Newton-Julia set for some polynomials related with number pi

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Abstract

Newton-Julia fractals and pi formulas

1. Sea $k,m \in \mathbb{N}$ y $p(z,k,m)$ el polinomio definido como sigue:

$$p(z,k,m) = z^{m+k} + z^m + z^k - 1 \quad , z \in \mathbb{C} \tag{1}$$

Algunos ejemplos particulares son:

$$p(z,1,2) = z^3 + z^2 + z - 1 \tag{2}$$

$$p(z,2,2) = z^4 + 2z^2 - 1 \tag{3}$$

$$p(z,1,3) = z^4 + z^3 + z - 1 \tag{4}$$

$$p(z,1,4) = z^5 + z^4 + z - 1 \tag{5}$$

$$p(z,2,3) = z^5 + z^3 + z^2 - 1 \tag{6}$$

$$p(z,1,5) = z^6 + z^5 + z - 1 \tag{7}$$

$$p(z,3,3) = z^6 + 2z^3 - 1 \tag{8}$$

$$p(z,3,4) = z^7 + z^4 + z^3 - 1 \tag{9}$$

$$p(z,5,7) = z^{12} + z^7 + z^5 - 1 \tag{10}$$

$$p(z,7,8) = z^{15} + z^8 + z^7 - 1 \tag{11}$$

$$p(z,11,20) = z^{31} + z^{20} + z^{11} - 1 \tag{12}$$

Para cada polinomio $p(z,k,m)$ definido por (1) , existe un número $\alpha(k,m)$ tal que:

$$0 < \alpha(k,m) < 1 \tag{13}$$

$$\pi = 4\tan^{-1}\left(\alpha\left(k,m\right)^{k}\right) + 4\tan^{-1}\left(\alpha\left(k,m\right)^{m}\right) \tag{14}$$

La ecuación (14) se puede escribir como:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2n+1}\left(\alpha^{2kn+k} + \alpha^{2mn+m}\right) \tag{15}$$

donde $\alpha \equiv \alpha\left(k,m\right)$.

la ecuación (15) se puede escribir como:

$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{n+k}\alpha^{n+k} \tag{16}$$

donde

$$c_0 = k \tag{17}$$

$$c_{-j} = 0 \quad , j \geq 1 \tag{18}$$

$$c_n = a_n - \left(c_{n-2k} + c_{n-2m} + c_{n-2m-2k}\right) \quad , n \in \mathbb{N} \tag{19}$$

$$a_0 = k\,, a_{2m} = k\,, a_{m-k} = m\,, a_{m+k} = m \tag{20}$$

$$a_j = 0\,,\ j \neq 0, 2m, m-k, m+k \tag{21}$$

Algunas fórmulas particulares son:

Para el polinomio $p(z,1,3) = z^4 + z^3 + z - 1$, se tiene:

$$z^4 + z^3 + z - 1 = 0 \Rightarrow \begin{cases} z_1 = \dfrac{\sqrt{5}-1}{2} \\ z_2 = -\dfrac{\sqrt{5}+1}{2} \\ z_3 = i = \sqrt{-1} \\ z_4 = -i \end{cases} \tag{22}$$

$$0 < z_1 < 1 \Rightarrow \alpha\left(1,3\right) = z_1 \tag{23}$$

$$\pi = 4\tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right) + 4\tan^{-1}\left(\left(\frac{\sqrt{5}-1}{2}\right)^{3}\right) \tag{24}$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2n+1}\left(z_1^{2n+1} + z_1^{6n+3}\right) \tag{25}$$

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n z_1^{6n+1}\left(\frac{1}{6n+1}+\frac{2z_1^2}{6n+3}+\frac{z_1^4}{6n+5}\right) \quad (26)$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{n+1} z_1^{n+1} \quad (27)$$

donde

$$c_{n+8} = -\left(c_{n+6}+c_{n+2}+c_n\right) \quad ,n = 0,1,2,3,... \quad (28)$$

$$c_0 = 1, c_1 = 0, c_2 = 2, c_3 = 0, c_4 = 1, c_5 = 0, c_6 = -1, c_7 = 0 \quad (29)$$

$$\frac{3}{4}\pi = -\tan^{-1}\left(z_2\right) - \tan^{-1}\left(z_2^3\right) \quad (30)$$

Para el polinomio $p(z,3,4) = z^7 + z^4 + z^3 - 1$, se tiene:

$$z^7 + z^4 + z^3 - 1 = 0 \Rightarrow \begin{cases} z_1 = 0.77612... \\ z_2 = 0.77576... + i\ 0.99466... \\ z_3 = 0.77576... - i\ 0.99466... \\ z_4 = -0.20139... + i\ 0.82146... \\ z_5 = -0.20139... - i\ 0.82146... \\ z_6 = -0.96243... + i\ 0.45351... \\ z_7 = -0.96243... - i\ 0.45351... \end{cases} \quad (31)$$

$$0 < z_1 < 1 \Rightarrow \alpha(3,4) = z_1 \quad (32)$$

$$\pi = 4\tan^{-1}\left(z_1^3\right) + 4\tan^{-1}\left(z_1^4\right) \quad (33)$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(z_1^{6n+3} + z_1^{8n+4}\right) \quad (34)$$

$$\pi = 4\sum_{n=0}^{\infty} z_1^{24n+3}\left(\frac{1}{8n+1}+\frac{(-1)^n z_1}{6n+1}-\frac{z_1^6}{8n+3}-\frac{(-1)^n z_1^9}{6n+3}+\frac{z_1^{12}}{8n+5}+\frac{(-1)^n z_1^{17}}{6n+5}-\frac{z_1^{18}}{8n+7}\right) \quad (35)$$

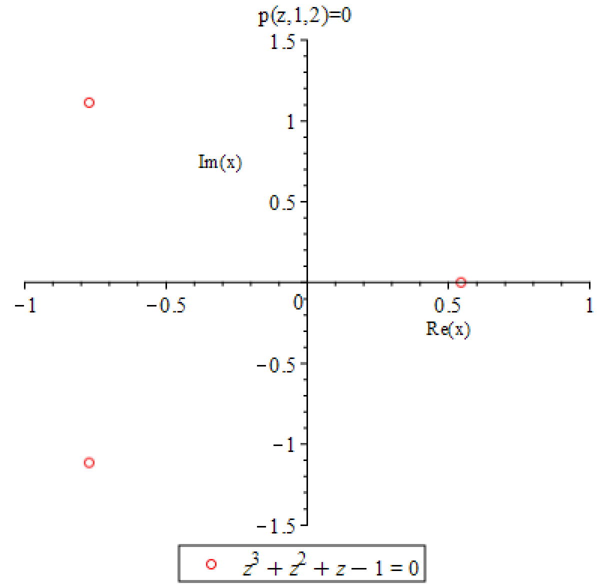
$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{n+3} z_1^{n+3} \quad (36)$$

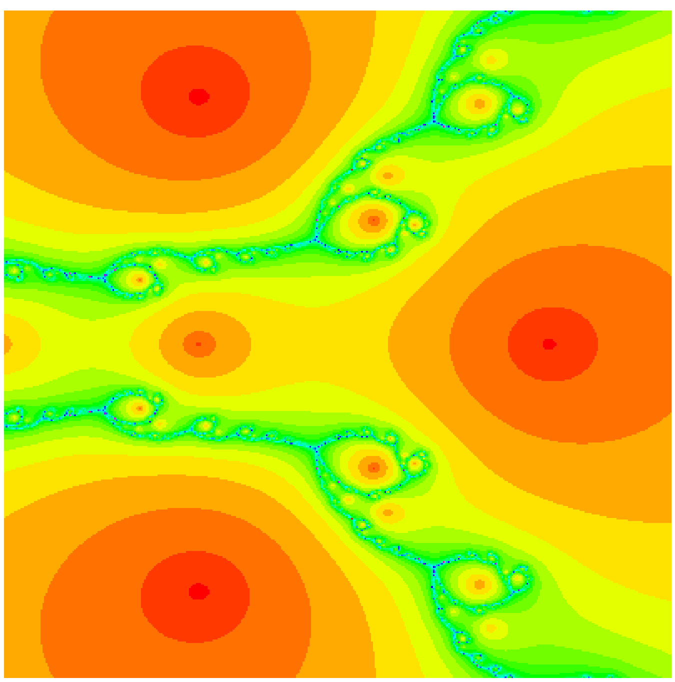
donde

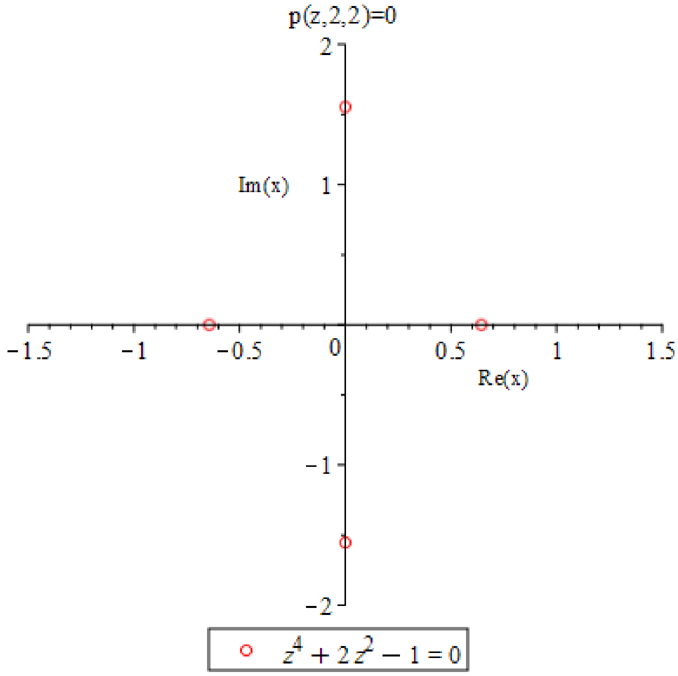
$$c_{n+14} = -\left(c_{n+8}+c_{n+6}+c_n\right) \quad ,n = 0,1,2,3,... \quad (37)$$

$$\begin{cases} c_0 = 3, c_1 = 4, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0, c_6 = -3, c_7 = 0 \\ c_8 = 0, c_9 = -4, c_{10} = 0, c_{11} = 0, c_{12} = 3, c_{13} = 0 \end{cases} \quad (38)$$

