A graphical representation of NP-completeness with one-dimensional cellular automata

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Abstract

This paper proves that for every instance of the (NP-complete or NP-hard, depending on the version) change-making problem (CMP) an one-dimensional cellular automaton can be constructed that solves it. As cellular automata are graphically represented as images or sequences of images - often resembling natural phenomena such as fluid flows - a possibly interesting graphical representation of NP-completeness can be obtained.

1 Introduction

Consider an arbitrary coinage consisting of n>0 types of coins of positive integer value – let $A=\{\alpha_0,\alpha_1,\ldots,\alpha_{n-1}\}$ be the set of these values – and a non-negative integer c.

1.1 The feasibility version of the change-making problem

Is the question of whether or not the following set is non-empty:

$$\left\{(x_0,x_1,\ldots,x_{n-1})\in\mathbb{N}^n \;\middle|\; \sum_{i=0}^{n-1}\alpha_i x_i \;=\; c\right\}$$

In other words, the question is whether there is a way to make change for the amount c using the available coins, assuming that the available number of coins of each type is as much as needed.

The problem is NP-complete for this version, as proved by Lueker (as cited in Martello & Toth, 1990).

1.2 The optimization version of the change-making problem

Is finding the minimum of the following set:

$$\left\{\sum_{i=0}^{n-1}x_i \;\middle|\; x_i\in\mathbb{N} \;\wedge\; \sum_{i=0}^{n-1}\alpha_i x_i \;=\; c\right\}$$

So, in this case we ask to make change for the amount c using the minimum number of coins.

The problem is NP-hard for this version (Martello & Toth, 1990)

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1.3 Example

Consider a coinage consisting of two coins whose value set is {5, 7}. The available number of coins of each type is as much as needed. Is there a way to make change for the amount c = 23, and if so what is the minimum number of coins needed to achieve this? As we will prove in section 3, the answer is that there is no way in this case.

2 Solving CMP with one-dimensional cellular automata

2.1 The algorithm

Consider an indexing of the square lattice of an one-dimensional cellular automaton as follows:

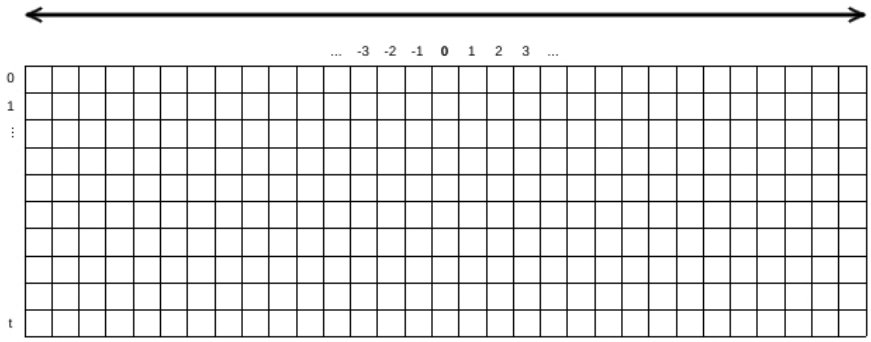

Figure 1

Denote a finite horizontal array of cells from the cell (y, x_1) to the cell (y, x_2) by $HA(y, x_1, x_2)$.

Denote the k-th cell of an array HA by HA_k.

Steps of the algorithm:

a) Define the following one-dimensional automaton:

-Terminal generation number: $t = \left\lfloor \frac{c}{min\, A} \right\rfloor$

- Color (states) set: {white, black}.

- Neighborhood radius: r = max A

- Initial configuration: black at (0,0) and white everywhere else.

- Transition function for n such that $0 \le n < t$ and $m \in \mathbb{Z}$:

$$F(HA(n, m - max\,A, m + max\,A)) = \begin{cases} \text{black}, \exists \alpha \in A : HA_{max\,A-a} \; is \; \text{black} \\ \text{white}, \; otherwise \end{cases}$$

b) Run the automaton up to generation t.

c) Decision of the problem

If there exists a black cell in column c of the lattice, then the answer to the feasibility version of the problem is “yes”, else is “no”. For the optimization problem, the answer is the minimum of the set $\{n \in \{0,1,2,\ldots,t\} \mid (n,c) \text{ is black }\}$.

2.2 Correctness proof

We will prove by induction on the generation n of the automaton the following theorem:

A cell (n,m) is black, iff there exists a way to make change for amount m using n coins.

Proof:

Inductive base:

For n=0:

If the cell (0, m) is black:

The only black cell is (0,0), therefore there must exist a way to make change for amount 0. There exist, indeed: selecting 0 of each type of coin.

If the cell (0, m) is white:

All cells except (0,0) are white, therefore there must not exist a way to make change for an amount different than 0 using 0 coins, which holds obviously.

Inductive assumption:

Let the theorem hold for the generation $n=k\geq 0$.

Inductive step:

We will prove that if the inductive assumption holds, then the theorem holds for the generation n=k+1.

