The problem on the existence and smoothness of the Navier–Stokes equations is resolved.

1. Problem description

The Navier–Stokes equations are thought to govern the motion of a fluid in \( \mathbb{R}^3 \), see [1]. Let \( u = u(x, t) \in \mathbb{R}^3 \), \( p = p(x, t) \in \mathbb{R} \), and \( f = f(x, t) \in \mathbb{R}^3 \) be the velocity, pressure, and given externally applied force respectively, each dependent on position \( x \in \mathbb{R}^3 \) and time \( t \geq 0 \). The fluid is here assumed to be incompressible with constant viscosity \( \nu > 0 \) and to fill all of \( \mathbb{R}^3 \). The Navier–Stokes equations can then be written as

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \nabla^2 u - \nabla p + f, \tag{1}
\]

\[
\nabla \cdot u = 0 \tag{2}
\]

with initial condition

\[
u_0(x) = \nu_0 \tag{3}
\]

where \( \nu_0 = \nu_0(x) \in \mathbb{R}^3 \). In these equations \( \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}) \) is the gradient operator and \( \nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \) is the Laplacian operator. When \( \nu = 0 \), equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

\[
u_0(x + e_j) = \nu_0(x), \quad f(x + e_j, t) = f(x, t) \quad \text{for} \quad 1 \leq j \leq 3 \tag{4}
\]

where \( e_1 = (1, 0, 0) \), \( e_2 = (0, 1, 0) \), \( e_3 = (0, 0, 1) \). The initial condition \( \nu_0 \) is a given \( C^\infty \) divergence-free vector field on \( \mathbb{R}^3 \) and

\[
|\partial_x^\alpha \partial_t^\beta f| \leq C_{\alpha \beta \gamma} (1 + |t|)^{-\gamma} \quad \text{on} \quad \mathbb{R}^3 \times [0, \infty) \quad \text{for any} \quad \alpha, \beta, \gamma. \tag{5}
\]

A solution of (1), (2), (3) would then be accepted to be physically reasonable if

\[
u(x + e_j, t) = \nu(x, t), \quad p(x + e_j, t) = p(x, t) \quad \text{on} \quad \mathbb{R}^3 \times [0, \infty) \quad \text{for} \quad 1 \leq j \leq 3 \tag{6}
\]

and

\[
u, p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \tag{7}
\]

I provide a proof of the following statement (D), see [2].

(D) Breakdown of Navier–Stokes Solutions on \( \mathbb{R}^3 / \mathbb{Z}^3 \).

Take \( \nu > 0 \). Then there exist a smooth, divergence-free vector field \( \nu_0 \) on \( \mathbb{R}^3 \) and a smooth \( f \) on \( \mathbb{R}^3 \times [0, \infty) \), satisfying (4), (5), for which there exist no solutions \( (\nu, p) \) of (1), (2), (3), (6), (7) on \( \mathbb{R}^3 \times [0, \infty) \).
2. Proof of statement (D)

Herein I take \( f = 0 \). I seek an approximation of the form

\[
  u = \sum_{L=-1}^{1} \sum_{l=0}^{n} \frac{\partial^l u_L}{\partial t^l} |_{t=0} t^l e^{jLx},
\]

(8)

\[
  p = \sum_{L=-1}^{1} \sum_{l=0}^{n-1} \frac{\partial^l p_L}{\partial t^l} |_{t=0} t^l e^{jLx}
\]

(9)

to the solution of (1), (2), (3), (4), (5), (6) in light of Theorem 1 and Theorem 2 in the Appendix. Here \( u_L = u_L(t), p_L = p_L(t), i = \sqrt{-1}, k = 2\pi, \) and \( \sum_{L=-H}^{H} \) denotes the sum over all \( L \in \mathbb{Z}^3 \) with \(-H \leq L_j \leq H\). Herein the smooth\(^1\) divergence-free initial condition \( u_0 \) on \( \mathbb{R}^3 \) is chosen to be

\[
  u_0 = \sum_{L=-1}^{1} L \times (L \times a_L) \delta_{jL} \sqrt{3} e^{jLx}
\]

(10)

where \( \delta_{i,j} \) is the Kronecker delta defined by

\[
  \delta_{i,j} = \begin{cases} 
  1, & i = j \\
  0, & i \neq j
\end{cases}
\]

(11)

and \( a_L \) are constant vectors that are chosen such that \( u_0 \in \mathbb{R}^3 \).