2. Gráficos de ceros y conjuntos de Newton-Julia para $p(z,k,m)$.



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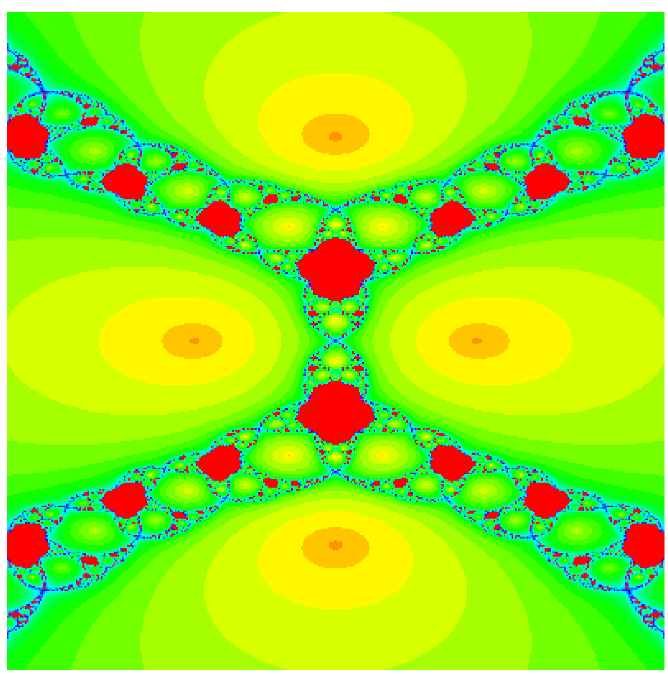
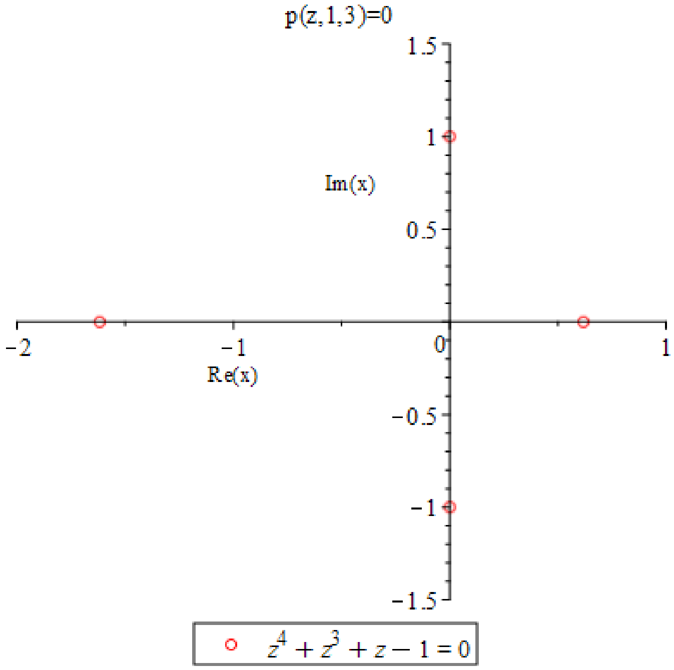
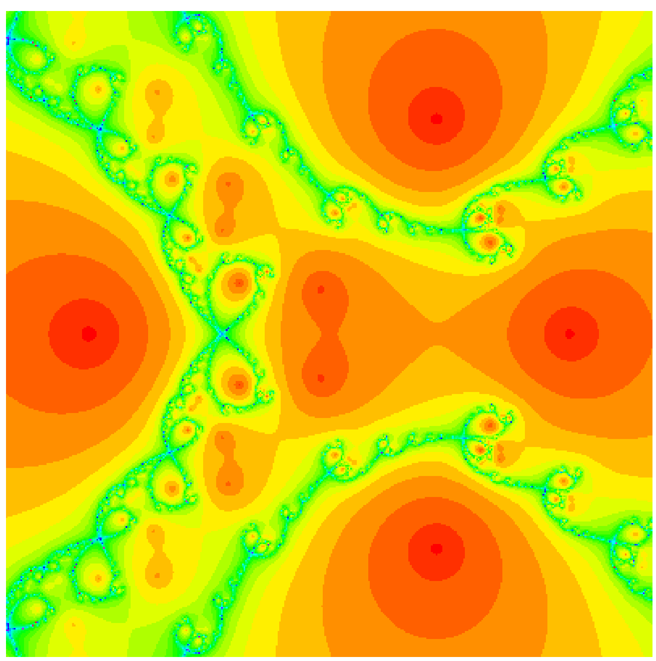
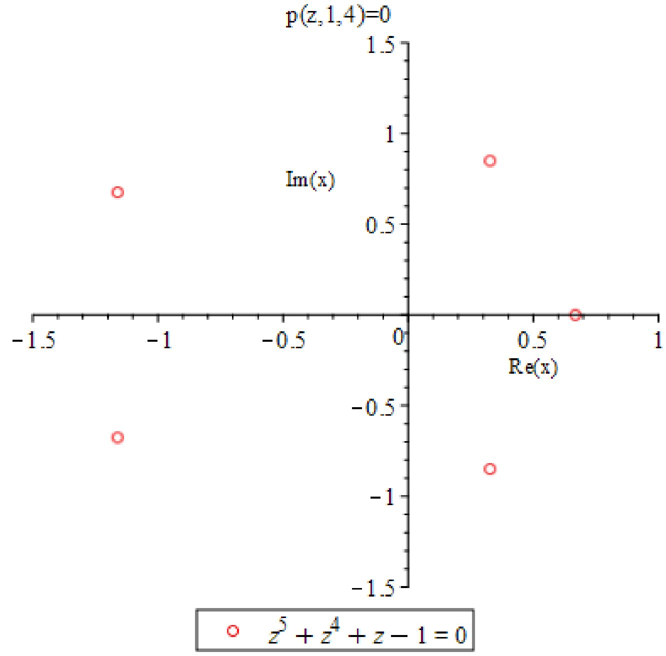
Im(x)
Re(x)
$z^4 + 2z^2 - 1 = 0$




p(z,1,3)=0
1.5
1
Im(x)
0.5
−2
−1
0
1
Re(x)
−0.5
−1
−1.5
$z^4 + z^3 + z - 1 = 0$

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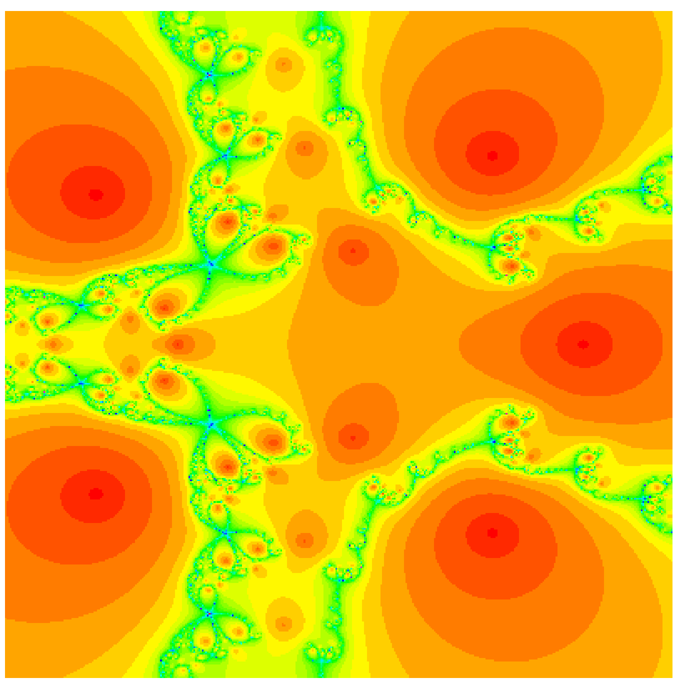
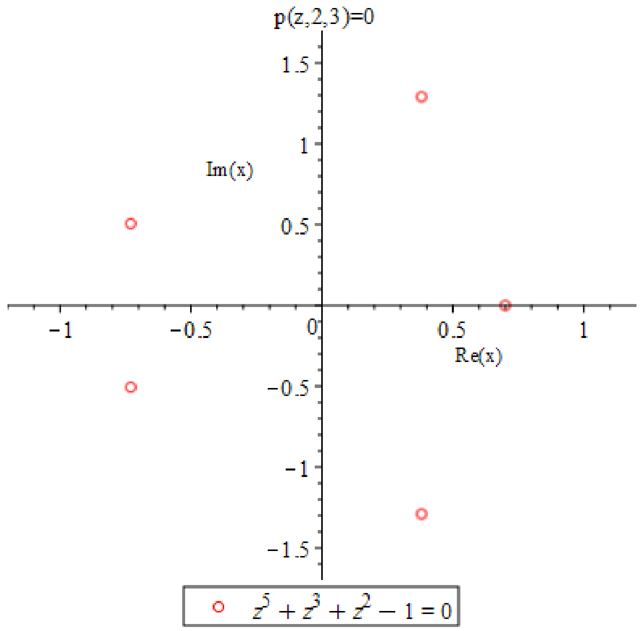
1.5
1
Im(x)
0.5
−1.5
−1
−0.5
0
0.5
1
Re(x)
−0.5
−1
−1.5
$z^5 + z^4 + z - 1 = 0$

