{ for all } x, y \in N\}$$
$$\cup \{6k_m - 1 \mid k_m \neq 6xy - x + y \text{ for all } x, y \in N\}$$

The total number of Primes less than a given number N can be calculated following this algorithm:

a. *Calculate the total number of elements in $A_n$ = Da*

b. *Calculate the total number of duplicates in $A_n$ = Daa*

c. *Calculate the total number of elements in $B_n$ = Db*

d. *Calculate the total number of duplicates in $B_n$ = Dbb*

e. *Calculate the total number of duplicates between $A$ and $B_n$ = Dab*

f. *Calculate the total number of Primes<N of the form ($6K_n+1$)*

$$\pi(P^+) = \frac{N}{6} - (Da - Daa) - (Db - Dbb) + Dab$$

g. *Calculate the total number of elements in $C_n$ = Dc*

h. *Calculate the total number of duplicates in $C_n$ = Dcc*

i. *Calculate the total number of Primes<N of the form ($6K_m$-1)*

$$\pi(P^-) = \frac{N}{6} - (Dc - Dcc)$$

j. *Calculate the total number of Primes<N:*

$$\pi(N) = \pi(P^+) + \pi(P^-) + 2$$

Where the additional (+2) comes from the fact that Primes {2,3} cannot be generated by either $(P^+)$ or $(P^-)$.

9.2. Calculation of the number of elements in sequences $A_n, B_n, and\ C_n$

Condition 1: $\boldsymbol{A_n = 6xy + x + y \quad x > 0, y > 0 \in N}$

For every x, the maximum value of y that makes the Prime $6A_n + \mathbf{1} \leq \boldsymbol{n}$ is:

$$6(6xy + x + y) + 1 \leq n$$

$$(6xy + x + y) \leq (n - 1)/6$$

$$y(6x + 1) + x \leq (n - 1)/6$$

$$y(6x + 1) \leq \frac{n - 1}{6} - x$$

$$y \leq (\tfrac{n-1}{6} - x)\ /(6x+1)$$

For y=1, we obtain the maximum value of x (number of rows):

$$xmax = \frac{n-7}{42}$$

$A_n$ is symmetric therefore $ymax = xmax$.

And the total number of non-generators for a given x, that we will call Da(x) can be calculated by:

$$Da = \sum_{x=1}^{xmax} \frac{\left(\frac{n-1}{6} - x\right)}{6\text{x}+1} - (\text{x}-1) \qquad (15)$$

Using the same logic to calculate the total number of elements in $B_n$ and $C_n$, that we will call respectively Db and Dc:

$$Db = \sum_{x=1}^{xmax} \frac{\left(\frac{n-1}{6} + x\right)}{6\text{x}-1} - (\text{x}-1) \qquad (16)$$

With:

$$xmax = \frac{n+5}{30}$$

$ymax = xmax$ because of the symmetry of $B_n$

And:

$$Dc = \sum_{x=1}^{xmax} \frac{\left(\frac{n+1}{6} - x\right)}{6\text{x}-1} \qquad (17)$$

With:

$$xmax = \frac{n+7}{42}$$

$$Ymax = \frac{n-5}{30}$$

Using these expressions, the calculated values of Da, Db, Dc, for different values of N are:

| N | Da | Db | Dc |
|---|---|---|---|
| **1.00E+02** | 2 | 3 | 4 |
| **1.00E+03** | 44 | 59 | 97 |
| **1.00E+04** | 743 | 896 | 1,626 |
| **1.00E+05** | 10,572 | 12,121 | 22,649 |
| **1.00E+06** | 137,523 | 153,049 | 290,411 |
| **1.00E+07** | 1,694,547 | 1,849,728 | 3,543,726 |

Table 16

9.3. Calculation of the number of duplicates Daa, Dbb, Dab, and Dab

It can easily be observed that non-generators An, Bn, Cn matrices have duplicates. If we set N=1000 as an example, the duplicates in these matrices (Daa, Dbb, Dab, Dcc) are shown in bold in the following print out:

N = 1000
N//6 = 166

Da = 44
Daa = 2 ---> **[106, 155]**
Na = 42
Db = 59
Dbb = 4 ---> **[64, 99, 119, 134]**
Nb = 55
Dab = 11 ---> **[29, 54, 64, 79, 99, 104, 119, 129, 134, 141, 154]**
-----------------------------------------------
Nab = 86 -> Pi(P+)= 80
-----------------------------------------------
Dc = 97
Dcc = 17 ---> **[41, 46, 71, 76, 76, 90, 96, 101, 111, 111, 121, 139, 141, 146, 146, 156, 156]**
-----------------------------------------------
Nc = 80 -> Pi(P-)= 86
-----------------------------------------------
Number of Primes less than 1000 CPI(X)= 168 $\pi$(X)= 168

The main problem is to find a close form expression to calculate the duplicates for a certain N, and to that end, we are going to use numeric analysis to calculate the number of duplicates as a function of N. In the following table we show a limited number of rows of the dataset built to analyze the number of duplicates computed for different values of N and values from [8]:

| N | Da | Daa | Db | Dbb | Dab | Dc | Dcc | π(P+) | π(P-) | π(N) |
|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 2 | 0 | 3 | 0 | 0 | 4 | 0 | 11 | 12 | 25 |
| 500 | 18 | 0 | 25 | 1 | 4 | 40 | 5 | 45 | 48 | 95 |
| 1000 | 44 | 2 | 59 | 4 | 11 | 97 | 17 | 80 | 86 | 168 |
| 5000 | 324 | 40 | 400 | 77 | 104 | 717 | 221 | 330 | 337 | 669 |
| 10000 | 743 | 119 | 896 | 214 | 251 | 1626 | 576 | 611 | 616 | 1229 |
| 25000 | 2170 | 440 | 2557 | 737 | 755 | 4695 | 1918 | 1371 | 1389 | 2762 |
| 50000 | 4811 | 1141 | 5585 | 1792 | 1686 | 10363 | 4605 | 2556 | 2575 | 5133 |
| 75000 | 7634 | 1941 | 8796 | 2985 | 2686 | 16386 | 7596 | 3682 | 3710 | 7394 |
| 100000 | 10572 | 2810 | 12121 | 4268 | 3733 | 22649 | 10789 | 4784 | 4806 | 9592 |

Table 17

We have decided to use as the independent variable $nmax$:

$$xmax = int(N / 42)$$

$$ymax = int(N / 5) - 1$$

$$nmax = ymax - xmax$$

The main reason to use $nmax$ is that provides a method to obtain all correlations in a linear or quadratic form using the logarithm of N in base 10 ($log(N, 10)$) as an input. We can now express the previous table introducing the independent variable $nmax$:

| nmax | Da | Daa | Db | Dbb | Dab | Dc | Dcc |
|---|---|---|---|---|---|---|---|
| 17 | 2 | 0 | 3 | 0 | 0 | 4 | 0 |
| 88 | 18 | 0 | 25 | 1 | 4 | 40 | 5 |
| 176 | 44 | 2 | 59 | 4 | 11 | 97 | 17 |
| 880 | 324 | 40 | 400 | 77 | 104 | 717 | 221 |
| 1761 | 743 | 119 | 896 | 214 | 251 | 1626 | 576 |
| 4404 | 2170 | 440 | 2557 | 737 | 755 | 4695 | 1918 |
| 8809 | 4811 | 1141 | 5585 | 1792 | 1686 | 10363 | 4605 |
| 13214 | 7634 | 1941 | 8796 | 2985 | 2686 | 16386 | 7596 |
| 17619 | 10572 | 2810 | 12121 | 4268 | 3733 | 22649 | 10789 |
| 22023 | 13599 | 3744 | 15539 | 5612 | 4791 | 29080 | 14137 |
| 26428 | 16699 | 4722 | 19026 | 7024 | 5886 | 35656 | 17596 |

Table 18

Then we can chart:

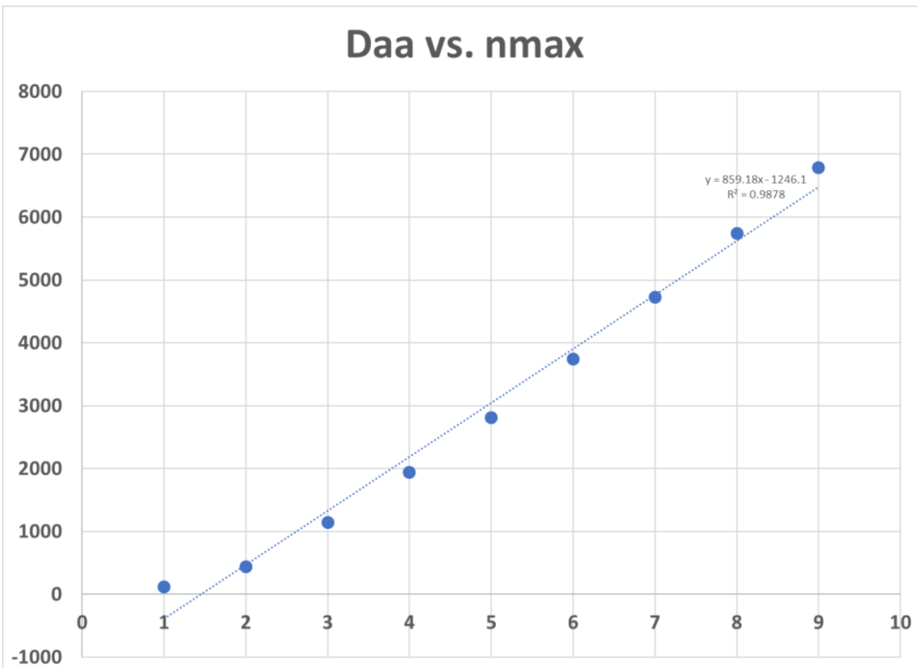


Fig 12

Where we can see that $Daa$ can be expressed as a linear regression over $nmax$.