Method 1

Let

\[
  u = \sum_{l=0}^{n} \frac{\partial^l u}{\partial t^l} |_{t=0} t^l,
\]

(12)

\[
  p = \sum_{l=0}^{n-1} \frac{\partial^l p}{\partial t^l} |_{t=0} t^l.
\]

(13)

Substituting (12), (13) into (1) and equating like powers of \( t \) in accordance with Theorem 1 yields

\[
  \frac{\partial^{l+1} u}{\partial t^{l+1}} |_{t=0} + \sum_{m=0}^{l} \frac{\partial^{l-m} u}{\partial t^{l-m}} |_{t=0} \cdot \nabla \frac{\partial^m u}{\partial t^m} |_{t=0} \left( \begin{array}{c} l \\ m \end{array} \right) = \nu \nabla^2 \frac{\partial^l u}{\partial t^l} |_{t=0} - \nabla \frac{\partial^l p}{\partial t^l} |_{t=0}
\]

(14)

where \( \left( \begin{array}{c} l \\ m \end{array} \right) = \frac{n!}{m!(l-m)!} \). Substituting (12) into (2) and equating like powers of \( t \) in accordance with Theorem 1 yields

\[
  \nabla \cdot \frac{\partial^l u}{\partial t^l} |_{t=0} = 0.
\]

(15)

\(^1\)In this paper, smooth functions and \( C^\infty \) functions will both mean continuous functions whose derivatives and integrals are all continuous.
Applying $\nabla \times \nabla \times$ to (14) and using the identities
\[ \nabla \times \nabla \times a = \nabla(\nabla \cdot a) - \nabla^2 a, \]
(16)
\[ \nabla \times \nabla a = 0 \]
(17)
along with (15) gives
\[ \nabla^2 \partial_{l+1} u \big|_{l=0} = \nabla \times \nabla \times \sum_{m=0}^{l} \left( \frac{\partial^{l-m} u}{\partial t^{l-m}} \big|_{l=0} \cdot \nabla \right) \frac{\partial^m u}{\partial t^m} \big|_{l=0} \left( \frac{1}{m} \right) + \nu \nabla^4 \frac{\partial^l u}{\partial t^l} \big|_{l=0}. \]
(18)
Applying the inverse Laplacian $\nabla^{-2}$ to (18) gives
\[ \frac{\partial_{l+1} u}{\partial t^{l+1}} \big|_{l=0} = \nabla^{-2} \nabla \times \nabla \times \sum_{m=0}^{l} \left( \frac{\partial^{l-m} u}{\partial t^{l-m}} \big|_{l=0} \cdot \nabla \right) \frac{\partial^m u}{\partial t^m} \big|_{l=0} \left( \frac{1}{m} \right) + \nu \nabla^2 \frac{\partial^l u}{\partial t^l} \big|_{l=0} + \Phi_l \]
(19)
where $\Phi_l$ must satisfy the Laplace equation
\[ \nabla^2 \Phi_l = 0. \]
(20)
The required solution to (20) is $\Phi_l = 0$ in light of (4), (6). Equation (19) is then solved for $\frac{\partial_{l+1} u}{\partial t^{l+1}} \big|_{l=0}$ where $l = 0, 1, \ldots, n - 1$. Applying $\nabla$ to (14) and noting (15) yields
\[ \nabla^2 \partial_{l} p \big|_{l=0} = -\nabla \cdot \sum_{m=0}^{l} \left( \frac{\partial^{l-m} u}{\partial t^{l-m}} \big|_{l=0} \cdot \nabla \right) \frac{\partial^m u}{\partial t^m} \big|_{l=0} \left( \frac{1}{m} \right) \]
(21)
Applying $\nabla^2$ to (21) gives
\[ \frac{\partial_{l} p}{\partial t} \big|_{l=0} = -\nabla^2 \nabla \cdot \sum_{m=0}^{l} \left( \frac{\partial^{l-m} u}{\partial t^{l-m}} \big|_{l=0} \cdot \nabla \right) \frac{\partial^m u}{\partial t^m} \big|_{l=0} \left( \frac{1}{m} \right) + \psi_l \]
(22)
where
\[ \nabla^2 \psi_l = 0. \]
(23)
Arbitrary constant $\psi_l \in \mathbb{R}$ is the solution to (23) in light of (4), (6). Equation (22) is then solved for $\frac{\partial_{l} p}{\partial t} \big|_{l=0}$ where $l = 0, 1, \ldots, n - 1$. After truncating (12), (13) in their modes, expressions for (8), (9) from Method 1 are then known in terms of given functions.
Note that for the Fourier series
\[ g = \sum_{L \neq 0} g_L e^{iL \cdot x} \]
(24)
where $\sum_{L \neq 0}$ denotes the sum over all $L \in \mathbb{Z}^3$ with $L \neq 0$, the $\nabla^{-2}$ operator is defined herein as
\[ \nabla^{-2} \sum_{L \neq 0} g_L e^{iL \cdot x} = \sum_{L \neq 0} \frac{g_L e^{iL \cdot x}}{-k^2 |L|^2}. \]
(25)
Method 2