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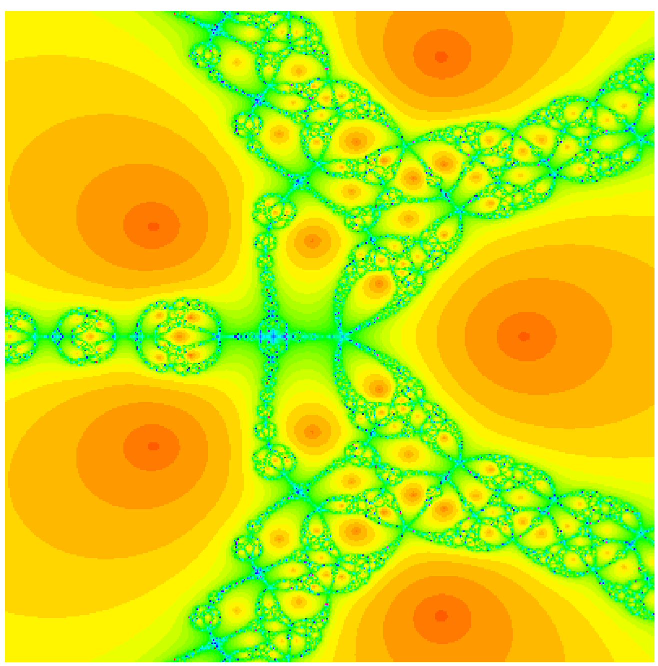
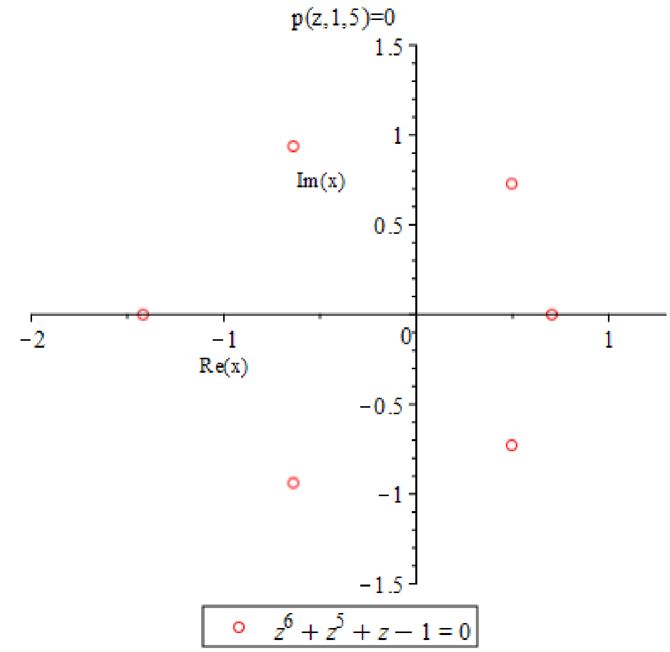
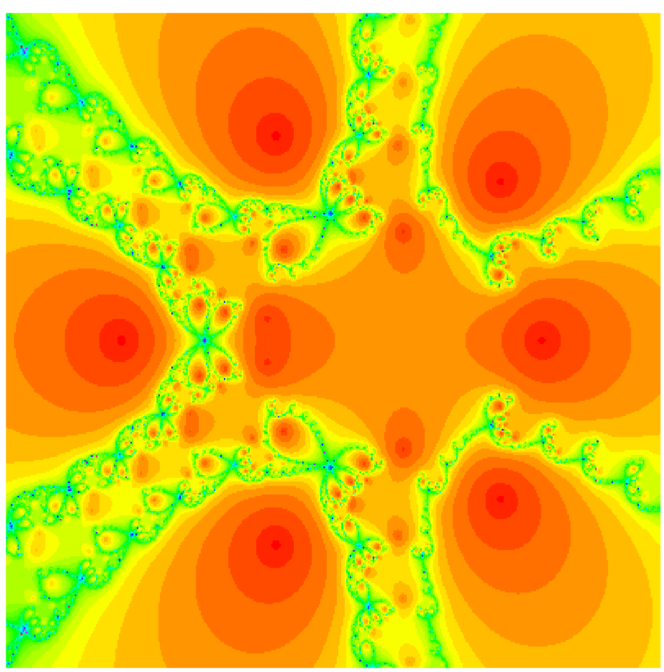
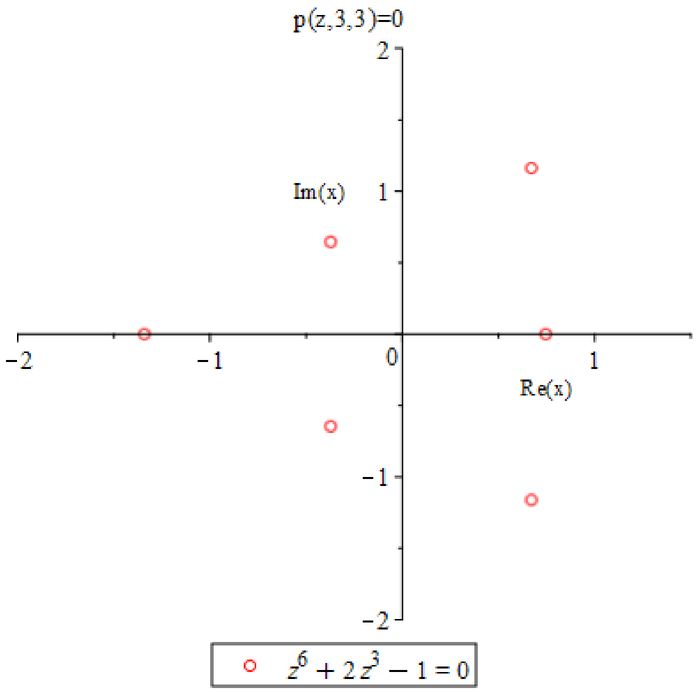
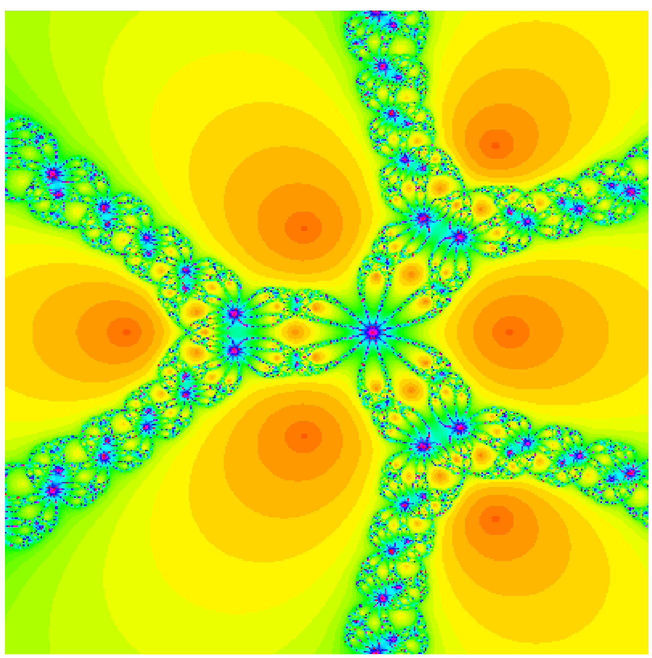
1.5
1
0.5
−0.5
−1
−1.5
Im(x)
−1
−0.5
0
0.5
1
Re(x)
$z^5 + z^3 + z^2 - 1 = 0$