9.4. The computation of all data vectors $\{Da\}, \ldots, \{Dcc\}$ as function of $nmax$

9.4.1. Computation of Dc, Dcc to obtain $\pi(P^{-})$

We define

$logx2 = log(N, 10) - 2$ where $log(N, 10)$ is the logarithm in base 10 of N

then the computation of all regressions for $R^2 > 0.999$ gives:

c[1] = 0.633480 - 26/1000000
c[2] = 0.193065 + 2/1000000
c[3] = 0.340500 - 34/1000000

c[4] = -(0.39520 +  4/1000000)
c[5] = -(0.01860 +  5/1000000)
c[6] = 0.260600   - 55/1000000
c[7] = 0.450000  + 50/1000000
c[8] = 0.997381368

$sx = c[1] * logx2 + c[2]$
$tx = c[3] * logx2 + c[4]$
$fx = (c[5] * logx2^2 + c[6] * logx2 + c[7])/c[8]$

$and:$

$Dc = int(nmax * sx)$

$Dcc = nmax * tx$

$\pi(P^-) = int((int(N/6) - (Dc - Dcc)) * 1/fx)$

The error of calculated $\pi(P^-)$ compared to actuals:


Fig 13

9.4.2. Computation of Da, Daa, Db, Dbb, Dab to obtain $\pi(P^+)$

We define:

$logx = log(N, 10)$

then the computation of all regressions for $R^2 > 0.9999$ gives:

$Da = int(nmax * sx)$
$Daa = nmax * tx$
$Db = int(nmax * ux)$
$Dbb = nmax * vx$
$Dab = nmax * wx$

Where:

$sx = b[0] * logx + b[1]$
$tx = b[2] * logx^2 + b[3] * logx + b[4]$

$$ux = b[5] * logx + b[6]$$
$$vx = b[7] * logx^2 + b[8] * logx + b[9]$$
$$wx = b[10] * logx^2 + b[11] * logx + b[12]$$

that can be simplified into one quadratic model:

$$D = c[0] + c[1] * logx + c[2] * logx ** 2$$

Where:

$c = 0.47956, 0.112818, 1/(-8.827 * logx - 39.927)$ for logx≥14

$c = [0.47956, 0.112818, 1/(-2.0238logx^3 + 41.114\, logx^2 - 290.89logx + 620.2)$ for logx<14

And:

$$\pi(P^+) = int((int(x/6) - D * nmax))$$

The error of calculated $\pi(P^+)$ compared to actuals:

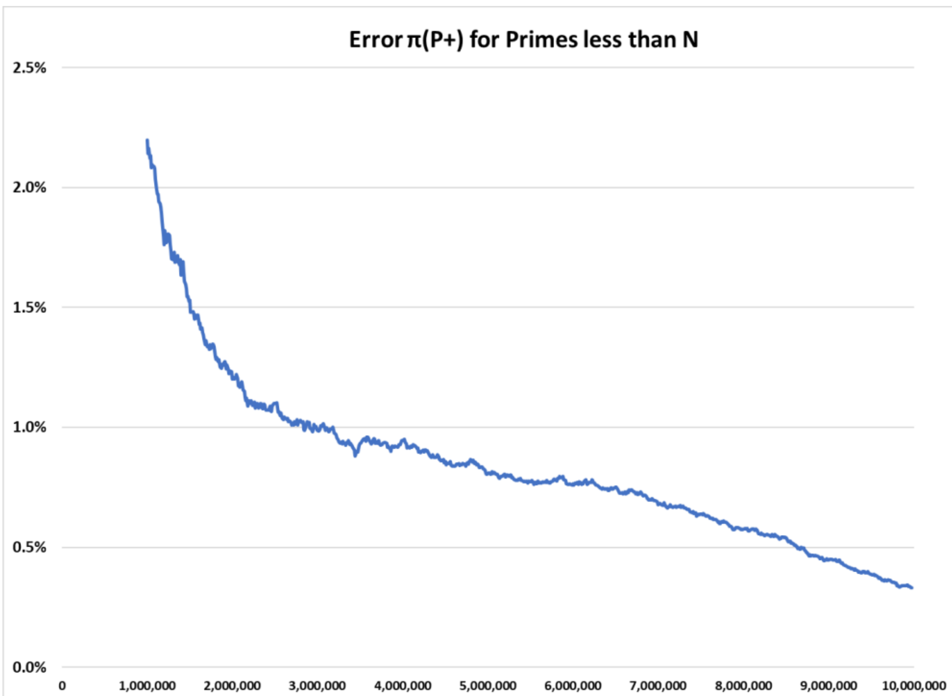


Fig 14

9.4.3. **[IDEA #]** METHOD CPIX1: Calculate values of CPIX using only counts over the series $P^-$

The sequences $P^-$ and $P^+$have similar number of elements. We have plotted these numbers in Fig. 1 and based on Fig. 3 we can say that:

$$\lim_{N\to\infty} \frac{\pi(P^-) - \pi(P^+)}{N} = 0$$

This can be used to simplify the algorithm described in 9.1:

a. *Calculate the total number of elements in* $C_n$ *= Dc*

b. *Calculate the total number of duplicates in* $C_n$ *= Dcc*

c. *Calculate the total number of Primes<N of the form (6*$K_m$*-1)*

$$\pi(P^-) = \frac{N}{6} - (Dc - Dcc)$$

d. *Calculate the total number of Primes<N:*

$$CPIX1 = 2 * \pi(P^-)$$

The error of the calculations obtained using this method are compared to the errors using Li(x) and x/ln(x) in the following table:

| x | π(x) | li(x) error | x / ln x % Error | CPIX1 Error |
|---|---|---|---|---|
| 10^2 | 25 | 20.40000000000000% | 13.200% | 4.000000000000% |
| 10^3 | 168 | 5.95238095238095% | 13.690% | 0.000000000000% |
| 10^4 | 1,229 | 1.38323840520749% | 11.640% | -0.732302685110% |
| 10^5 | 9,592 | 0.39616346955798% | 9.450% | -0.625521267723% |
| 10^6 | 78,498 | 0.16560931488701% | 7.790% | -0.405105862570% |
| 10^7 | 664,579 | 0.05100973699139% | 6.640% | -0.307412662753% |
| 10^8 | 5,761,455 | 0.01308697195412% | 5.780% | -0.226314359828% |
| 10^9 | 50,847,534 | 0.00334529497537% | 5.100% | -0.162411022725% |
| 10^10 | 455,052,511 | 0.00068211907966% | 4.560% | -0.116388106251% |
| 10^11 | 4,118,054,813 | 0.00028139499171% | 4.130% | -0.082868372447% |
| 10^12 | 37,607,912,018 | 0.00010174188874% | 3.770% | -0.058232969673% |
| 10^13 | 346,065,536,839 | 0.00003148854433% | 3.470% | -0.039101919895% |
| 10^14 | 3,204,941,750,802 | 0.00000982513957% | 3.210% | -0.022973864527% |
| 10^15 | 29,844,570,422,669 | 0.00000352700336% | 2.990% | -0.007120543713% |
| 10^16 | 279,238,341,033,925 | 0.00000115121441% | 2.790% | 0.014604542693% |
| 10^17 | 2,623,557,157,654,230 | 0.00000030327485% | 2.630% | 0.086124038326% |
| 10^18 | 24,739,954,287,740,860 | 0.00000008872107% | 2.480% | -0.141772447230% |
| 10^19 | 234,057,667,276,344,607 | 0.00000004267229% | 2.340% | -0.048090380881% |
| 10^20 | 2,220,819,602,560,918,840 | 0.00000001002984% | 2.220% | -0.028947325069% |
| 10^21 | 21,127,269,486,018,731,928 | 0.00000000282760% | 2.110% | -0.019832875445% |
| 10^22 | 201,467,286,689,315,906,290 | 0.00000000095914% | 2.020% | -0.014235299182% |
| 10^23 | 1,925,320,391,606,803,968,923 | 0.00000000037657% | 1.930% | -0.010370932566% |
| 10^24 | 18,435,599,767,349,200,867,866 | 0.00000000009301% | 1.840% | -0.007528339373% |
| 10^25 | 176,846,309,399,143,769,411,680 | 0.00000000003120% | 1.770% | -0.005356412793% |
| 10^26 | 1,699,246,750,872,437,141,327,603 | 0.00000000000917% | 1.700% | -0.003656237399% |
| 10^27 | 16,352,460,426,841,680,446,427,399 | 0.00000000000311% | 1.640% | -0.002303853847% |