Let

\[ u = \sum_{L=-1}^{1} u_L e^{iL \cdot x}, \quad (26) \]
\[ p = \sum_{L=-1}^{1} p_L e^{iL \cdot x}. \quad (27) \]

Substituting (26), (27) into (1) and equating like powers of \( e \) in accordance with Theorem 2 yields

\[ \frac{\partial u_L}{\partial t} + \sum_{M} (u_{L-M} \cdot i k M) u_M = -\nu k^2 |L|^2 u_L - i k |L|^2 p_L. \quad (28) \]

Substituting (26) into (2) and equating like powers of \( e \) in accordance with Theorem 2 yields

\[ L \cdot u_L = 0. \quad (29) \]

Applying \( L \times \) to (28) and noting the vector identity

\[ a \times (b \times c) = (c \cdot a) b - (b \cdot a) c \quad (30) \]

along with (29) yields

\[ |L|^2 \frac{\partial u_L}{\partial t} = \sum_{M} L \times (L \times (u_{L-M} \cdot i k M) u_M) - \nu k^2 |L|^4 u_L. \quad (31) \]

Equation (31) implies

\[ \frac{\partial u_L}{\partial t} = \sum_{M} \hat{L} \times (\hat{L} \times (u_{L-M} \cdot i k M) u_M) - \nu k^2 |L|^2 u_L \quad (32) \]

where the right hand side of (32) is 0 when \( L = 0 \) and \( \hat{L} = L/|L| \) is the unit vector in the direction of \( L \). Applying \( L \cdot \) to (28) and noting (29) gives

\[ i k |L|^2 p_L = -\sum_{M} (u_{L-M} \cdot i k M)(u_M \cdot L) \quad (33) \]

implying that

\[ p_L = -\sum_{M} (u_{L-M} \cdot \hat{L})(u_M \cdot \hat{L}) \quad (34) \]

where \( p_0 \in \mathbb{R} \) is an arbitrary function of \( t \). Let

\[ u_L = \sum_{l=0}^{n} \frac{\partial^l u_L}{\partial t^l} |_{t=\frac{l}{T}}, \quad (35) \]
\[ p_L = \sum_{l=0}^{n-1} \frac{\partial^{l+1} p_L}{\partial t^{l+1}} |_{t=\frac{l}{T}}. \quad (36) \]
Substituting (35) into (32) and equating like powers of $t$ in accordance with Theorem 1 yields

$$\frac{\partial\hat{u}_L}{\partial t} |_{t=0} = \sum_{m=0}^{l} \hat{L} \times \left( \hat{L} \times (\hat{L}^{l-m} u_{L-M} |_{l=0} \cdot ikM) \frac{\partial^m u_M}{\partial t^m} |_{l=0} \right) \left( \frac{1}{m} \right) - \nu k^2 |L|^2 \frac{\partial^l u_L}{\partial t^l} |_{l=0}. \tag{37}$$