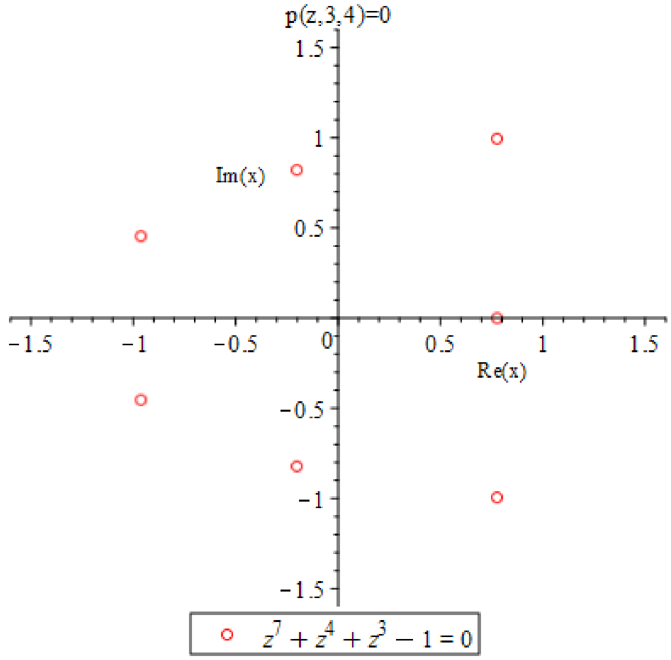


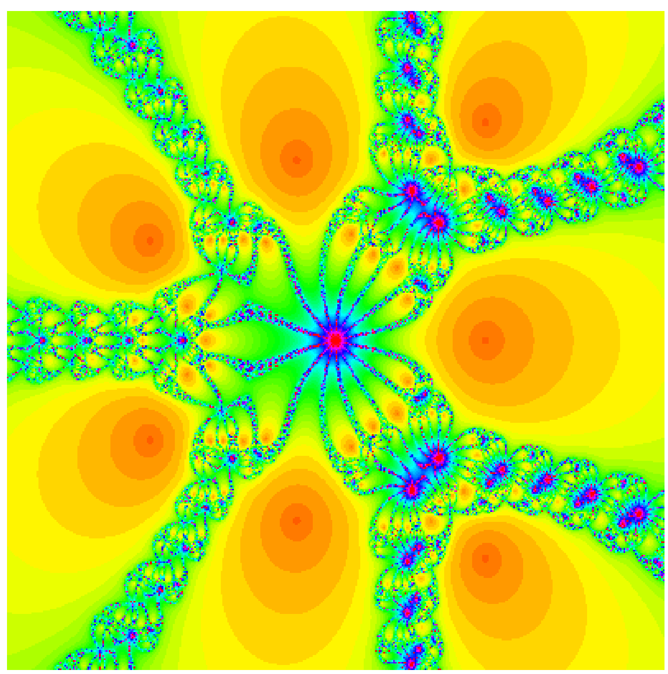

p(z,1,5)=0
1.5
1
Im(x)
0.5
−2
−1
0
1
Re(x)
−0.5
−1
−1.5
$z^6 + z^5 + z - 1 = 0$

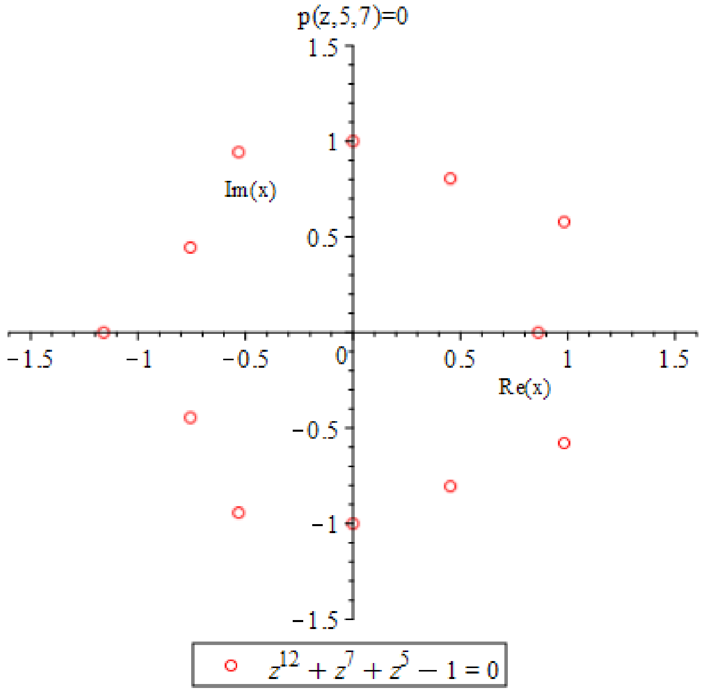


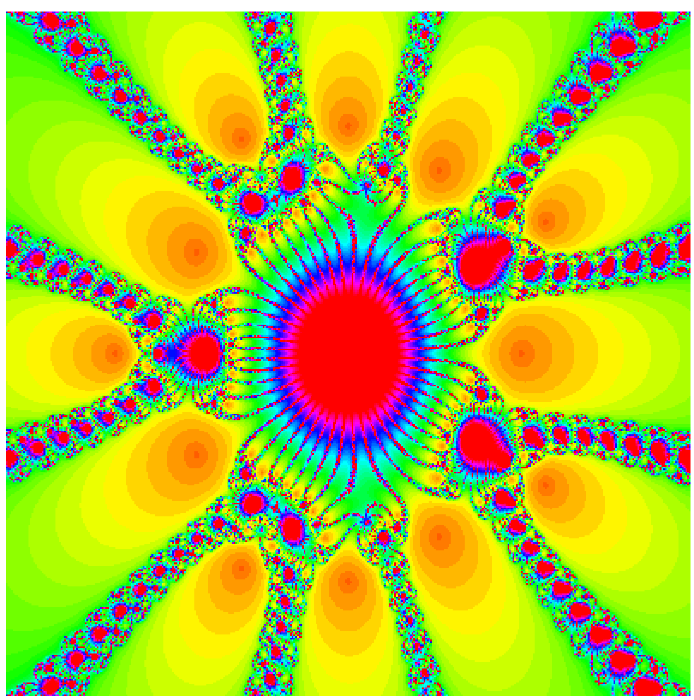

p(z,3,3)=0
Im(x)
Re(x)
2
1
0
−1
−2
−2
−1
1
$z^6 + 2z^3 - 1 = 0$

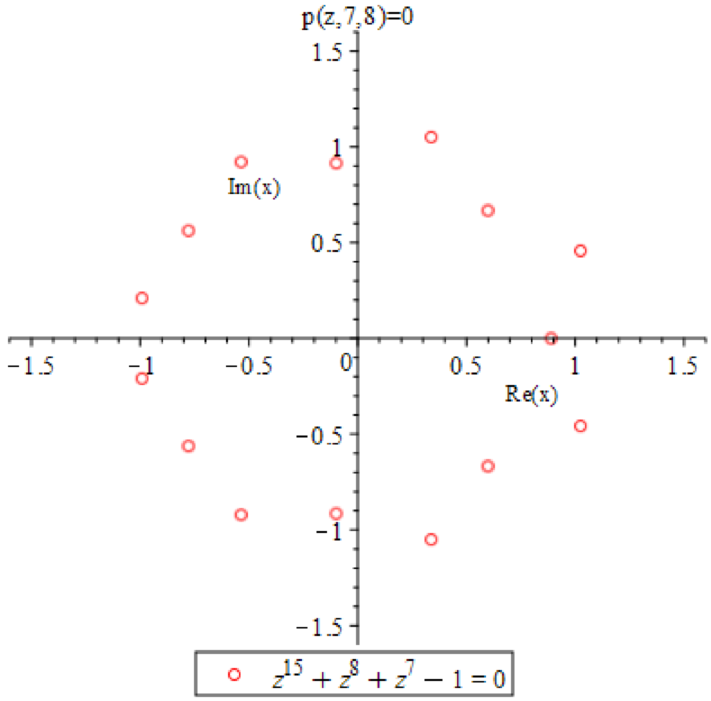



p(z,3,4)=0
1.5
1
Im(x)
0.5
−1.5
−1
−0.5
0
0.5
1
1.5
Re(x)
−0.5
−1
−1.5
$z^7 + z^4 + z^3 - 1 = 0$

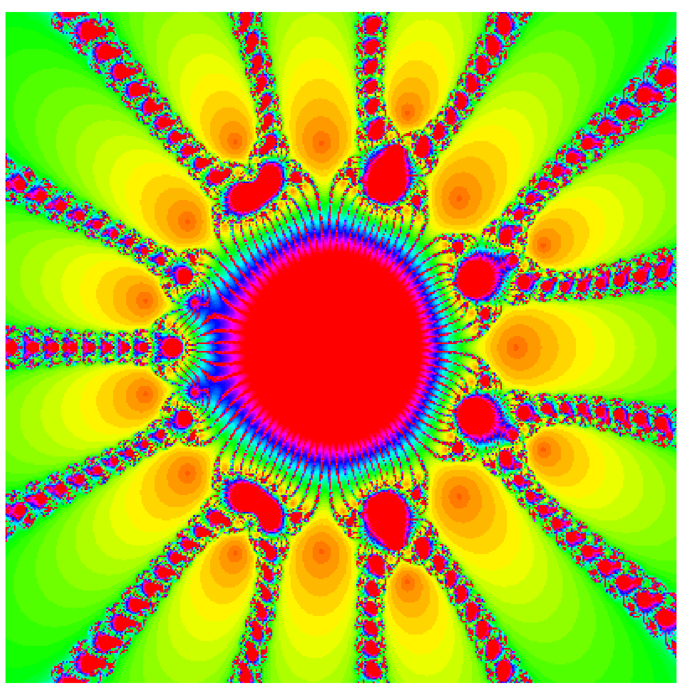


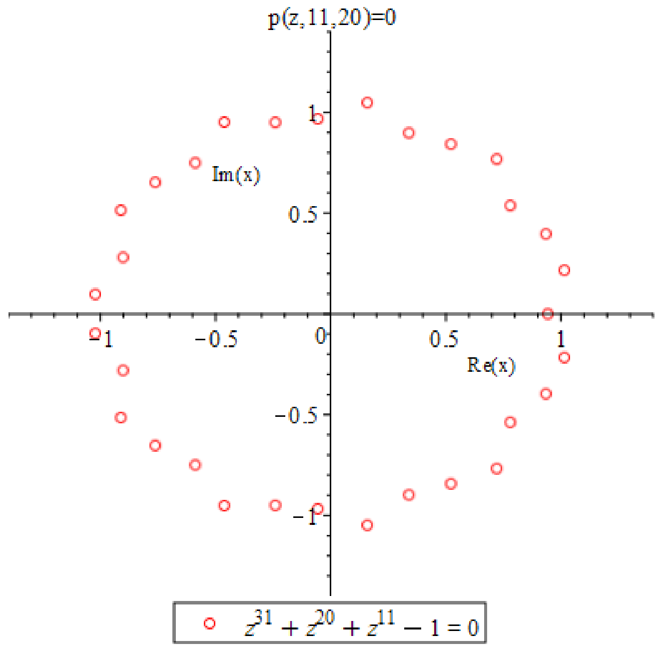

p(z,5,7)=0
1.5
1
0.5
−0.5
−1
−1.5
−1.5
−1
−0.5
0
0.5
1
1.5
Im(x)
Re(x)
$z^{12} + z^7 + z^5 - 1 = 0$

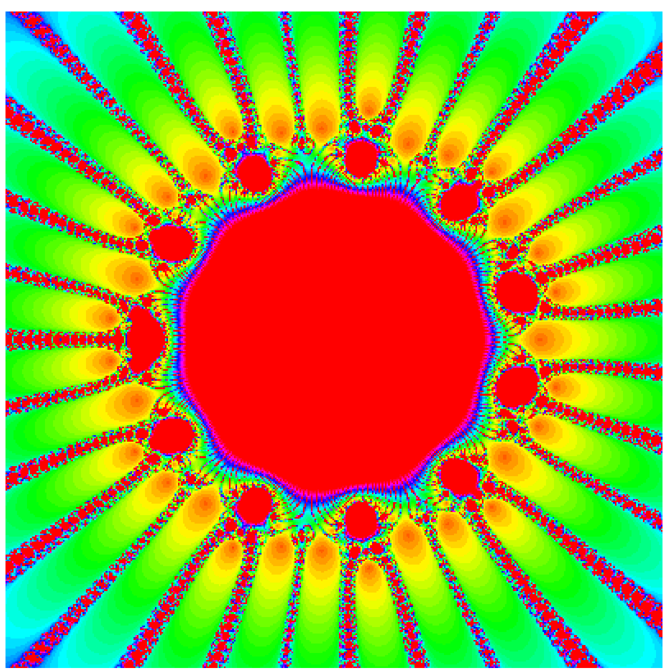




p(z,7,8)=0
1.5
1
Im(x)
0.5
−1.5
−1
−0.5
0
0.5
1
1.5
Re(x)
−0.5
−1
−1.5
$z^{15} + z^8 + z^7 - 1 = 0$




p(z,11,20)=0
Im(x)
Re(x)
1
0.5
−0.5
−1
−1
−0.5
0
0.5
1
$z^{31} + z^{20} + z^{11} - 1 = 0$



3. Sea $k, m \in \mathbb{N}$ y $q(z,k,m)$ el polinomio definido como sigue:

$$q(z,k,m) = z^{2k+2m} - 8z^{k+m} - 3z^{2m} - 3z^{2k} + 1 \ , z \in \mathbb{C} \tag{39}$$

Algunos ejemplos particulares son:

$$q(z,1,1) = z^4 - 14z^2 + 1 \tag{40}$$

$$q(z,1,2) = z^6 - 3z^4 - 8z^3 - 3z^2 + 1 \tag{41}$$

$$q(z,3,4) = z^{14} - 3z^8 - 8z^7 - 3z^6 + 1 \tag{42}$$

$$q(z,6,7) = z^{26} - 3z^{14} - 8z^{13} - 3z^{12} + 1 \tag{43}$$

Para cada polinomio $q(z,k,m)$ existe un número $\beta(k,m)$ tal que:

$$0 < \beta(k,m) < 1 \tag{44}$$

$$\pi = 6\tan^{-1}\left(\beta(k,m)^k\right) + 6\tan^{-1}\left(\beta(k,m)^m\right) \tag{45}$$

La fórmula (45) se puede escribir como:

$$\pi = 6\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\beta^{2kn+k} + \beta^{2mn+m}\right) \tag{46}$$

donde $\beta \equiv \beta(k,m)$.

la fórmula (46) se puede escribir como:

$$\pi = 6\sum_{n=0}^{\infty} \frac{c_n}{n+k}\beta^{n+k} \tag{47}$$

donde los coeficientes c_n se obtienen por las fórmulas (17)-(21).