Table 19

And plotting the differences graphically:

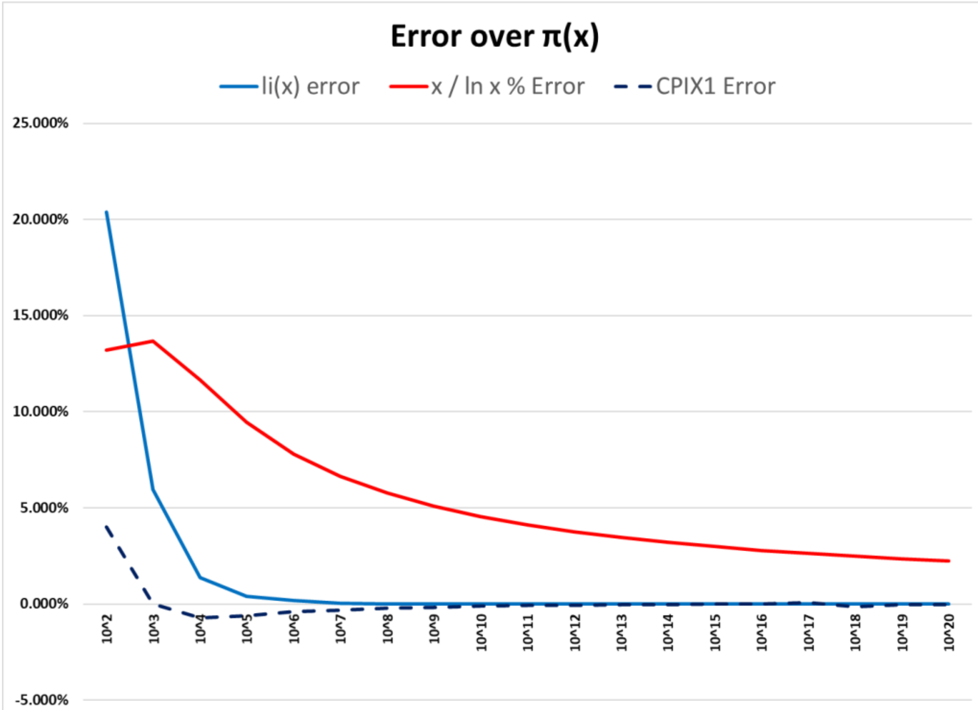


Fig 15

9.4.4. **[IDEA #24]** METHOD CPIX2: Calculate values of CPIX2 using counts from the series $P^-$ and $P^+$

Using 8.4.1 and 8.4.2. we can calculate $\mathrm{CPIX2(N)} = \pi(P^+) + \pi(P^-) + 2$

| x | π(x) | li(x) error | x / ln x % Error | CPIX2 Error |
|---|---|---|---|---|
| 10^2 | 25 | 20.40000000000000% | 13.200% | 0.000000000000% |
| 10^3 | 168 | 5.95238095238095% | 13.690% | 0.595238095238% |
| 10^4 | 1,229 | 1.38323840520749% | 11.640% | 4.963384865745% |
| 10^5 | 9,592 | 0.39616346955798% | 9.450% | 9.288990825688% |
| 10^6 | 78,498 | 0.16560931488701% | 7.790% | 0.016560931489% |
| 10^7 | 664,579 | 0.05100973699139% | 6.640% | 0.000000000000% |
| 10^8 | 5,761,455 | 0.01308697195412% | 5.780% | 0.004356538409% |
| 10^9 | 50,847,534 | 0.00334529497537% | 5.100% | 0.000000000000% |
| 10^10 | 455,052,511 | 0.00068211907966% | 4.560% | 0.000005713626% |
| 10^11 | 4,118,054,813 | 0.00028139499171% | 4.130% | 0.000000000000% |
| 10^12 | 37,607,912,018 | 0.00010174188874% | 3.770% | 0.000000109020% |
| 10^13 | 346,065,536,839 | 0.00003148854433% | 3.470% | 0.000000169332% |
| 10^14 | 3,204,941,750,802 | 0.00000982513957% | 3.210% | 0.000000222500% |
| 10^15 | 29,844,570,422,669 | 0.00000352700336% | 2.990% | 0.000000266394% |
| 10^16 | 279,238,341,033,925 | 0.00000115121441% | 2.790% | 0.000000385323% |
| 10^17 | 2,623,557,157,654,230 | 0.00000030327485% | 2.630% | 0.000000386950% |
| 10^18 | 24,739,954,287,740,860 | 0.00000008872107% | 2.480% | 0.000000350235% |
| 10^19 | 234,057,667,276,344,607 | 0.00000004267229% | 2.340% | 0.000000564029% |
| 10^20 | 2,220,819,602,560,918,840 | 0.00000001002984% | 2.220% | 0.000000774968% |

Table 20

CPIX2 method is more precise than CPIX1. The comparison is shown in the following chart:

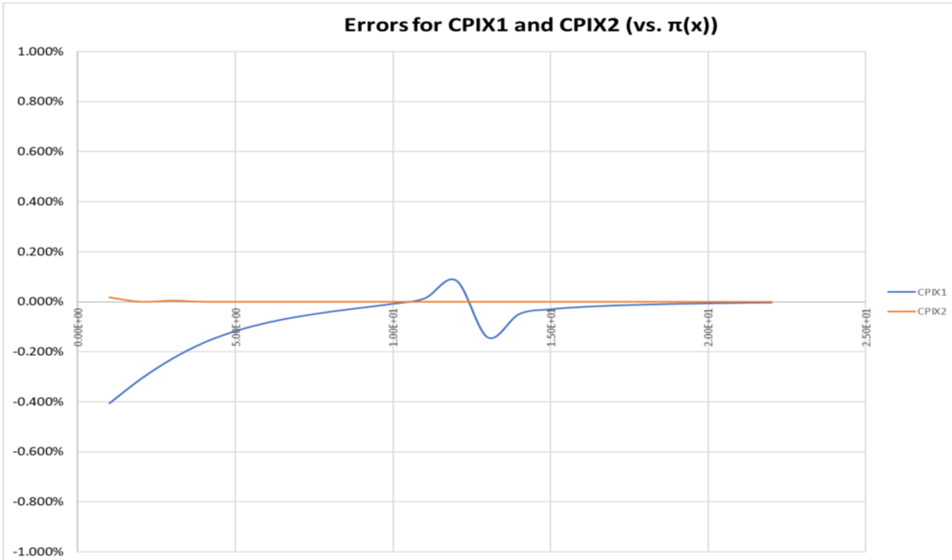


Fig 16

## 10. [IDEA #25] Counting Primes less than a given number N using Prime generators $K_{1n}$

10.1. Based on:

**{Primes} =** $\{\boldsymbol{k_{1n}+1} \mid \boldsymbol{k_n} \neq \boldsymbol{xy+x+y} \quad \boldsymbol{for\ all\ x, y} \in \boldsymbol{N}\}$

The total number of Primes less than a given number N can be calculated following this algorithm:

a. *Calculate the total number of elements in* $A_n$ *= Da*
b. *Calculate the total number of duplicates in* $A_n$ *= Daa*
c. *Calculate the total number of Primes<N of the form (*$K_{1n}$*+1)*

$$\pi(P^+) = N - (Da - Daa)$$

10.2. Calculation of *Da* the number of elements in sequence $A_n = xy + x + y$

Condition 1: $\boldsymbol{A_{1n} = xy + x + y} \qquad \boldsymbol{x > 0, y > 0 \in N}$

For every x, the maximum value of y that makes the Prime $A_{1n} + \mathbf{1} \leq \boldsymbol{n}$ is:

$$y \leq \frac{n}{x+1} - 1$$

For x=1, we obtain:

$$y_{max} = \frac{n}{2} - 1$$

$A_n$ is symmetric therefore $ymax = xmax$.