Equation (37) is then solved for $\frac{\partial^j u_L}{\partial t^j} |_{t=0}$ where $l = 0, 1, \ldots, n - 1$ and $-1 \leq L_j \leq 1$. Substituting (35), (36) into (34) and equating like powers of $t$ in accordance with Theorem 1 yields

$$\frac{\partial^j p_L}{\partial t^j} |_{t=0} = - \sum_{m=0}^{l} \sum_{M} \left( \frac{\partial^{j-m} u_{L-M} |_{l=0} \cdot \hat{L}}{\partial t^{j-m}} \frac{\partial^m u_M}{\partial t^m} |_{l=0} \right) \left( \frac{1}{m} \right) \tag{38}$$

Equation (38) is then solved for $\frac{\partial^j p_L}{\partial t^j} |_{t=0}$ where $l = 0, 1, \ldots, n - 1$ and $-1 \leq L_j \leq 1$. Expressions for (8), (9) from Method 2 are then known in terms of given functions.

At $l = 0$ in (37) it is found that

$$\frac{\partial u_L}{\partial t} |_{t=0} = \sum_{M} \hat{L} \times (\hat{L} \times (u_{L-M} |_{l=0} \cdot ikM) u_M |_{l=0}) - \nu k^2 |L|^2 u_L |_{l=0}. \tag{39}$$

In (39) with $1 \leq |L|^2 \leq 3$, $u_M |_{l=0} = 0$ unless $|M|^2 = 3$ and $u_{L-M} |_{l=0} = 0$ unless $|L - M|^2 = 3$. With $|L|^2 = 3$ and $|M|^2 = 3$ the equation $|L - M|^2 = 3$ then implies $2L \cdot M = 3$ which is not possible as an even number can not be equal to an odd number. Likewise, with $|L|^2 = 1$ and $|M|^2 = 3$ the equation $|L - M|^2 = 3$ then implies $2L \cdot M = 1$ which is not possible as an even number can not be equal to an odd number. With $|L|^2 = 2$ and $|M|^2 = 3$ the equation $|L - M|^2 = 3$ then implies $|L \cdot M| = 1$ which is not possible as in this instance $|L \cdot M| \in \{0, 2\}$ when $-1 \leq L_j \leq 1, -1 \leq M_j \leq 1$. Therefore

$$\frac{\partial u_L}{\partial t} |_{t=0} = -3k^2 \nu u_L |_{l=0}. \tag{40}$$

At $O(t)$, I find that Method 2 gives the same result for (8) as given by Method 1. At $l = 1$ in (37) it is found that

$$\frac{\partial^2 u_L}{\partial t^2} |_{t=0} = \sum_{M} \hat{L} \times (\hat{L} \times ((\frac{\partial u_{L-M}}{\partial t} |_{l=0} \cdot ikM) u_M |_{l=0} + (u_{L-M} |_{l=0} \cdot ikM) \frac{\partial u_M}{\partial t} |_{l=0}) \tag{41}$$

By a similar argument as that applied to (39) it is found in Method 2 that

$$\frac{\partial^2 u_L}{\partial t^2} |_{t=0} = -3k^2 \nu \frac{\partial u_L}{\partial t} |_{t=0} = 9k^4 \nu^2 u_L |_{l=0}. \tag{42}$$

In fact for $l \geq 0$ it is found in Method 2 that

$$\frac{\partial^{l+1} u_L}{\partial t^{l+1}} |_{t=0} = (-3k^2 \nu)^{l+1} u_L |_{l=0}. \tag{43}$$
With Method 1 for $\nu = 0$, I find that $u_{tt}|_{t=0} \neq 0$ when truncated onto the modes with $-1 \leq L_j \leq 1$. Therefore at $O(t^2)$, the approximation (8) found from Method 1 is different to the approximation (8) found from Method 2. Because of this nonuniqueness at least one of the assumptions used was invalid.

