There exists no algorithm for solving the quadratic equation with real coefficients neither in $\mathbb{R}$ nor in $\mathbb{C}$

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Abstract

This paper proves that the problem of the equality of two real constants is undecidable and as a consequence there exists no algorithm for solving the (univariate) quadratic equation with real coefficients neither in $\mathbb{R}$ nor in $\mathbb{C}$.

1. Definitions and assumptions

a) An *algorithm* is a Turing machine.

b) Assume that an algorithm **(A1)** for solving the quadratic equation with real coefficients in $\mathbb{R}$ necessarily gives an output that implies the decidability of the decision problem “is there a real solution for the given equation?” **(P1)**, e.g:

a list, empty if there exists no real solution for the given equation, with up to two (distinct) elements, otherwise.

c) Assume that an algorithm **(A2)** for solving the quadratic equation with real coefficients in $\mathbb{C}$ necessarily gives an output that implies the decidability of the decision problem “has the given equation exactly one complex solution?” **(P2)**, e.g:

a list, with one element if the given equation has exactly one complex solution, with two elements, if the given equation has exactly two (distinct) complex solutions.

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2. The problem of the equality of two real constants is undecidable (Thm1)

Proof

Let C(TM) be a string that encodes a Turing machine TM. For every Turing machine TM and for every input string (on the input alphabet of TM) x consider the following series:

$$\sum_{k=1}^{\infty} 10^{-k} h_{C(TM),x}(k), \textit{wherein } h_{C(TM),x} \textit{ a sequence of natural numbers},$$
$$\textit{for } n \geq 1:$$
$$h_{C(TM),x}(n) = \begin{cases} 1, & TM - \textit{encoded by } C(TM) - \textit{ on input } x \textit{ halts after} \\ & n \textit{ steps} \text{ or } \textit{less} \\ 0, & \textit{otherwise} \end{cases}$$

The series above converges in ℝ as $10^{-k}h_{C(TM),x}(k) \leq 10^{-k}$ holds for every natural number $k \geq 1$ and the series of the sequence 10^{-k} converges as geometric series with common ratio 10^{-1} for which it holds that $|10^{-1}|<1$. Therefore, for every Turing machine TM and for every input string (on the input alphabet of TM) x the following lemma holds:

$$\sum_{k=1}^{\infty} 10^{-k} h_{C(TM),x}(k)\, \textit{constant}, \in \mathbb{R} \quad (\boldsymbol{L1})$$

Obviously, for every Turing machine TM and for every input string (on the input alphabet of TM) x the following lemma also holds:

$$\sum_{k=1}^{\infty} 10^{-k} h_{C(TM),x}(k) = 0 \;\; \textit{iff} \;\; TM \textit{ on input } x \textit{ does not halt} \quad (\boldsymbol{L2})$$

Consider a Turing machine TM_1 on input (on the input alphabet of TM_1) x_1. Assuming that there exists an algorithm **(A3)** that decides whether or not two real constants are equal, it follows that there exists another algorithm **(A4)** that:

a) simulating the algorithm **A3** decides whether or not it holds that (the first member of the equality below is a real constant by **L1**):

$$\sum_{k=1}^{\infty} 10^{-k} h_{C(TM_1),x_1}(k)=0$$

b) via the previous result and **L2** decides whether or not the Turing machine TM_1 on input x_1 halts.

In other words, the algorithm **A4** decides the halting problem, a result that contradicts Turing's “halting theorem” (Halatsis, 2003). We reached a contradiction because we assumed that there exists an algorithm such **A3**. Therefore, there exists no such algorithm. ▪

3. There exists no algorithm such A1

Proof

Assuming that there exists an algorithm such **A1**, via the assumption b) in section 1, it follows that there exists another algorithm **(A5)** that simulating the former, given any quadratic equation with real coefficients, decides the decision problem **P1**. If the form of output of the algorithm **A1** is identical with the one in the example of the assumption b) in section 1, then the algorithm **A5** decides **P1** as follows:

- If the algorithm **A1** gives as output a non-empty list – i.e there exists a real solution for the given equation – then gives as output “True”.

- Otherwise, if the algorithm **A1** gives as output an empty list – i.e there exists no real solution for the given equation – then gives as output “False”.