If the cell (k+1, m) is black, then by the transition function it follows:

$\exists a \in A : (k, m-a) \; is \; \text{black}$

Therefore, via the inductive assumption we get that there exists α element of A such that there exist a way to make change for the amount m-α using k coins. Therefore, there exists a way to make change for the amount m-α+α = m using k+1 coins.

If the cell (k+1, m) is white, then by the transition function it follows:

$\nexists a \in A : (k, m-a) \; is \; \text{black}$

Therefore, via the inductive assumption we get that there does not exist α element of A such that there exists a way to make change for amount m - α using k coins **(1)**.

Assume that there exist a way to make change for the amount m using k+1 coins. Then, removing one coin whose value is α it follows that there exist a way to make change for the amount m-α using k coins, which contradicts with **(1)**. Therefore, there does not exist a way to make change for the amount m using k+1 coins. ∎

Finally, in order to complete the correctness proof, it suffices to show that there does not exist a solution consisting of a number of coins greater than $\left\lfloor \frac{c}{min\,A} \right\rfloor$.

Proof:

Assume there exists an l-tuple $(x_{0,} x_1 \ldots x_{l-1})$ such that $\sum_{i=0}^{l-1} \alpha_i x_i = c$ and $\sum_{i=0}^{l-1} x_i > \left\lfloor \frac{c}{min\,A} \right\rfloor$.

From the last relation it follows that there exist a positive integer b such that:

$$\sum_{i=0}^{l-1} x_i = \left\lfloor \frac{c}{min\,A} \right\rfloor + b \quad .$$

If there exists $k \in \{0,1,\ldots,l-1\}$ such that $x_k \neq 0 \ \wedge \ a_k > min\,A$, then:

$$\sum_{i=0}^{l-1} \alpha_i x_i > \sum_{i=0}^{l-1} min\,A \cdot x_i = min\,A \sum_{i=0}^{l-1} x_i = min\,A \left(\left\lfloor \frac{c}{min\,A} \right\rfloor + b \right) = min\,A \left\lfloor \frac{c}{min\,A} \right\rfloor + b\,min\,A > c \quad .$$

A contradiction, as $\sum_{i=0}^{l-1} \alpha_i x_i = c$ by assumption.

If there does not exist $k \in \{0,1,\ldots,l-1\}$ such that $x_k \neq 0 \ \wedge \ a_k > min\,A$, then letting $\mu \in \{0,1,\ldots,l-1\}$ such that $x_\mu = min\,A$ we get:

$$(\forall i \in \{0,1,\ldots,l-1\} : x_i = 0) \vee ((\forall i \in \{0,1,\ldots,l-1\} \setminus \{\mu\} : x_i = 0) \wedge x_\mu \neq 0)$$

If the first proposition of the disjunction above is true, we get c=0, therefore:

$$\sum_{i=0}^{l-1} 0 > \left\lfloor \frac{0}{min\,A} \right\rfloor \Rightarrow 0 > 0$$, a contradiction.

If the second proposition of the disjunction above is true, we get:

$$\sum_{i=0}^{l-1} x_i = x_\mu = \frac{c}{min\,A} > \left\lfloor \frac{c}{min\,A} \right\rfloor = \frac{c}{min\,A}$$, a contradiction. ∎

3 Graphical representations of sample instances of CMP with the automaton described in subsection 2.1

Consider the example in subsection 1.3. The evolution of the automaton described in subsection 2.1 up to the terminal generation number $t = \left\lfloor \frac{23}{5} \right\rfloor = 4$ is shown below:

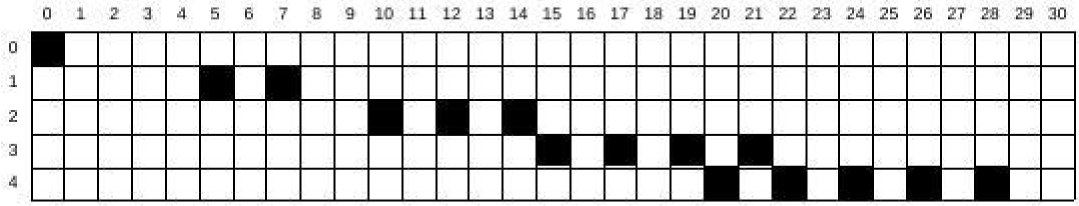

Figure 2

Applying the step c) of the algorithm described in subsection 2.1, we observe that there does not exist any black cell in column 23. Therefore, there exists no solution for the given instance of the problem.

Examples of the automaton for other (pseudo-) randomly generated instances of CMP are shown below.