**An exact solution**

Herein I denote $u = (u, v, w)$ and $x = (x, y, z)$. Let the initial condition be

$$u_0 = (\cos(k(x + y - z)), \cos(k(x - y - z)), \cos(k(x + y - z)) - \cos(k(x - y - z))) \quad (44)$$

which is consistent with (10). I used Maple to find the Maclaurin series of the solution $(u, p)$ to (1), (2), (3), (4), (5), (6) using (44). The nonuniqueness of results found with Method 1 and Method 2 does occur when using (44). It appeared from the Maclaurin series of the solution $(u, p)$ that

$$v = \cos(k(x - y - z))e^{\nu \lambda t}, \quad (45)$$
$$w = u - \cos(k(x - y - z))e^{\nu \lambda t}, \quad (46)$$
$$p = 0 \quad (47)$$

where $\lambda = -3k^2$. On substitution of (45), (46), (47) into (1), (2), (6), I found that $u$ must satisfy

$$\frac{\partial u}{\partial t} + (\frac{\partial u}{\partial y} - \frac{\partial u}{\partial z})e^{\nu \lambda t} \cos(k(x - y - z)) - \nu \nabla^2 u = 0, \quad (48)$$
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} = 0, \quad (49)$$
$$u(x + e_j, t) = u(x, t), \quad for \ 1 \leq j \leq 3. \quad (50)$$

For $\nu = 0$, I used Maple to find that the exact general solution of (48) is

$$u = F(x, y + z, \frac{t \cos(k(x - y - z)) - y}{\cos(k(x - y - z))}) \quad (51)$$

where $F$ is an arbitrary function. On matching (51) with (44) at $t = 0$, I then deduced that

$$u = \cos(2tk \cos(k(x - y - z)) - k(x + y - z)). \quad (52)$$

The solution (52) also satisfies (49), (50). The resulting $(u, p)$ was then verified to be an exact solution to (1), (2), (3), (4), (5), (6) for $\nu = 0$. Integrating (52) with respect to $t$ yields

$$\int u dt = \frac{\sin(2tk \cos(k(x - y - z)) - k(x + y - z))}{2k \cos(k(x - y - z))} \quad (53)$$

which is undefined for some values of $x \in \mathbb{R}^3$ and $t \geq 0$.

For $\nu > 0$, it is found that for the small time $O(t)$ solution the equation (48) for $u$ is

$$\frac{\partial u}{\partial t} + (\frac{\partial u}{\partial y} - \frac{\partial u}{\partial z})e^{\nu \lambda t} \cos(k(x - y - z)) - \nu \lambda u = 0. \quad (54)$$
Equation (54) implies
\[ \frac{\partial}{\partial t}(ue^{-\nu \lambda t}) + \left( \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z} \right) \cos(k(x - y - z)) = 0. \] (55)

Then a change of variables
\[ \tau = \frac{e^{\nu \lambda t} - 1}{\nu \lambda}, \] (56)
\[ u(x, t) = a(x, \tau) \frac{\partial \tau}{\partial t} \] (57)
yields
\[ \frac{\partial a}{\partial \tau} + \left( \frac{\partial a}{\partial y} - \frac{\partial a}{\partial z} \right) \cos(k(x - y - z)) = 0. \] (58)