And the total number of non-generators for a given x, that we will call Da(x) can be calculated by:

$$Da = \sum_{x=1}^{xmax} \frac{n}{x+1} - 1 \qquad (18)$$

With values:

| N | Da | Da-odds |
|---:|---:|---:|
| 100 | 283 | 56 |
| 1,000 | 5,070 | 1,111 |
| 10,000 | 73,669 | 16,877 |
| 100,000 | 966,751 | 226,337 |
| 1,000,000 | 11,970,035 | 2,839,095 |
| 10,000,000 | 142,725,365 | 34,147,078 |
| 100,000,000 | 1,657,511,569 | 399,035,115 |

Table 21

Computing a polynomial regression, Da can be defined as a function of n:

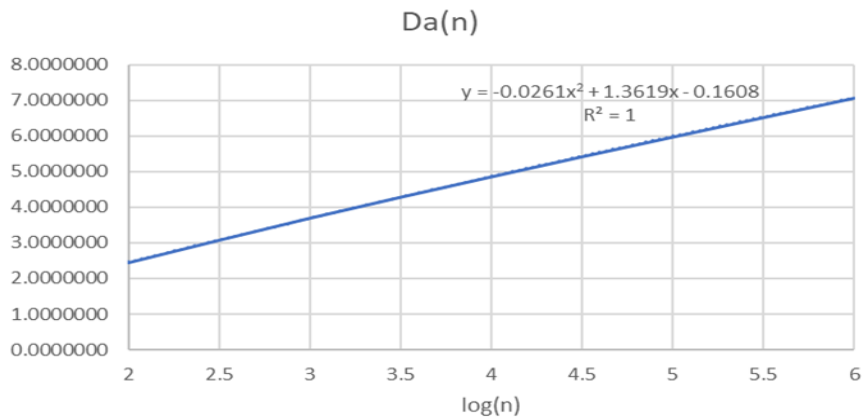


Fig 17

10.3. Calculation of *Daa,* the repeated numbers in sequence $A_n = xy + x + y$

The number of duplicated elements Daa and its behavior can be observed in the following table:

| | Calculated | | | |
|---|---|---|---|---|
| N | Daa Total | Daa Unique | Daa computed | Error |
| 100 | 276 | 203 | 208 | -0.0240384615 |
| 400 | 2211 | 1339 | 1347 | -0.0059391240 |
| 900 | 6833 | 3723 | 3731 | -0.0021441973 |
| 1600 | 14785 | 7507 | 7520 | -0.0017287234 |
| 2500 | 26582 | 12815 | 12828 | -0.0010134082 |
| 3600 | 42779 | 19766 | 19764 | 0.0001011941 |
| 4900 | 63475 | 28366 | 28369 | -0.0001057492 |
| 6400 | 89171 | 38724 | 38733 | -0.0002323600 |
| 8100 | 120146 | 50902 | 50897 | 0.0000982376 |
| 10000 | 156489 | 64891 | 64898 | -0.0001078616 |
| 12100 | 198705 | 80806 | 80804 | 0.0000247512 |
| 14400 | 246772 | 98627 | 98627 | 0.0000000000 |
| 16900 | 300797 | 118376 | 118405 | -0.0002449221 |

Table 22

The calculation of the duplicate composite numbers less than N in the previous table correspond to the following formula **[IDEA #26]**, which gives the approximations shown in the Error column in Table 22:

$$Daa = \sum_{k=3}^{xmax} \frac{N}{2k} + \sum_{k=4}^{xmax} \frac{N}{3k} + \frac{f(N)}{\log(N)} \sum_{j>3\ prime}^{xmax} \sum_{k=j+1}^{xmax} \frac{N}{ij} \quad (19)$$

Where f(N) is an algebraic function.

And the calculated values for $CPIX1(n) = N - Da(n) + Daa(n)$ are:

| N | Da | Daa | CPIX1(N) | π(N) | Error |
|---|---|---|---|---|---|
| 100 | 283 | 203 | 20 | 25 | -20.000% |
| 400 | 1669 | 1339 | 70 | 78 | -10.256% |
| 900 | 4477 | 3723 | 146 | 154 | -5.195% |
| 1600 | 8869 | 7507 | 238 | 251 | -5.179% |
| 2500 | 14961 | 12815 | 354 | 367 | -3.542% |
| 3600 | 22861 | 19766 | 505 | 503 | 0.398% |
| 4900 | 32615 | 28366 | 651 | 654 | -0.459% |
| 6400 | 44299 | 38724 | 825 | 834 | -1.079% |
| 8100 | 57979 | 50902 | 1023 | 1018 | 0.491% |
| 10000 | 73669 | 64891 | 1222 | 1229 | -0.570% |
| 12100 | 91457 | 80806 | 1449 | 1447 | 0.138% |
| 14400 | 111341 | 98627 | 1686 | 1686 | 0.000% |
| 16900 | 133357 | 118376 | 1919 | 1948 | -1.489% |
| 25000 | 207037 | 184710 | 2673 | 2762 | -3.222% |
| 30000 | 253926 | 227051 | 3125 | 3245 | -3.698% |

Table 23

The error with this method is larger than the methods developed using series $P^{6+}$and $P^{6-}$.

## 11. Ideas on Legendre's Conjecture

11.1. Legendre's Conjecture states that there is always a prime number between $n^2$ and $(n+1)^2$ provided that n$\neq -1$ or 0.

The following chart shows the number of Primes between $n^2$ and $(n+1)^2$.

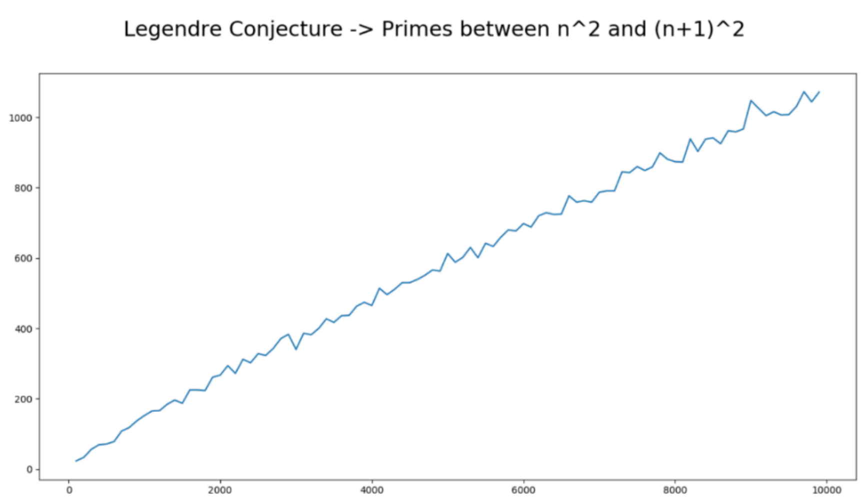


Fig 18

The prime number theorem suggests that the actual number of primes between $n^2$ and $(n + 1)^2$ is asymptotic to n/ln(n).

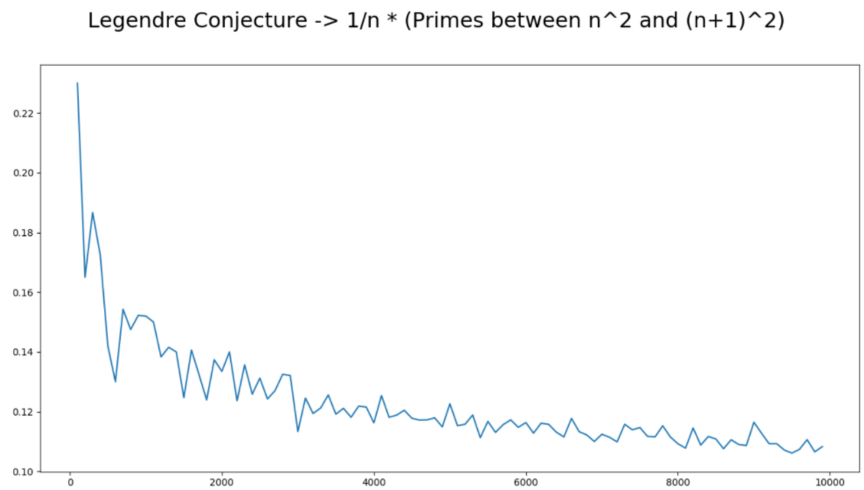


Fig 19

A way to understand this conjecture can be based on the definition of Primes in series $P^{1+}$:

{Primes} = $\{k_{1n} + 1 \mid k_{1n} \neq xy + x + y\}$

Based on this definition, all composite numbers are of the form $c = xy + x + y$+1.