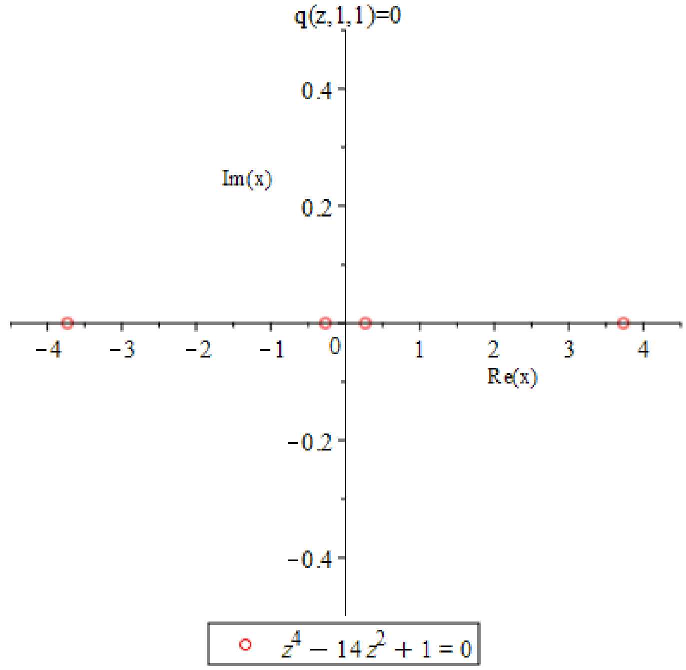
Un ejemplo particular es:

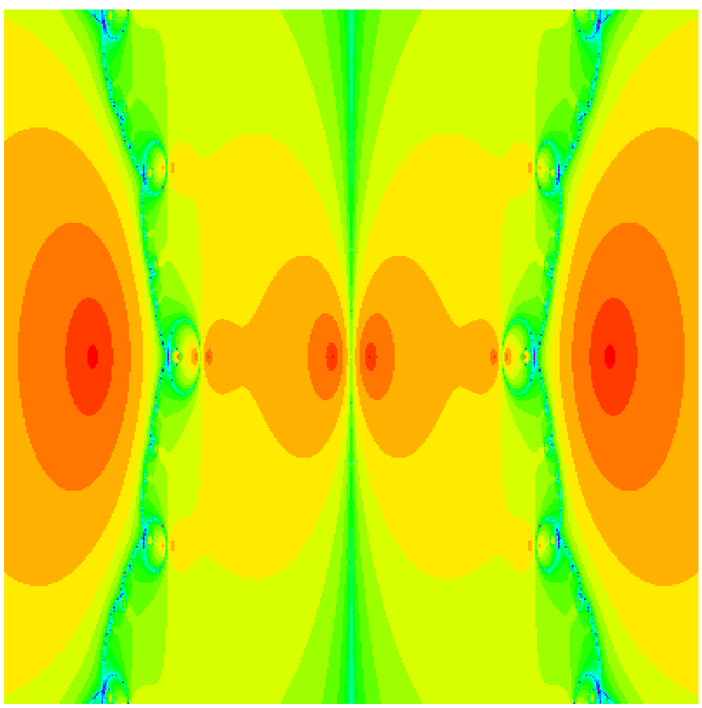
$$q(z,1,2) = z^6 - 3z^4 - 8z^3 - 3z^2 + 1 = 0 \Rightarrow \begin{cases} z_1 = 0.39049036... \\ z_2 = 2.56088267... \\ z_{3,4} = -0.4144... \pm i\,0.4677... \\ z_{5,6} = -1.0612... \pm i\,1.1977... \end{cases} \tag{48}$$

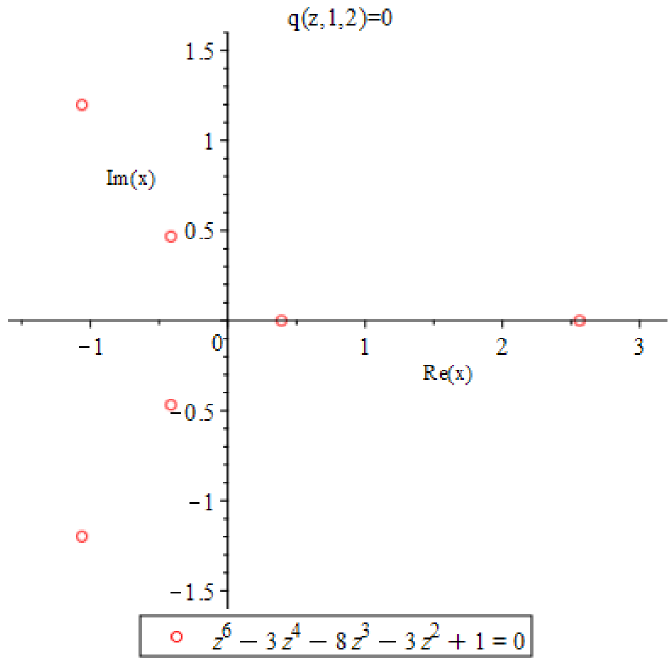
Para $\beta = z_1$ se tiene:

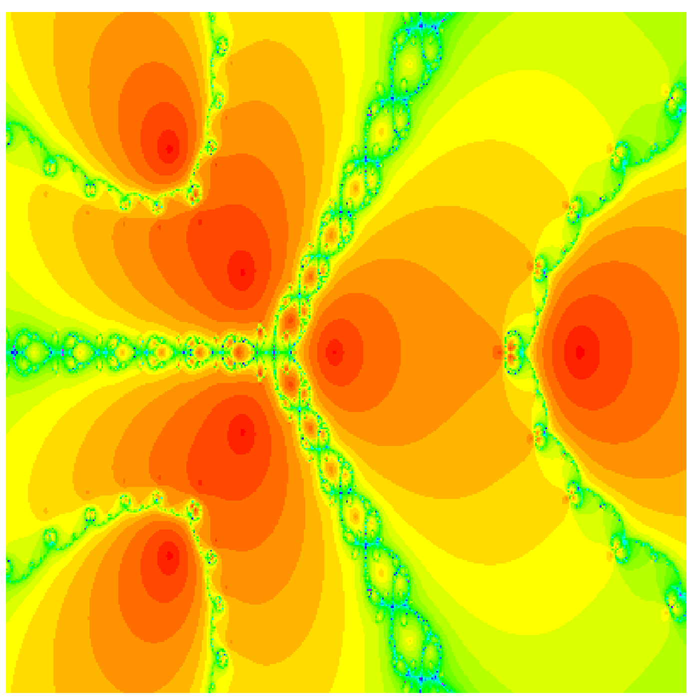
$$\pi = 6\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\beta^{2n+1} + \beta^{4n+2}\right) \tag{49}$$

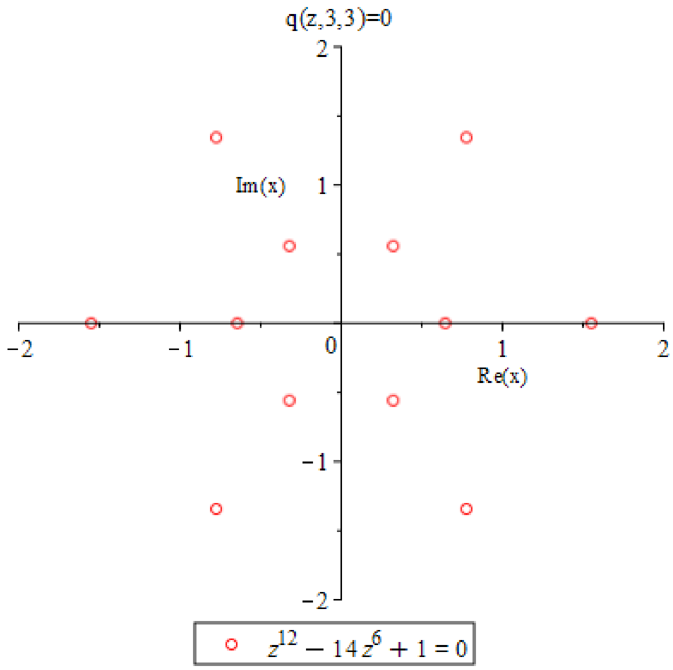
4. Gráficos de ceros y conjuntos de Newton-Julia para $q(z,k,m)$