Equation (49) becomes
\[ \frac{\partial a}{\partial x} + \frac{\partial a}{\partial z} = 0, \] (59)
the initial condition (44) implies
\[ a(x, 0) = \cos(k(x + y - z)), \] (60)
and the spatially periodic boundary conditions (50) imply
\[ a(x + e_j, \tau) = a(x, \tau) \text{ for } 1 \leq j \leq 3. \] (61)

Equations (58), (59), (60), (61) define an Euler problem. In light of this and (52), it is then clear that
\[ u = e^{\nu \lambda t} \cos(\frac{2k}{\nu \lambda}(e^{\nu \lambda t} - 1) \cos(k(x - y - z)) - k(x + y - z)) \] (62)
is valid for small time when \( \nu > 0 \). Integrating (62) with respect to \( t \) yields
\[ \int^t u \, dt = \frac{\sin\left(\frac{2k}{\nu \lambda}(e^{\nu \lambda t} - 1) \cos(k(x - y - z)) - k(x + y - z)\right)}{2k \cos(k(x - y - z))} \] (63)
which is undefined for some values of \( x \in \mathbb{R}^3 \) and \( t \geq 0 \).
Therefore statement (D) is true. \( \square \)

Appendix

Theorem 1

Providing that the Maclaurin series
\[ \tilde{A} = \sum_{l=0}^n \frac{\partial^l \tilde{A}}{\partial t^l} \bigg|_{t=0} t^l = \sum_{l=0}^n \frac{\partial^l \tilde{A}}{\partial t^l} \bigg|_{t=0} \frac{t^l}{l!} \] (64)
of the exact general solution to a $Q^{th}$ order partial differential equation

$$\frac{\partial^Q A}{\partial t^Q} = \Psi$$  \hspace{1cm} (65)$$

exists, it will solve the coefficients of $t^l$ for all $l = 0, 1, \ldots, n - Q$ in (65) with $A = \tilde{A}$ provided $\Psi|_{A=\tilde{A}}$ is expandable in Maclaurin series as

$$\Psi|_{A=\tilde{A}} = \sum_{l=0}^{m} \frac{\partial^l \Psi|_{A=\tilde{A}}}{\partial t^l}|_{t=0} \frac{t^l}{l!}$$  \hspace{1cm} (66)$$

where $m \geq n$. Here all of the partial derivatives of $A$ with respect to $t$ are defined at $t = 0$.

**Proof of Theorem 1**

Since the Maclaurin series of $A$ exists and all of the partial derivatives of $A$ with respect to $t$ are defined at $t = 0$, one can integrate (65) $Q$ times with respect to $t$ and then substitute the result into (64) to find

$$\tilde{A} = \sum_{l=0}^{n} \frac{\partial^l Q \Psi|_{A=\tilde{A}}}{\partial t^l Q}|_{t=0} \frac{t^l}{l!} = \sum_{l=0}^{n} \frac{\partial^l \int_Q \Psi dt|_{A=\tilde{A}}}{\partial t^l}|_{t=0} \frac{t^l}{l!}$$  \hspace{1cm} (67)$$

where $\int_Q \Psi dt$ denotes the $Q^{th}$ integral of $\Psi$ with respect to $t$. Substituting $A = \tilde{A}$ into the residual $r$ of (65) then gives

$$r = \sum_{l=0}^{n} \frac{\partial^l Q \Psi|_{A=\tilde{A}}}{\partial t^l Q}|_{t=0} \frac{t^l Q}{(l - Q)!} - \sum_{l=0}^{m} \frac{\partial^l \Psi|_{A=\tilde{A}}}{\partial t^l}|_{t=0} \frac{t^l}{l!}$$  \hspace{1cm} (68)$$

providing $\Psi|_{A=\tilde{A}}$ is expanded in Maclaurin series as in (66). Collecting like powers of $t$ in (68) yields

$$r = \sum_{l=0}^{n} \frac{\partial^l Q \Psi|_{A=\tilde{A}}}{\partial t^l Q}|_{t=0} \frac{t^l Q}{(l - Q)!} - \sum_{l=0}^{m} \frac{\partial^l \Psi|_{A=\tilde{A}}}{\partial t^l}|_{t=0} \frac{t^l}{l!}$$  \hspace{1cm} (69)$$