Figure 3
A clip of the automaton for the value set {86, 87, 168, 179}

5

Figure 4
A clip of the automaton for the value set {125,180,187,215}

Figure 5
A clip of the automaton for the value set {50, 55, 105, 158}

Figure 6
A clip of the automaton for the value set {8,9,17}

Figure 7
A clip of the automaton for the value set {29, 97, 137, 244}

Figure 8
A clip of the automaton for the value set {75, 84, 122}

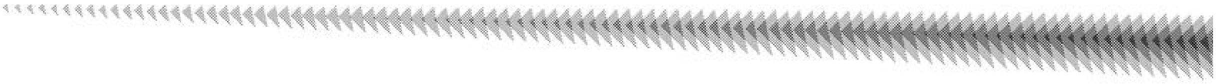

Figure 9
A clip of the automaton for the value set {18,35,107}

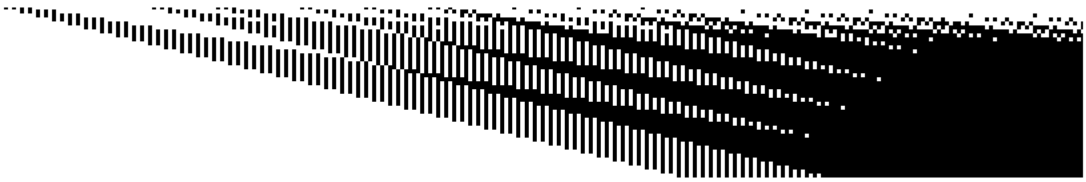

Figure 10
A clip of the automaton for the value set {4,6,41,57}

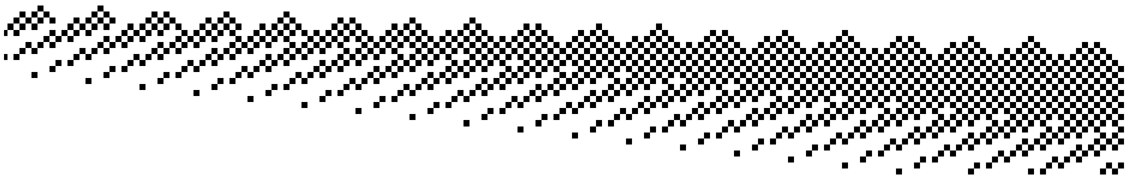

Figure 11
A clip of the automaton for the value set {9,21,31}

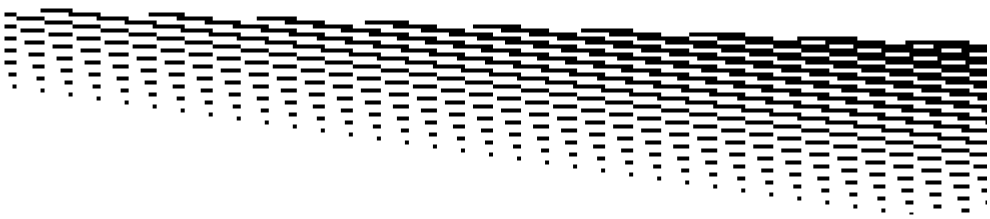

Figure 12
A clip of the automaton for the value set {7,27,28}

Figure 13
A clip of the automaton for the value set {12,43,54,56}

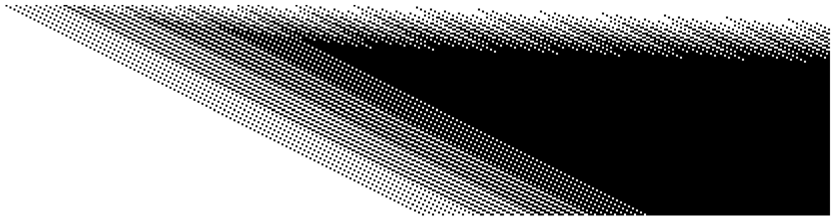

Figure 14
A clip of the automaton for the value set {2,7,31}

Figure 15
A clip of the automaton for the value set {6,12,29}

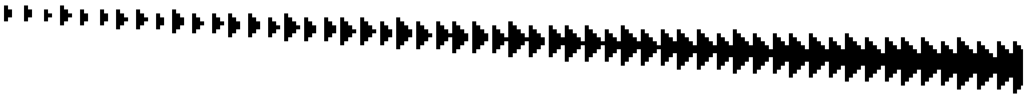

Figure 16
A clip of the automaton for the value set {14,19,28}

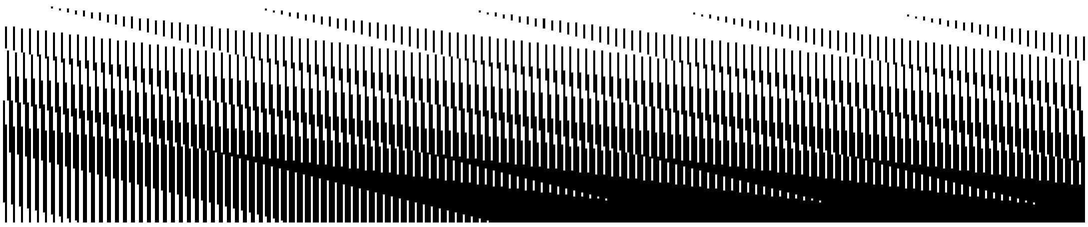

Figure 17
A clip of the automaton for the value set {4,8,107}

Figure 18
A clip of the automaton for the value set {15,16,77}

References

Martello S. & Toth P. (1990). *KNAPSACK PROBLEMS: Algorithms and Computer Implementations*. Retrieved from: http://www.or.deis.unibo.it/kp/Chapter5.pdf