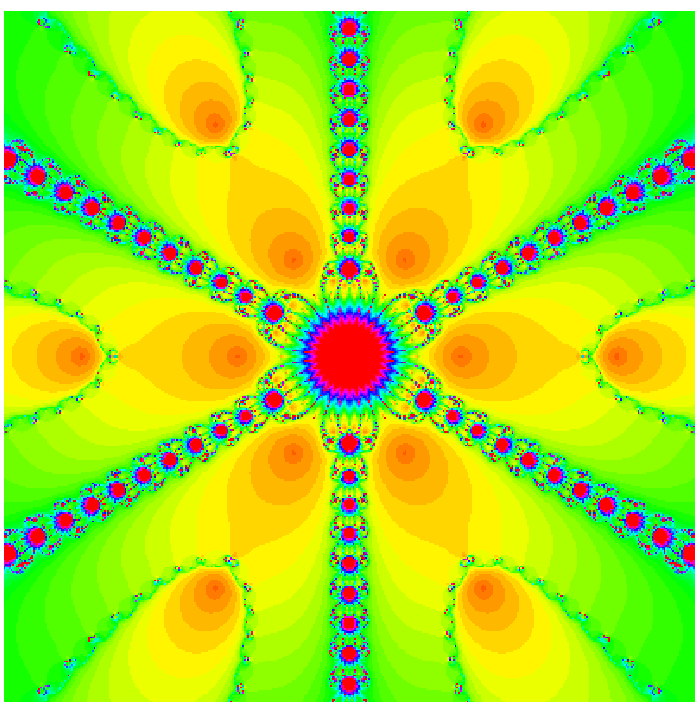


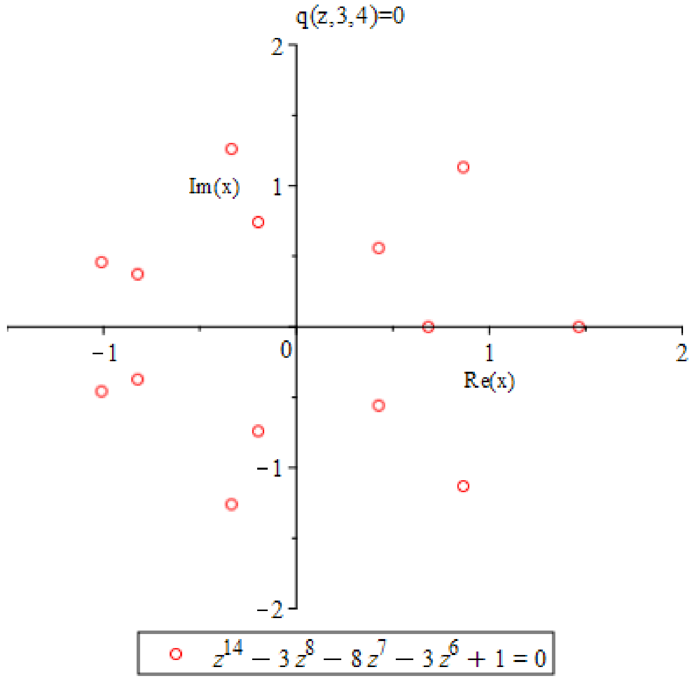

q(z,1,2)=0
1.5
1
Im(x)
0.5
−1
0
1
2
3
Re(x)
−0.5
−1
−1.5
$z^6 - 3z^4 - 8z^3 - 3z^2 + 1 = 0$

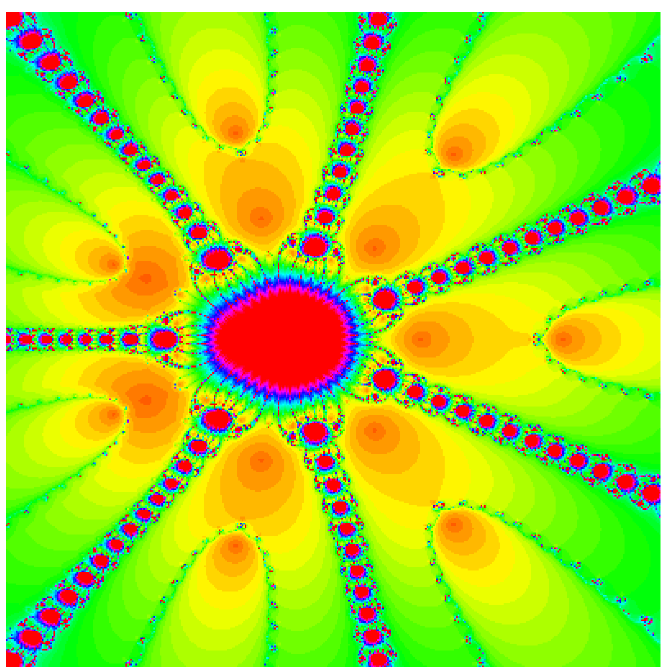


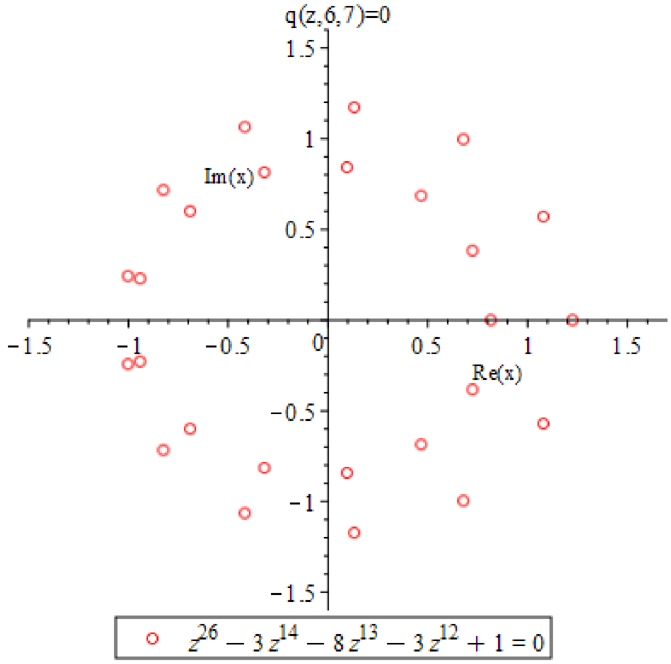


q(z,3,3)=0
Im(x)
Re(x)
2
1
−1
−2
0
$z^{12} - 14z^6 + 1 = 0$

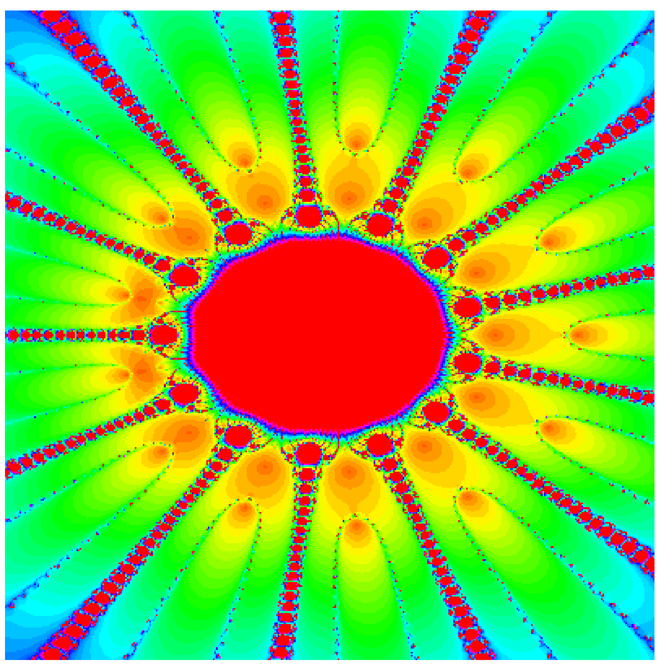



q(z,3,4)=0
2
1
Im(x)
−1
0
1
2
Re(x)
−1
−2
$z^{14} - 3z^8 - 8z^7 - 3z^6 + 1 = 0$





q(z,6,7)=0
Im(x)
Re(x)
1.5
1
0.5
0
−0.5
−1
−1.5
−1.5
−1
−0.5
0.5
1
1.5
$z^{26} - 3z^{14} - 8z^{13} - 3z^{12} + 1 = 0$



5. Sea $k,m\in\mathbb{N}$ y $r(z,k,m)$ el polinomio definido como sigue:

$$r(z,k,m)=z^{2k+2m}+2z^{k+2m}+2z^{2k+m}-4z^{k+m}-z^{2k}-z^{2m}-2z^{k}-2z^{m}+1 \tag{50}$$

$$z\in\mathbb{C}$$

Algunos ejemplos particulares son:

$$r(z,1,1)=z^4+4z^3-6z^2-4z+1 \tag{51}$$

$$r(z,1,2)=z^6+2z^5+z^4-4z^3-3z^2-2z+1 \tag{52}$$

$$r(z,2,3)=z^{10}+2z^8+2z^7-z^6-4z^5-z^4-2z^3-2z^2+1 \tag{53}$$

Para cada polinomio $r(z,k,m)$ definido por (47) , existe un número $\gamma\left(k,m\right)$ tal que:

$$0<\gamma\left(k,m\right)<1 \tag{54}$$

$$\pi=8\tan^{-1}\left(\gamma\left(k,m\right)^{k}\right)+8\tan^{-1}\left(\gamma\left(k,m\right)^{m}\right) \tag{55}$$

La fórmula (52) se puede escribir como:

$$\pi=8\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{2n+1}\left(\gamma^{2kn+k}+\gamma^{2mn+m}\right) \tag{56}$$

donde $\gamma\equiv\gamma\left(k,m\right)$.

la fórmula (56) se puede escribir como:

$$\pi=8\sum_{n=0}^{\infty}\frac{c_n}{n+k}\gamma^{n+k} \tag{57}$$

donde los coeficientes c_n se obtienen por las fórmulas (17)-(21).