which shows that Theorem 1 is true. □

**Theorem 2**

Providing that the Fourier series

$$\tilde{A} = \sum_{L=-N}^{N} P(A, e^{iL\cdot x}) e^{iL\cdot x} = \sum_{L=-N}^{N} P(\tilde{A}, e^{iL\cdot x}) e^{iL\cdot x}$$  \hspace{1cm} (70)$$

of the exact general solution to a $Q^{th}$ order partial differential equation

$$\frac{\partial^Q A}{\partial t^Q} = \Psi$$  \hspace{1cm} (71)$$
exists, it will solve the coefficients of $e^{i\mathbf{k} \cdot \mathbf{x}}$ for all $-N \leq L_j \leq N$ in (71) with $\mathbf{A} = \tilde{\mathbf{A}}$ provided $\Psi|_{\mathbf{A} = \tilde{\mathbf{A}}}$ is expandable in Fourier series as

$$
\Psi|_{\mathbf{A} = \tilde{\mathbf{A}}} = \sum_{L = -M}^{M} P(\Psi|_{\mathbf{A} = \tilde{\mathbf{A}}}, e^{i\mathbf{k} \cdot \mathbf{x}}) e^{i\mathbf{k} \cdot \mathbf{x}} \quad (72)
$$

where $M \geq N$. Here $\mathbf{A}$ is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, $k > 0$ is a constant, and $P(\mathbf{h}, e^{i\mathbf{k} \cdot \mathbf{x}})$ denotes the projection of $\mathbf{h}$ onto $e^{i\mathbf{k} \cdot \mathbf{x}}$.

**Proof of Theorem 2**

Since the Fourier series of $\mathbf{A}$ exists where $\mathbf{A}$ is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, one can integrate (71) $Q$ times with respect to $t$ and then substitute the result into (70) to find

$$
\tilde{\mathbf{A}} = \sum_{L = -N}^{N} P(\int_{Q} \Psi \, dt, e^{i\mathbf{k} \cdot \mathbf{x}}) e^{i\mathbf{k} \cdot \mathbf{x}} = \sum_{L = -N}^{N} P(\int_{Q} \Psi \, dt|_{\mathbf{A} = \tilde{\mathbf{A}}}, e^{i\mathbf{k} \cdot \mathbf{x}}) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (73)
$$

Substituting $\mathbf{A} = \tilde{\mathbf{A}}$ into the residual $\mathbf{r}$ of (71) then gives

$$
\mathbf{r} = \frac{\partial Q}{\partial t} \sum_{L = -N}^{N} P(\int_{Q} \Psi \, dt|_{\mathbf{A} = \tilde{\mathbf{A}}}, e^{i\mathbf{k} \cdot \mathbf{x}}) e^{i\mathbf{k} \cdot \mathbf{x}} - \sum_{L = -M}^{M} P(\Psi|_{\mathbf{A} = \tilde{\mathbf{A}}}, e^{i\mathbf{k} \cdot \mathbf{x}}) e^{i\mathbf{k} \cdot \mathbf{x}} \quad (74)
$$

providing $\Psi|_{\mathbf{A} = \tilde{\mathbf{A}}}$ is expanded in Fourier series as in (72). Equation (74) can be written as

$$
\mathbf{r} = \sum_{L = -N}^{N} P(\Psi|_{\mathbf{A} = \tilde{\mathbf{A}}}, e^{i\mathbf{k} \cdot \mathbf{x}}) e^{i\mathbf{k} \cdot \mathbf{x}} - \sum_{L = -M}^{M} P(\Psi|_{\mathbf{A} = \tilde{\mathbf{A}}}, e^{i\mathbf{k} \cdot \mathbf{x}}) e^{i\mathbf{k} \cdot \mathbf{x}} \quad (75)
$$

which shows that Theorem 2 is true. □

**References**