In this case, we can observe that:

$n^2$ = (n-1) * (n-1) + (n-1) + (n-1) +1 $\qquad x = y = (n - 1)$

and $(n + 1)^2$ = n*n +n +n +1 $\qquad x = y = n$

Any number between $n^2$ and $(n + 1)^2$ that can't be represented by $c = xy + x + y + 1 = (\mathrm{x} + 1) * (\mathrm{x} - 1)$ would be Prime. Between $n^2$ and $(n + 1)^2$ there are $2n + 1$ numbers:

$n^2 \rightarrow$ $\{n^2$+1, $n^2$+2, …, $n^2 + 2n\}$ $\rightarrow (n + 1)^2$

We can see those numbers (in orange) in the following matrix with each element $c_{ij} = xy + x + y + 1$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 |
| 2 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 | 33 | 36 | 39 | 42 | 45 | 48 | 51 | 54 | 57 | 60 |
| 3 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 44 | 48 | 52 | 56 | 60 | 64 | 68 | 72 | 76 | 80 |
| 4 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 | 90 | 95 | 100 |
| 5 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 | 66 | 72 | 78 | 84 | 90 | 96 | 102 | 108 | 114 | 120 |
| 6 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 | 77 | 84 | 91 | 98 | 105 | 112 | 119 | 126 | 133 | 140 |
| 7 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 | 88 | 96 | 104 | 112 | 120 | 128 | 136 | 144 | 152 | 160 |
| 8 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 | 99 | 108 | 117 | 126 | 135 | 144 | 153 | 162 | 171 | 180 |
| 9 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 130 | 140 | 150 | 160 | 170 | 180 | 190 | 200 |
| 10 | 22 | 33 | 44 | 55 | 66 | 77 | 88 | 99 | 110 | 121 | 132 | 143 | 154 | 165 | 176 | 187 | 198 | 209 | 220 |
| 11 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | 108 | 120 | 132 | 144 | 156 | 168 | 180 | 192 | 204 | 216 | 228 | 240 |
| 12 | 26 | 39 | 52 | 65 | 78 | 91 | 104 | 117 | 130 | 143 | 156 | 169 | 182 | 195 | 208 | 221 | 234 | 247 | 260 |
| 13 | 28 | 42 | 56 | 70 | 84 | 98 | 112 | 126 | 140 | 154 | 168 | 182 | 196 | 210 | 224 | 238 | 252 | 266 | 280 |
| 14 | 30 | 45 | 60 | 75 | 90 | 105 | 120 | 135 | 150 | 165 | 180 | 195 | 210 | 225 | 240 | 255 | 270 | 285 | 300 |
| 15 | 32 | 48 | 64 | 80 | 96 | 112 | 128 | 144 | 160 | 176 | 192 | 208 | 224 | 240 | 256 | 272 | 288 | 304 | 320 |
| 16 | 34 | 51 | 68 | 85 | 102 | 119 | 136 | 153 | 170 | 187 | 204 | 221 | 238 | 255 | 272 | 289 | 306 | 323 | 340 |
| 17 | 36 | 54 | 72 | 90 | 108 | 126 | 144 | 162 | 180 | 198 | 216 | 234 | 252 | 270 | 288 | 306 | 324 | 342 | 360 |
| 18 | 38 | 57 | 76 | 95 | 114 | 133 | 152 | 171 | 190 | 209 | 228 | 247 | 266 | 285 | 304 | 323 | 342 | 361 | 380 |
| 19 | 40 | 60 | 80 | 100 | 120 | 140 | 160 | 180 | 200 | 220 | 240 | 260 | 280 | 300 | 320 | 340 | 360 | 380 | 400 |

Fig 24

For example, if $n = 9$, with $n^2 = 81$,  $x = y = (n-1) = 8$ with $8 * 8 + 8 + 8 + 1 = 81$

The matrix is symmetric, with the upper area values calculated between:

$$x = 1 \quad and \quad y = \frac{n^2 - 2}{2}$$

and:

$$x = n - 1 \quad and \quad y = n - 1$$

We can observe that for every row $x$ in the chart, the values are the multiples of $(x + 1)$.

If we prove that the set of values between $n^2$ and $(n+1)^2$ in the matrix does not contain all the $(2n + 1)$ elements between $n^2$ and $(n+1)^2$ $\{n^2+1, n^2+3, n^2+5\ldots, n^2+2n-1\}$, then Legendre's conjecture will be true.

Based on (18) the number of composite numbers c(x) less than $n^2$ is:

$$c(n^2) = Da(n^2) - Daa(n^2) = \sum_{x=1}^{\frac{n^2-2}{2}} \left(\frac{n^2}{x+1} - 1\right) - Daa(n^2)$$

And:

$$c((n+1)^2) = Da((n+1)^2) - Daa((n+1)^2) = \sum_{x=1}^{\frac{(n+1)^2-2}{2}} \left(\frac{(n+1)^2}{x+1} - 1\right) - Daa((n+1)^2)$$

We are trying to prove that :

$c((n+1)^2) - c(n^2) < 2n + 1$ linearly increasing as it can be observed in the next chart:

Which can easily be observed graphically:

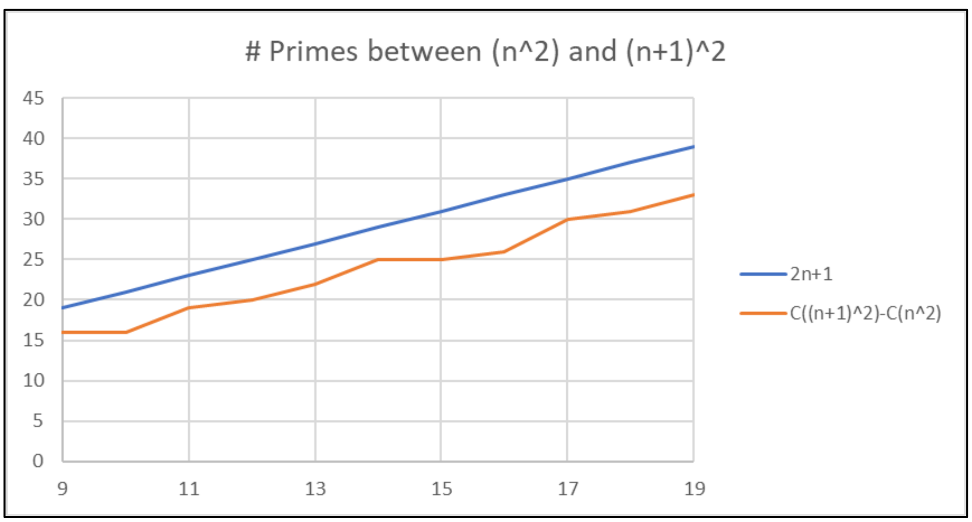


Fig 20

And numerically:

| N | N^2 | Da | Daa Unique | Daa Total | Calc #Primes | Actual # Primes | Actual Primes between N^2 and (N+1)^2 |
|---|---|---|---|---|---|---|---|
| 9 | 81 | 110 | 42 | 52 | 3 | 3 | [83, 89, 97] |
| 10 | 100 | 146 | 56 | 72 | 5 | 5 | [101, 103, 107, 109, 113] |
| 11 | 121 | 187 | 73 | 97 | 4 | 4 | [127, 131, 137, 139] |
| 12 | 144 | 235 | 91 | 126 | 5 | 5 | [149, 151, 157, 163, 167] |
| 13 | 169 | 286 | 114 | 157 | 5 | 5 | [173, 179, 181, 191, 193] |
| 14 | 196 | 346 | 142 | 195 | 4 | 4 | [197, 199, 211, 223] |
| 15 | 225 | 413 | 171 | 237 | 6 | 6 | [227, 229, 233, 239, 241, 251] |
| 16 | 256 | 485 | 201 | 284 | 7 | 7 | [257, 263, 269, 271, 277, 281, 283] |
| 17 | 289 | 564 | 234 | 337 | 5 | 5 | [293, 307, 311, 313, 317] |
| 18 | 324 | 651 | 270 | 394 | 6 | 6 | [331, 337, 347, 349, 353, 359] |
| 19 | 361 | 743 | 311 | 455 | 6 | 6 | [367, 373, 379, 383, 389, 397] |
| 20 | 400 | 844 | 355 | 523 | | | [401, 409, 419, 421, 431, 433, 439] |

Table 25

With the number of primes between $n^2$ and $(n+1)^2$ calculated following:

$$Cal\#Primes \ = \ (2n+1) - Delta(Da) \ + \ Delta(Daa)$$

The analytical prove of Legendre's conjecture will have to show that, for every n, there are more elements in $(2n+1)$ than in the set $\{c((n+1)^2 - c(n)^2\}$.

**[IDEA #27]** At the core of the prove, we need to formulate $Daa(n)$ and demonstrate that for every $n$:

Condition 1: $Daa(n) \leq Da(n)$
which is obvious because Da(n) contains all elements of the c-matrix and Daa is a subset of c-matrix
and
Condition 2: $Daa((n+1)^2) > \ Daa(n^2)$

Numerically, a good regression for Daa(n) for $n < 10^{18}$ is shown in the next picture. Algebraic approximations for Prime related functions only work on limited intervals. Logarithmic approximations always work better for general purposes:

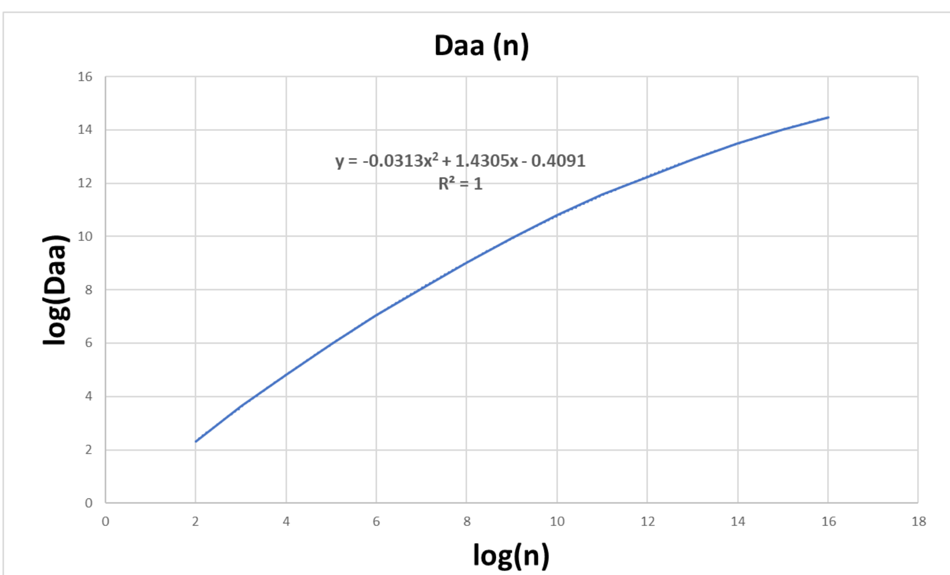


Fig 21

From (17), and for $\pi(n)$ the Prime Counting function:

$$\pi(n) = N - Da(n) + Daa(n)$$

With:

$$Daa(n) = \pi(n) - N + Da(n)$$

And:

$$Daa(n+1) - Daa(n) = \pi(n+1) - \pi(n) - N + 1 - N + Da(n+1) - Da = \pi(n+1) - \pi(n) + Da(n+1) - Da + 1$$

And we know that for n>1:

$$\pi((n+1)^2) > \pi(n^2)$$

And:

$$Da((n+1^2)) > Da(n^2)$$

Therefore:

$Daa((n+1^2)) > Daa(n^2)$ and Daa(n) is strictly increasing which proves Legendre's conjecture.

11.2. What is the minimum value of x such that there is always, at least, one prime between $n^x$ and $(n+1)^x$?

Let's observe the problem numerically and let's define $c(n) = floor(n^x)$ where $floor(a)$ is the highest integer less than $a$.

For x=1 the calculation rapidly shows that there are no new primes between n=3 and n=4.

| N | N^1 | Da | Daa | π(x)=N-Da+Daa |
|---:|---:|---:|---:|---:|
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 0 | 0 | 1 |
| 3 | 3 | 0 | 0 | 2 |
| 4 | 4 | 1 | 0 | 2 |

Table 26

The lowest x that verifies that the count of primes increases at least by one with an increase in n is 1.60:

π(n^x) for x= 1.00 ---> Repeats at N=4^1 = 4
π(n^x) for x= 1.01 ---> Repeats at N=4^1.01 = 4
π(n^x) for x= 1.02 ---> Repeats at N=4^1.02 = 4
…
…
π(n^x) for x= 1.58 ---> Repeats at N=21^1.58 = 122
π(n^x) for x= 1.59 ---> Repeats at N=21^1.59 = 126
π(n^x) for x= 1.60 ---> NO REPEATS
π(n^x) for x= 1.61 ---> Repeats at N=20^1.61 = 124

So, we can conclude that there is a Prime p between $n^{1.6}$ and $(n+1)^{1.6}$.

## 12. [IDEA #28] Ideas on the Prime Conjecture that there is infinite Primes of the form $n^2 + 1$

We are looking for odd values of $n^2 + 1$ that only happens when $n$ is even. If $n$ is odd, then $n^2$ is odd, and $n^2 + 1$ is even and never prime.

If we use $n = 2k$ for $k \in N$, then using generators $k_{1n}$:

$n^2 = xy + x + y + 1$ which is true for all composite numbers.

Then, if $n^2 + 1$ to be prime, cannot be expressed, for any $a, b \in N$, as:

$$n^2 + 1 = \mathrm{ab} + \mathrm{a} + \mathrm{b} + 1$$

Or:

$$4k^2 = \mathrm{ab} + \mathrm{a} + \mathrm{b}$$

Or, if we call $k_{np}$ the values of k that make $(2k_{np})^2 + 1$ no prime, then:

$$k_{np} = \frac{1}{2}\sqrt{(\mathrm{ab} + \mathrm{a} + \mathrm{b})}$$

If we plot the values of $k_{np}$ for $a \leq 1000$ and $b \leq 1000$:

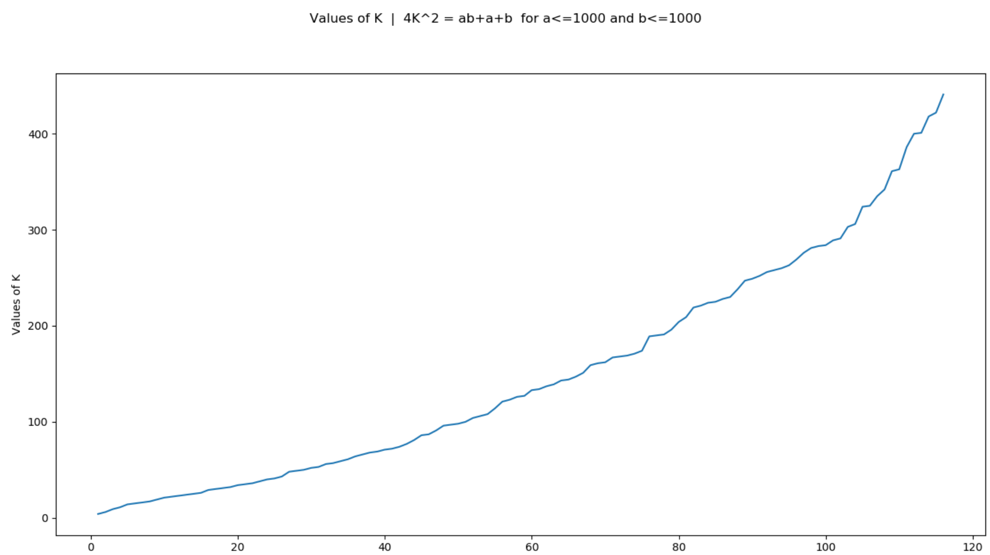


Fig 22

Of the $a * b = 1{,}000{,}000$ possible combinations of a & b, only 116 of those combinations make $(2k_{np})^2 + 1$ no Prime. The first elements of this sequence are:

$$\{4, 6, 9, 11, 14, 15, 16, 17, 19, 21, 22, 23, 24, 25, 26, 29, 30, 31, 32, 34, 35, 36, 38, 40, 41, 43, 48, 49, \ldots\}$$

For example, the element {23} is in the sequence because:

$4 * 23^2 = 2116 = 28 * 72 + 28 + 72$ with a=28 and b=72

and $n = 2k_{np} = 46$

The following histogram shows that the number of $k_{np}$ decreases as $a, b$ grow

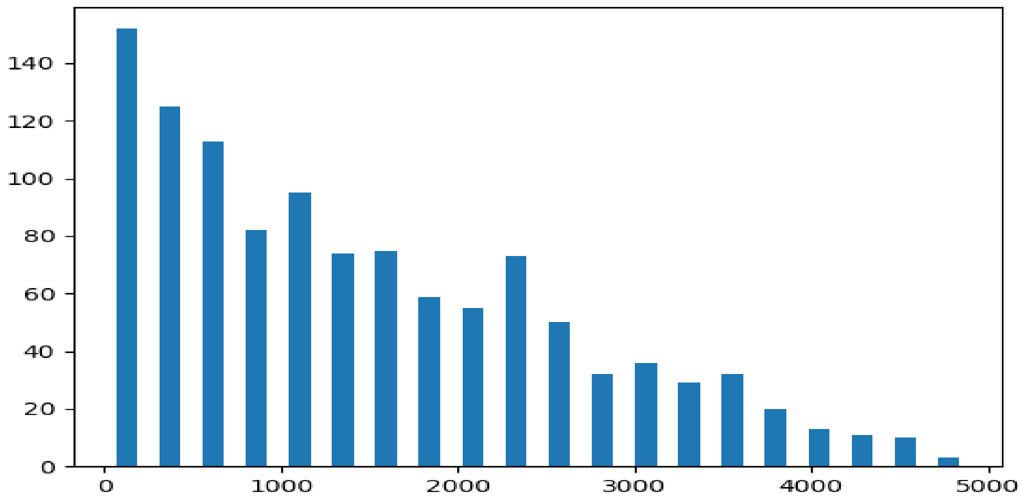


Fig 23

And the following chart shows how the histogram of values of n such that $n^2 + 1$ is prime increases:

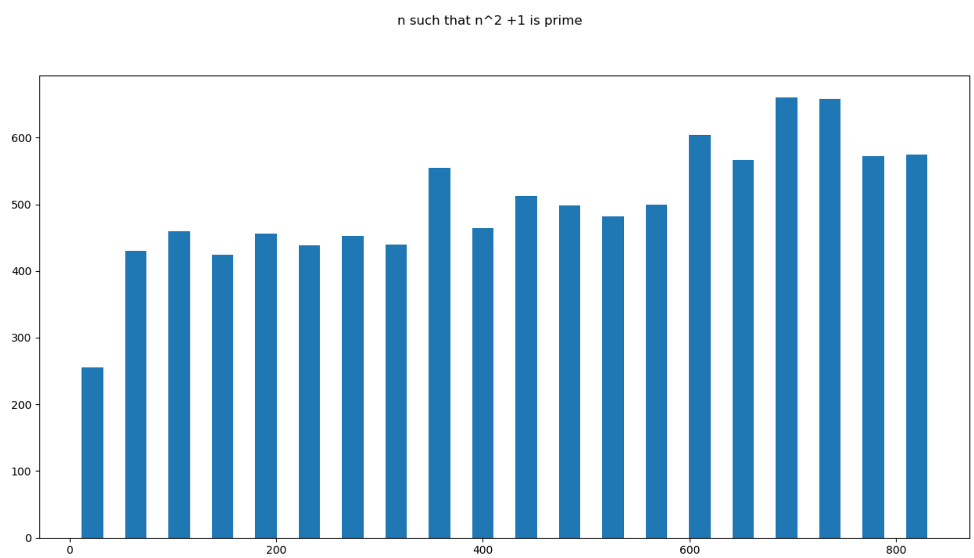


Fig 24

## 13. [IDEA #29] Ideas on Goldbach's Conjecture using $P^{6+}$ and $P^{6-}$

Golbach's conjecture says that every even integer greater than 2 can be expressed as the sum of two Primes. [12] From the definition of the two DNA-Prime sequences we know that any Prime can be expressed as:

$$p^{+} = 6k_n + 1 \qquad k_n \in N$$

$$p^{-} = 6k_m - 1 \qquad k_m \in N$$

The addition of two odd Prime numbers will always be even.

If N=2q is any even number, for it to be the addition of two Primes, the following needs to be true:

$$N = 2q = p_1 + p_2$$

To illustrate the problem, we can build a simple table for $N = 18$

| N | p1 | p2 |
|---|---|---|
| 18 | 1 | 17 |
| | 3 | 15 |
| | 5 | 13 |
| | 7 | 11 |
| | 9 | 9 |
| | 11 | 7 |
| | 13 | 5 |
| | 15 | 3 |
| | 17 | 1 |

We can observe that:

- there are N/2 combinations of two odd numbers that add up to N.
- there are 2 combinations involving 1 and 1 is not a Prime.
- the option $N/2 + N/2 = N$ does not involve addition of Primes and we can disregard it.

Of the remaining combinations, they repeat themselves due to the commutative property of the addition in N.

So, the net number of potential valid combinations of two odd numbers with one of them at least being Prime is:

$$(N/2 - 3)/2$$

If p is Prime, based on the Prime number theorem, we can see that for N>76 the number of combinations is larger than the number of Primes <N as:

$$(N/2 - 3)/2 \; > N/\ln N \quad \text{for } N>76$$

So, the number of Primes that meet Golbach's conjecture for any even number N are proportionally less than the number of combinations of odd numbers as N grows.

It is easily observable that any even number N belongs to one of the following sets:

$$\{N \;==\; 0\;(mod\;6)\}$$

$$\{(N\;+\;2)\;==\;0\;(mod\;6)\}$$

$$\{(N\;-\;2)\;==\;0\;(mod\;6)\}$$

Let's chart the number of potential representations of even integers as the sum of two Primes separating these three sets of even numbers which let's us propose a conjecture: In any combination of three consecutive even numbers >= 48, the one of the form N == 0 (mod 6) will have the largest number of decompositions into 2 Prime numbers. This sequence contains those local maxima for every set of three consecutive even numbers. This sequence forms the upper envelope of Goldbach's comet chart.

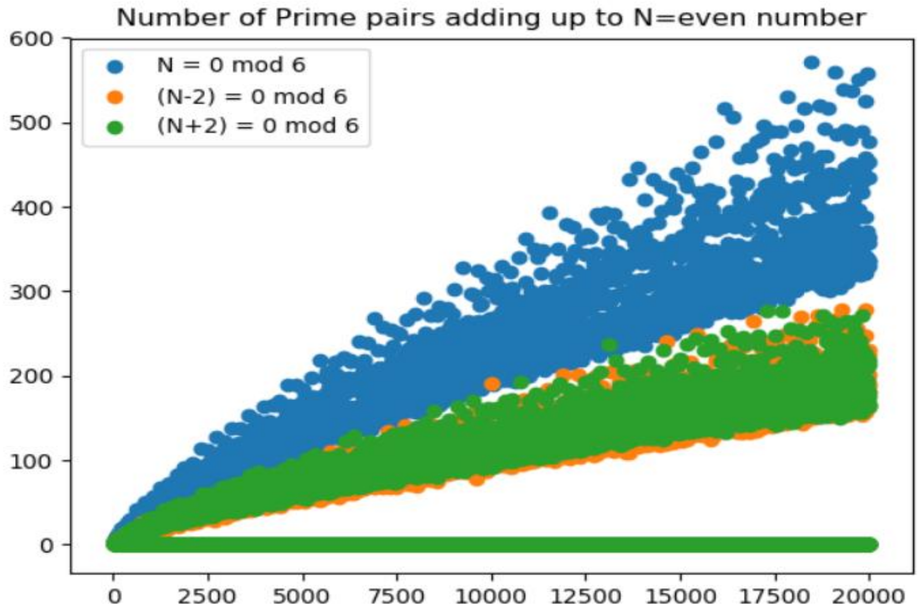


Fig 25

The first terms of this sequence (OEIS A322921 Caceres) are:

$$\{1, 1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 6, 7, 8, 9, 7, 8, 8, 10, 12, 10, 9, 8, 11, 12, 11, 10, 13 \dots\}$$

We know that if $p_1$ and $p_2$ are Primes, we can use the DNA-Prime series to say:

$$p_1 = 6 * k_1 \pm 1$$

$$p_2 = 6 * k_2 \pm 1$$

And there are three possibilities:

$$N = p_1 + p_2 = 6 * (k_1 + k_2) - 2$$

$$N = p_1 + p_2 = 6 * (k_1 + k_2)$$

$$N = p_1 + p_2 = 6 * (k_1 + k_2) + 2$$

Based on this, and assuming that $p_1$ and $p_2$ exist, we can affirm that:

| | | |
|---|---|---|
| If | N mod 6=0 | N is the addition of a $p_1 \in P^+$ and $p_2 \in P^-$ |
| If | (N-2) mod 6=0 | N is the addition of a $p_1 \in P^+$ and $p_2 \in P^+$ |
| If | (N+2) mod 6=0 | N is the addition of a $p_1 \in P^-$ and $p_2 \in P^-$ |

given that for any even number, there is a $q \in N$ such that N=2q and the previous expressions are equivalent to:

q mod 3 =0

or (q-1) mod 3 =0

or (q+1) mod 3 =0

Which is obviously true as for any 3 consecutive numbers (q-1), q, (q+1), one of them must necessarily be divisible by 3. The three possible combinations of Primes mentioned earlier can be also reformulated as follows:

**[IDEA #30]** If q mod 3 =0 $\frac{q}{3} = k_1 + k_2$ $p_1 = 6k_1 + 1$ and $p_2 = 6k_2 - 1$

If (q-1) mod 3 =0 $\frac{q-1}{3} = k_1 + k_2$ $p_1 = 6k_1 + 1$ and $p_2 = 6k_2 + 1$

If (q+1) mod 3 =0 $\frac{q+1}{3} = k_1 + k_2$ $p_1 = 6k_1 - 1$ and $p_2 = 6k_2 - 1$

As examples of these expressions:

| | | | | Potential Primes | | $k_1$ | $k_2$ |
|---|---|---|---|---|---|---|---|
| N=54 | q=27 | q=3*9 | q/3=9 | $2P^+$ | $7P^-$ | 13 | 41 |
| N=64 | q=32 | q=3*11-1 | (q+1)/3=11 | $3P^-$ | $8P^-$ | 17 | 47 |
| N=68 | q=34 | q=3*11+1 | (q-1)/3=11 | $5P^+$ | $6P^+$ | 31 | 37 |

To prove Golbach's conjecture we must prove that for any $n \in N$ we can find combinations of $k_n$ and $k_m$ in the DNA-Prime generator series such that:

$$q = k_n + k_m = R_n^1 + R_m^2 - 2$$

$$\text{or } q = k_{n1} + k_{n2} = R_n^1 + R_n^2 - 2$$

$$\text{or } q = k_{m1} + k_{m2} = R_m^1 + R_m^2 - 2$$

$$\text{where } R_m = k_m + 1 \text{ and } R_n = k_n + 1$$

In other words, that for any $q \in N$, we can find two elements of $R_n$, or two elements of $R_m$, or one element of $R_n$ and one element of $R_m$, that add up to q, as all even numbers are of the form: N=2q, with $q \in N$

To prove it, we are going to use an induction proof.

We will define a condition that is observable and met for a certain k=k*, we will assume that the condition is met at k=n-1 and then we will prove that this means the condition is also true at k=n for any element n of the generator series.