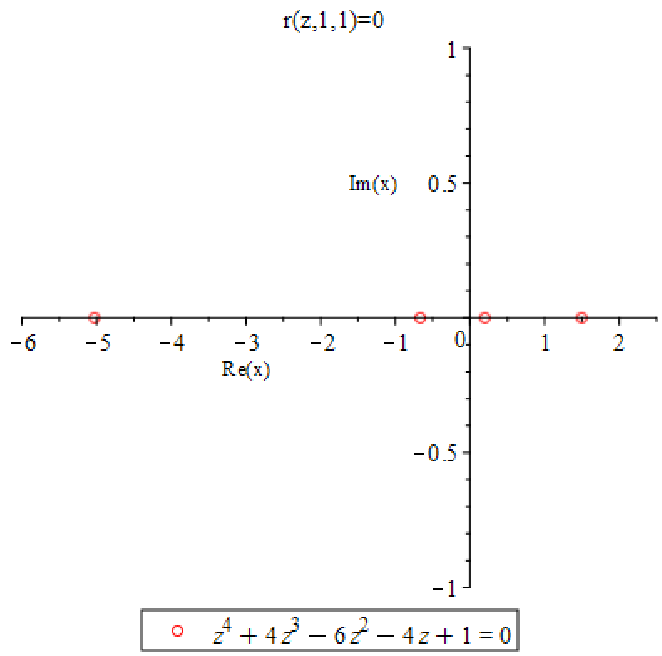
Un ejemplo particular es:

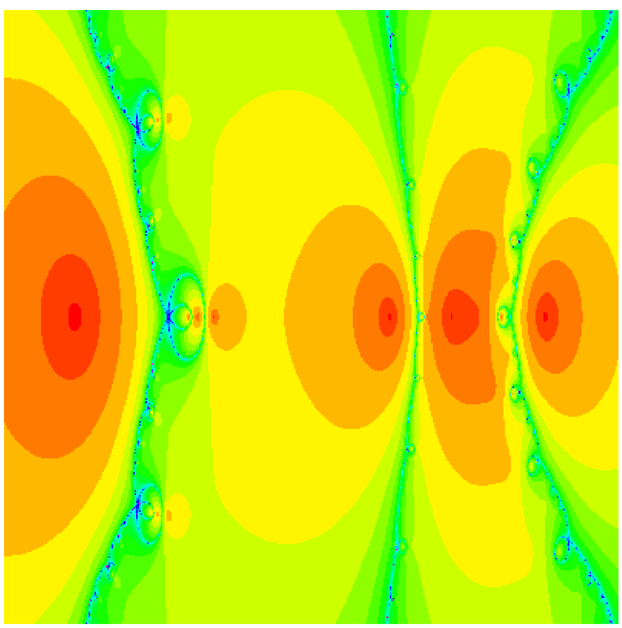
$$r\left(z,1,2\right)=z^6+2z^5+z^4-4z^3-3z^2-2z+1=0\Rightarrow\begin{cases} z_1=0.30756557...\\ z_2=1.31146706...\\ z_{3,4}=-0.4486...\pm i\,0.7491...\\ z_{5,6}=-1.3608...\pm i1.1829...\end{cases} \tag{58}$$

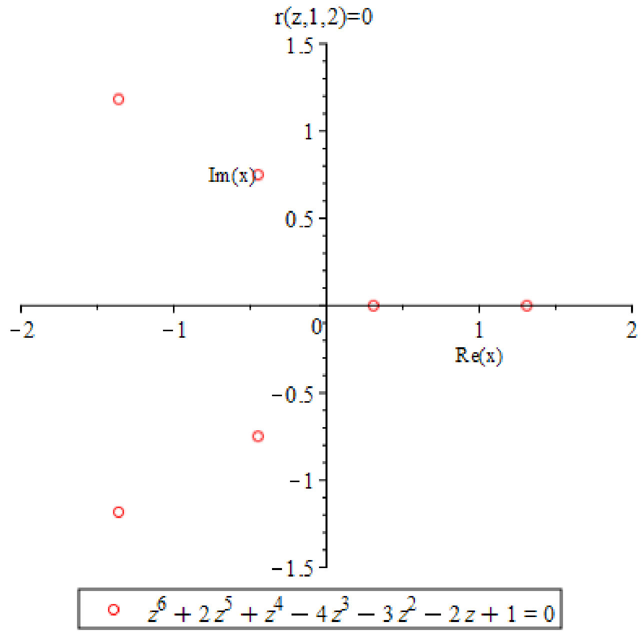
Para $\gamma=z_1$ se tiene:

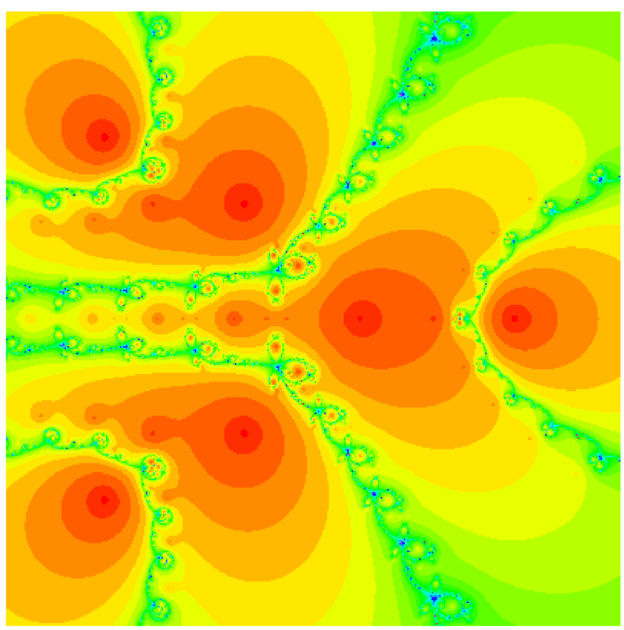
$$\pi=8\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{2n+1}\left(\gamma^{2n+1}+\gamma^{4n+2}\right) \tag{59}$$

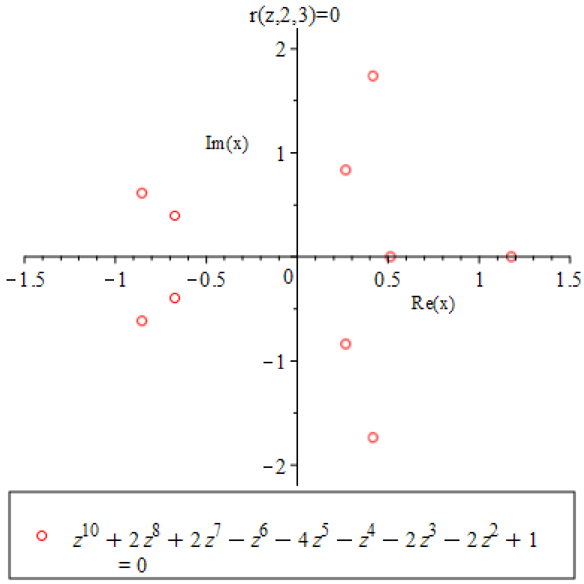
6. Gráficos de ceros y conjuntos de Newton-Julia para $r(z,k,m)$.

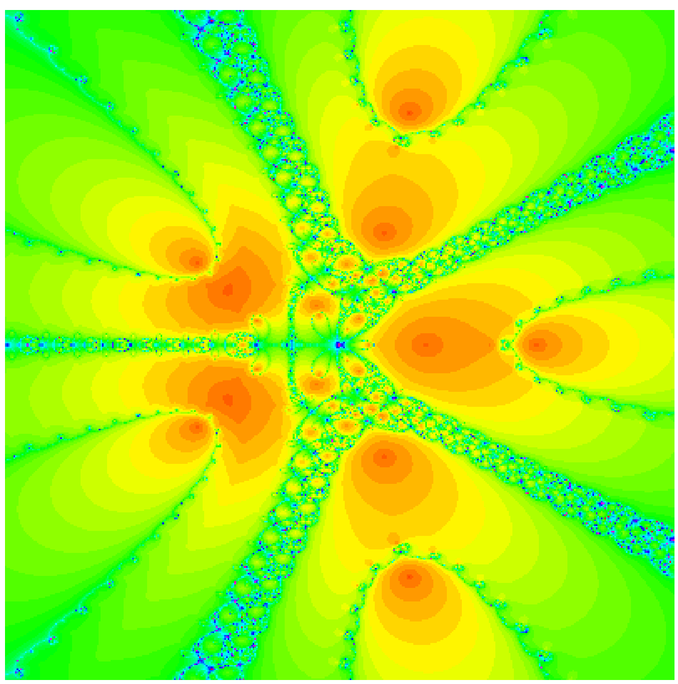





r(z,1,2)=0
1.5
1
Im(x)
0.5
−2
−1
0
1
2
Re(x)
−0.5
−1
−1.5
$z^6 + 2z^5 + z^4 - 4z^3 - 3z^2 - 2z + 1 = 0$




r(z,2,3)=0
Im(x)
Re(x)
2
1
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
$z^{10} + 2z^8 + 2z^7 - z^6 - 4z^5 - z^4 - 2z^3 - 2z^2 + 1 = 0$



7. Sea $k,m \in \mathbb{N}$ y $s(z,k,m)$ el polinomio definido como sigue:

$$s(z,k,m) = z^{2k+2m} + 4z^{k+2m} + 4z^{2k+m} + z^{2m} + z^{2k} - 4z^{m} - 4z^{k} + 1 \quad , z \in \mathbb{C} \tag{60}$$

Algunos ejemplos particulares son:

$$s(z,1,1) = z^4 + 8z^3 + 2z^2 - 8z + 1 \tag{61}$$

$$s(z,1,2) = z^6 + 4z^5 + 5z^4 - 3z^2 - 4z + 1 \tag{62}$$

Para cada polinomio $s(z,k,m)$ definido por (54) , existe un número $\delta\left(k,m\right)$ tal que:

$$0 < \delta\left(k,m\right) < 1 \tag{63}$$

$$\pi = 12\tan^{-1}\left(\delta\left(k,m\right)^{k}\right) + 12\tan^{-1}\left(\delta\left(k,m\right)^{m}\right) \tag{64}$$

La fórmula (58) se puede escribir como:

$$\pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\delta^{2kn+k} + \delta^{2mn+m}\right) \tag{65}$$

donde $\delta \equiv \delta\left(k,m\right)$.

la fórmula (65) se puede escribir como:

$$\pi = 12\sum_{n=0}^{\infty}\frac{c_n}{n+k}\delta^{n+k} \tag{66}$$

donde los coefientes c_n se obtienen por las fórmulas (17)-(21).