Let r,s be real constants and d=|s-r|. Consider the following quadratic equation with real coefficients:

$$x^2+\sqrt{d}\,x+d=0 \qquad (\boldsymbol{E1})$$

Given an instance of the problem **P1** consisting of **E1**, the algorithm **A5** gives the following output:

- Either “True”, i.e there exists a real solution for **E1** – therefore, letting Δ be the discriminant of **E1**:

$$\Delta \geq 0 \Rightarrow \left(\sqrt{d}\right)^2 - 4d \geq 0 \Rightarrow d - 4d \geq 0 \Rightarrow -3d \geq 0 \Rightarrow d \leq 0 \Rightarrow$$
$$|s-r| \leq 0 \Rightarrow |s-r| = 0 \Rightarrow s-r=0 \lor r-s=0 \Rightarrow s=r$$

- Or “False”, i.e there exists no real solution for **E1** – therefore, letting Δ be the discriminant of **E1**:

$$\Delta < 0 \Rightarrow \left(\sqrt{d}\right)^2 - 4d < 0 \Rightarrow d - 4d < 0 \Rightarrow -3d < 0 \Rightarrow d > 0 \Rightarrow$$
$$|s-r| > 0 \Rightarrow s > r \lor s < r \Rightarrow s \neq r$$

The real constants r and s have no particular properties. The same is true for the algorithm **A5**, except that it uses the output of the algorithm **A1**, which has also no particular properties, except the assumed form of its output. Therefore, for all constants s, r $\in \mathbb{R}$ the equation **E1** can be constructed and given as input to the algorithm **A5** whose output then implies that either s=r or s≠*r*, which contradicts with **Thm1**. We reached a contradiction because we assumed that there exists an algorithm such **A1**. Therefore, there exists no such algorithm. ▪

4. There exists no algorithm such A2

Proof

Assuming that there exists an algorithm such **A2**, via the assumption c) in section 1, it follows that there exists another algorithm **(A6)** that simulating the former, given any quadratic equation with real coefficients, decides the decision problem **P2**. If the form of output of the algorithm **A2** is identical with the one in the example of the assumption c) in section 1, then the algorithm **A6** decides **P2** as follows:

- If the algorithm **A2** gives as output a list with one element, then gives as output “True”.

- Otherwise, if the algorithm **A2** gives as output a list with two elements, then gives as output “False”.

Let r,s be real constants and d=|s-r|. Consider the quadratic equation with real coefficients **E1**.

Given an instance of the problem **P2** consisting of **E1**, the algorithm **A6** gives the following output:

- Either "True", i.e there exists exactly one complex solution for **E1** in ℂ– therefore, letting Δ be the discriminant of **E1**:

$$\Delta = 0 \Rightarrow \left(\sqrt{d}\right)^2 - 4d = 0 \Rightarrow d - 4d = 0 \Rightarrow -3d = 0 \Rightarrow d = 0 \Rightarrow$$
$$|s-r| = 0 \Rightarrow s-r=0 \vee r-s=0 \Rightarrow s=r$$

- Or "False", i.e there exist exactly two (distinct) complex solutions for **E1** in ℂ– therefore, letting Δ be the discriminant of **E1**:

$$\Delta \neq 0 \Rightarrow \left(\sqrt{d}\right)^2 - 4d \neq 0 \Rightarrow d - 4d \neq 0 \Rightarrow -3d \neq 0 \Rightarrow d \neq 0 \Rightarrow$$
$$|s-r| \neq 0 \Rightarrow s-r \neq 0 \vee r-s \neq 0 \Rightarrow s \neq r$$

The real constants r and s have no particular properties. The same is true for the algorithm **A6**, except that it uses the output of the algorithm **A2**, which has also no particular properties, except the assumed form of its output. Therefore, for all constants s, r ∈ ℝ the equation **E1** can be constructed and given as input to the algorithm **A6** whose output then implies that either s=r or s≠*r*, which contradicts with **Thm1**. We reached a contradiction because we assumed that there exists an algorithm such **A2**. Therefore, there exists no such algorithm. ▪

References

Halatsis C. (2003). Themeleioseis Epistimis I/Y, Volume II - Theoria Ypologismou. Patras, Hellenic Open University.