Let's observe that in the following chart of $(Rm + Rm)$, the square $Rm * Rm$ contains at least all naturals up to $Rm$.

| Rm x Rm table | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 11 | 13 | 14 | 16 | 17 | 18 |
| 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 | 14 | 15 | 17 | 18 | 19 |
| 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 | 13 | 15 | 16 | 18 | 19 | 20 |
| 3 | 4 | 5 | 6 | 7 | 9 | 10 | 11 | 12 | 14 | 16 | 17 | 19 | 20 | 21 |
| 4 | 5 | 6 | 7 | 8 | 10 | 11 | 12 | 13 | 15 | 17 | 18 | 20 | 21 | 22 |
| 6 | 7 | 8 | 9 | 10 | 12 | 13 | 14 | 15 | 17 | 19 | 20 | 22 | 23 | 24 |
| 7 | 8 | 9 | 10 | 11 | 13 | 14 | 15 | 16 | 18 | 20 | 21 | 23 | 24 | 25 |
| 8 | 9 | 10 | 11 | 12 | 14 | 15 | 16 | 17 | 19 | 21 | 22 | 24 | 25 | 26 |
| 9 | 10 | 11 | 12 | 13 | 15 | 16 | 17 | 18 | 20 | 22 | 23 | 25 | 26 | 27 |
| 11 | 12 | 13 | 14 | 15 | 17 | 18 | 19 | 20 | 22 | 24 | 25 | 27 | 28 | 29 |
| 13 | 14 | 15 | 16 | 17 | 19 | 20 | 21 | 22 | 24 | 26 | 27 | 29 | 30 | 31 |
| 14 | 15 | 16 | 17 | 18 | 20 | 21 | 22 | 23 | 25 | 27 | 28 | 30 | 31 | 32 |
| 16 | 17 | 18 | 19 | 20 | 22 | 23 | 24 | 25 | 27 | 29 | 30 | 32 | 33 | 34 |
| 17 | 18 | 19 | 20 | 21 | 23 | 24 | 25 | 26 | 28 | 30 | 31 | 33 | 34 | 35 |
| 18 | 19 | 20 | 21 | 22 | 24 | 25 | 26 | 27 | 29 | 31 | 32 | 34 | 35 | 36 |

Table 27

For example, the square for $(R_{11}\ x\ R_{11})$ of dimension (11x11) contains up to the natural number up to 11, i.e. contains [1],2,3,4,5,6,7,8,9,10,11 We could use any other cardinal to observe that this true.

Let's assume now that the condition is true for $R_{m-1}$, which means that the square $R_{m-1}$ x $R_{m-1}$ contains all naturals up to $R_{m-1}$ generated by the addition of two given $R_m^1$ and $R_m^2$ both < $R_{m-1}$ and let's prove that the condition is met for the square $R_m$x $R_m$, which means that the square $R_m$x $R_m$ must contain all naturals up to $R_m$.

The set of natural numbers between $R_{m-1}$ and $R_m$ are, by definition of the matrix

$R_m$x $R_m$:

$D_n$ = { $R_m$- $R_{m-1}$}= {$R_{m-1}$+1, $R_{m-1}$+2, $R_{m-1}$+3 …, $R_{m-1}$+($R_m$- $R_{m-1}$)}

We assumed that all naturals up to $R_{m-1}$ exist and for each even number $n < R_{m-1}$ there are two $R_{m-1}^j$ and $R_{m-1}^k$ such that n=$R_{m-1}^j$+$R_{m-1}^k$

We can express this as:

$$\{R_{m-1}^j + R_{m-1}^k\} = \{2,3,4,5, \dots R_{m-1}\}$$

with $R_{m-1}^j = \{1,2,3,4,5,6,8,9, \dots R_{m-1}\}$

If we add $R_{m-1}$ to any of the two $R_{m-1}^j + R_{m-1}^k$ we can say that:

$$\{R_{m-1}+R_{m-1}^j\}=\{R_{m-1}+1, R_{m-1}+2 \dots, R_{m-1}+R_{m-1}\}$$

$\{2 * R_{m-1}\}$ is the last diagonal term of the defined and known matrix $R_{m-1}xR_{m-1}$ which is contained in $R_mxR_m$

We can use Bertrand's postulate [13] to affirm that $R_m$- $R_{m-1}$< $2 * R_{m-1}$ therefore $D_n$ is contained in the matrix $R_mxR_m$. [QED ]

Same proof works for ($R_n$+$R_n$) and ($R_n$+$R_m$), which proves Golbach's conjecture.

As an example, to find the Primes that add up to 180:

N=180 N mod 6=0 so one Prime belongs to $P^+$ and the other to $P^-$

and the addition of the two Prime generators is kn + km = 30. That give all these potential combinations:

| Kn | Km | P+ | P- | P+ + P- |
|---|---|---|---|---|
| 2 | 28 | 13 | 167 | 180 |
| 5 | 25 | 31 | 149 | 180 |
| 7 | 23 | 43 | 137 | 180 |
| 11 | 19 | 67 | 113 | 180 |
| 12 | 18 | 73 | 107 | 180 |
| 13 | 17 | 79 | 101 | 180 |
| 16 | 14 | 97 | 83 | 180 |
| 18 | 12 | 109 | 71 | 180 |
| 21 | 9 | 127 | 53 | 180 |
| 23 | 7 | 139 | 41 | 180 |
| 25 | 5 | 151 | 29 | 180 |
| 26 | 4 | 157 | 23 | 180 |
| 27 | 3 | 163 | 17 | 180 |

Table 28

## 14. Conclusion

The sequence of Prime numbers can be defined using the fact that Primes are one away from a multiple of $a = 1, 2, 3, 4, \text{and } 6$. In general, we can say that a Prime $p = a * k(a) \pm 1$ for some $k(a) \neq a * x * y \pm x \pm y$.

In the case of a=1:

**{Primes} =** $\{\boldsymbol{k_{1n} + 1 \mid k_{1n} \neq xy + x + y \quad for\ x, y \in N}\}$

$k_{1n} = \{1, 2, 4, 6, 10, 12, 16, 18, 22, 28, 30, 36, 40, 42, 46, 52, 58, 60, 66, 70, 72 \dots\}$

In the case of a=2:

**{Primes} = {2}** $\cup \{\boldsymbol{2k_{2n} + 1 \mid k_{2n} \neq xy + x + y \quad for\ x, y \in N}\}$

$k_{2n} = \{1, 2, 3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 21, 23, 26, 29, 30, 33, 35, 36, 39, 41, \dots\}$

In case of a=3:

**{Primes} = {3}** $\cup \{\boldsymbol{3k_{3n} + 1 \mid k_{3n} \neq 3xy + x + y\ and\ k_{3n} \neq 3xy - x - y\ for\ x, y \in N}\}$
$\cup \{\boldsymbol{3k_{3m} - 1 \mid k_{3m} \neq 3xy - x + y\ for\ x, y \in N}\}$

$k_{3n} = \{2, 4, 6, 10, 12, 14, 20, 22, 24, 26, 32, 34, 36, 42, 46, 50, 52, 54, 60, \dots\}$
$k_{3m} = \{1, 2, 4, 6, 8, 10, 14, 16, 18, 20, 24, 28, 30, 34, 36, 38, 44, 46, 50, 56, 58, \dots\}$

In case of a=4:

**{Primes} = {2}** $\cup \{\boldsymbol{4k_{4n} + 1 \mid k_{4n} \neq 4xy + x + y\ and\ k_{4n} \neq 4xy - x - y\ for\ x, y \in N}\}$
$\cup \{\boldsymbol{4k_{4m} - 1 \mid k_{4m} \neq 4xy - x + y\ for\ x, y \in N}\}$

$k_{4n} = \{1, 3, 4, 7, 9, 10, 13, 15, 18, 22, 24, 25, 27, 28, 34, 37, 39, 43, 45, 48, 49, \dots\}$
$k_{4m} = \{1, 2, 3, 5, 6, 8, 11, 12, 15, 17, 18, 20, 21, 26, 27, 32, 33, 35, 38, 41, 42, \dots\}$

In case of a=6:

**{Primes} = {2,3}** $\cup \{\boldsymbol{6k_{6n} + 1 \mid k_{6n} \neq 6xy + x + y\ and\ k_{6n} \neq 6xy - x - y\ for\ x, y \in N}\}$
$\cup \{\boldsymbol{6k_{6m} - 1 \mid k_{6m} \neq 6xy - x + y\ for\ x, y \in N}\}$

$k_{6n} = \{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 16, 17, 18, 21, 23, 25, 26, 27, 30, 32, 33, \dots\}$
$k_{6m} = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 14, 15, 17, 18, 19, 22, 23, 25, 28, 29, 30, 32, \dots\}$

These expressions have been used to formulate factorization methods, primality tests, methods to count Primes less than a number (CPIX), and different ideas about some open conjectures regarding Prime numbers. In the process, several conjectures are proposed as well as new numerical and computational methods.

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