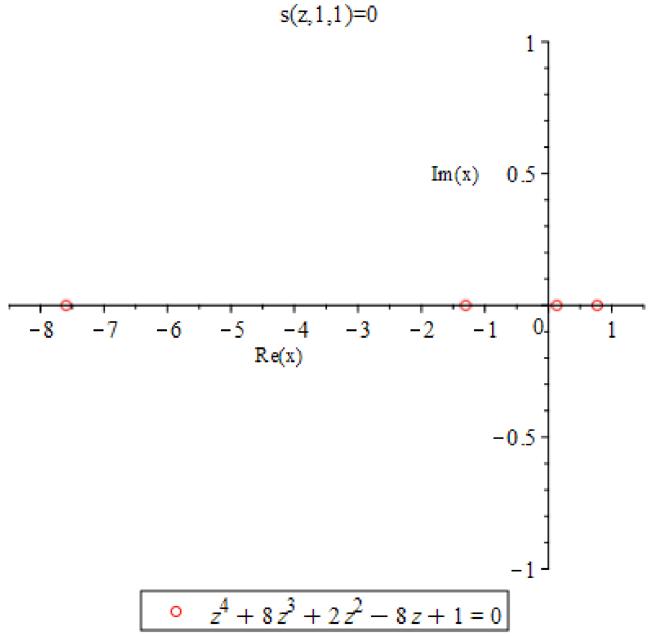
Un ejemplo particular es:

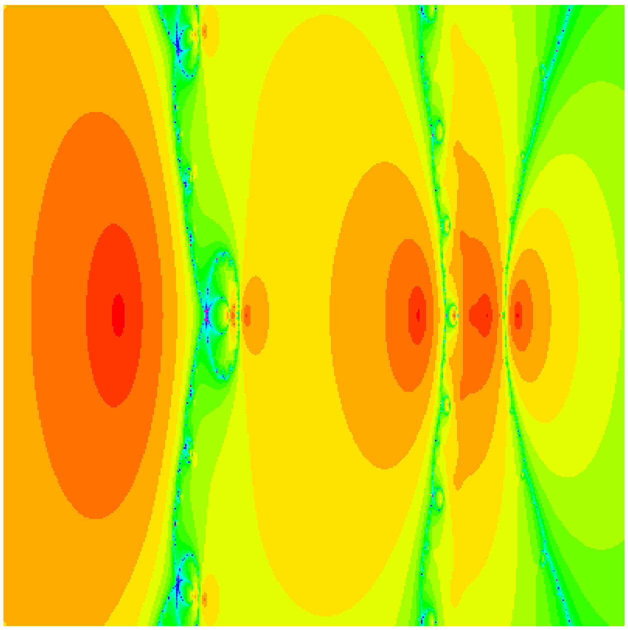
$$s\left(k,1,2\right) = z^6 + 4z^5 + 5z^4 - 3z^2 - 4z + 1 = 0 \Rightarrow \begin{cases} z_1 = 0.21776197... \\ z_2 = 0.83757924... \\ z_{3,4} = -0.5527... \pm i\,0.9425... \\ z_{5,6} = -1.9749... \pm i\,0.8318... \end{cases} \tag{67}$$

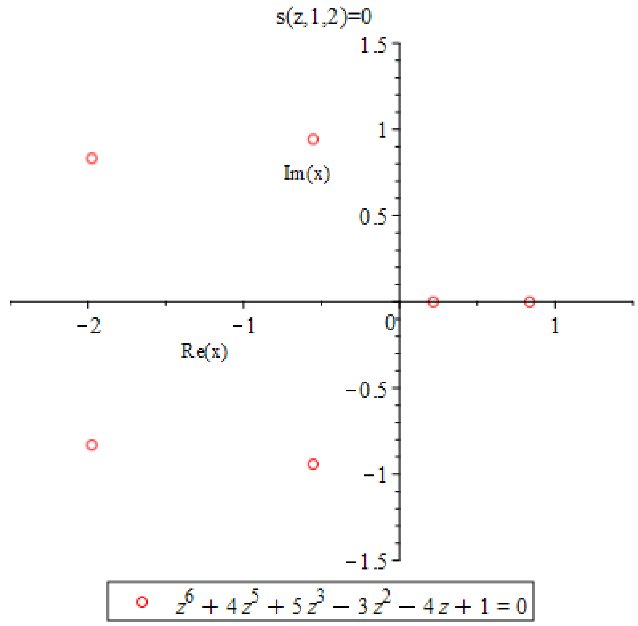
Para $\delta = z_1$ se tiene:

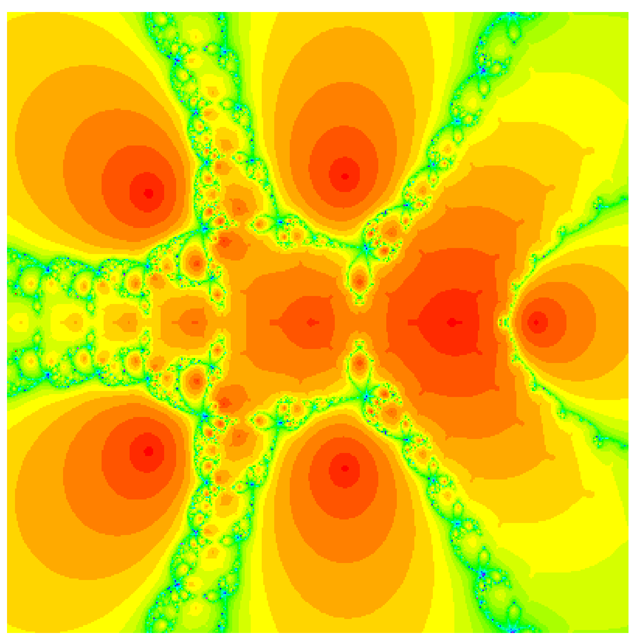
$$\pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\delta^{2n+1} + \delta^{4n+2}\right) \tag{68}$$

8. Gráficos de ceros y conjuntos de Newton-Julia para $s(z,k,m)$






s(z,1,2)=0
Im(x)
Re(x)
$z^6 + 4z^5 + 5z^3 - 3z^2 - 4z + 1 = 0$



9. Sea $k,m\in\mathbb{N}$ y $t(z,k,m)$ el polinomio definido como sigue:

$$t(z,k,m)=\left(z^k+z^m\right)^4+4\left(z^k+z^m\right)^3\left(1-z^{k+m}\right)-6\left(z^k+z^m\right)^2\left(1-z^{k+m}\right)^2-4\left(z^k+z^m\right)\left(1-z^{k+m}\right)^3+\left(1-z^{k+m}\right)^4 \quad ,z\in\mathbb{C} \tag{69}$$

Un ejemplo particular es:

$$t(z,1,3)=z^{16}+4z^{15}-6z^{14}-15z^{12}-24z^{11}+10z^{10}-20z^9+36z^8+20z^7+10z^6+24z^5-15z^4-6z^2-4z+1 \quad ,z\in\mathbb{C} \tag{70}$$

Para cada polinomio $t(z,k,m)$ definido por (69) existe un número $\rho(k,m)$ tal que:

$$0<\rho(k,m)<1 \tag{71}$$

$$\pi=16\tan^{-1}\left(\rho(k,m)^k\right)+16\tan^{-1}\left(\rho(k,m)^m\right) \tag{72}$$

Para el polinomio $t(z,1,3)$ se tiene:

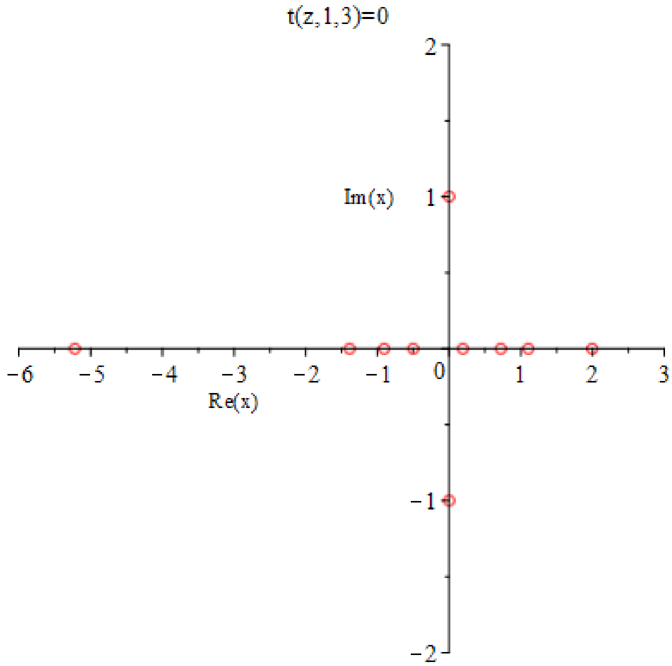
$$t(z,1,3)=\left(z^8+4z^7-10z^6-16z^5+19z^4+16z^3-10z^2-4z+1\right)\left(1+z^2\right)^4 \tag{73}$$

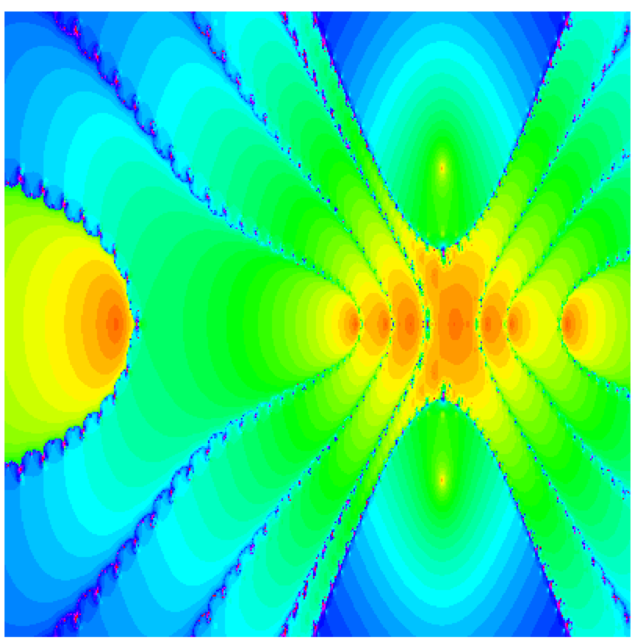
$$t(z,1,3)=0\Rightarrow\begin{cases} z_1=0.19160946... \\ z_2=0.72024254... \\ z_3=1.10438977... \\ z_4=1.99728534... \\ z_5=-0.50067958... \\ z_6=-0.90547741... \\ z_7=-1.38842117... \\ z_8=-5.21894895... \\ z_{9,10,11,12}=i \\ z_{13,14,15,16}=-i \end{cases} \tag{74}$$

Para $\rho=z_1$ se tiene:

$$\pi=16\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\rho^{2n+1}+\rho^{6n+3}\right) \tag{75}$$

10. Gráficos de ceros y conjuntos de Newton-Julia para $t(z,k,m)$





$$\rho \equiv \rho(1,3) = z_1 = \frac{\sqrt{11+4\sqrt{2}+4\sqrt{2+\sqrt{2}}+2\sqrt{2}\sqrt{2+\sqrt{2}}} - \sqrt{2}\sqrt{2+\sqrt{2}} - \sqrt{2} - 1}{2} \quad (76)$$

$$\pi = 16\tan^{-1}(\rho) + 16\tan^{-1}(\rho^3) \quad (77)$$

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