Closed forms, (including Gelfond's Constant), using MKB constant like integrals.

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Abstract

First we will consider

\[
\int_1^\infty \cos(\pi i x) x^{1/x} \, dx \quad \text{and} \quad \int_1^\infty \sin(\pi i x) x^{1/x} \, dx,
\]

where it appears that the limit of the ratio of \(a\) to \(a - 1\), as \(a\) goes to infinity,

\[
\frac{\text{integral}(\sin(\pi i x) x^{1/x}, \{x, 1, a\})}{\text{integral}(\sin(\pi i x) x^{1/x}, \{x, 1, a - 1\})} \quad \text{and}
\]

\[
\frac{\text{integral}(\cos(\pi i x) x^{1/x}, \{x, 1, a\})}{\text{integral}(\cos(\pi i x) x^{1/x}, \{x, 1, a - 1\})}
\]

is Gelfond's Constant, \((e^{\pi})\). We will consider that the hypothesis and provide hints for a proof using L’Hospital’s Rule, since we have indeterminate forms as \(a\) goes to infinity.

We shall compare the ratios of

\[
\int_1^\infty \cos(\pi i x) x^{1/x} \, dx \quad \text{and} \quad \int_1^\infty \sin(\pi i x) x^{1/x} \, dx
\]

with

\[
\int_1^\infty \cos(\pi x) x^{1/x} \, dx \quad \text{and} \quad \int_1^\infty \sin(\pi x) x^{1/x} \, dx
\]

In later parts of this paper we will look at the ratio of three deceptively similar looking integrals

which have the terms of MKB for integrands. (For reference we call these the "later integrals.")

Finally after finding no closed form for the ratios of the later integrals, we will settle for closed forms of their differences.

Preliminaries

Below, and throughout this paper, we will show the best known (to the author) Mathematica 11 options, found to give the desired results. Many of the computations took several minutes and produced a few warning messages which are not displayed.

In 1999\(^1\), the constant referred to at https://oeis.org/A037077, Limit[Sum[(-1)^n n^(1/n),{n,1,2x}],x->Infinity] \(^2\), was named the MRB constant, (after its original investigator), by Simon Plouffe\(^3\). Then on Feb 23, 2009, Marvin Ray Burns named
https://oeis.org/A157852 “MKB constant” (MKB) after his wife at the time. Technically, A157852 is the integer sequence of the digits of MKB, (the integral analog of the MRB constant\textsuperscript{4}). Hence MKB was named after one with a close relationship to the person the MRB constant was named after.

MKB= \[ \lim_{n \to \infty} \int_1^n e^{\pi x} x^{1/3} \, dx \].

It appears that

\[ \lim_{n \to \infty} \int_1^n e^{\pi x} x^{1/3} \, dx = \lim_{n \to \infty} \int_1^n \cos(\pi x) \, x^{1/3} \, dx + \]

\[ I \lim_{n \to \infty} \int_1^n \sin(\pi x) \, x^{1/3} \, dx \]

and

\[ \lim_{n \to \infty} \int_1^n e^{\pi x} x^{1/3} \, dx = \lim_{n \to \infty} \int_1^n \cos(\pi x) \, x^{1/3} \, dx + \]

\[ I \lim_{n \to \infty} \int_1^n \sin(\pi x) \, x^{1/3} \, dx, \quad a \in \mathbb{N}. \]

They are indicated to be true in the result to the following Mathematica code.

\[ \text{Needs} \] NumericalCalculus`

\[ \text{Table} \{ \text{NIntegrate} \left[ x\left(1/x\right) \exp[I \pi x], \{x, 1, 2 n\}, \text{WorkingPrecision} \to d \right] - \]

\[ (\text{NIntegrate} \left[ x\left(1/x\right) \cos(\pi x), \{x, 1, 2 n\}, \text{WorkingPrecision} \to d \right] + \]

\[ I \text{NIntegrate} \left[ x\left(1/x\right) \sin(\pi x), \{x, 1, 2 n\}, \text{WorkingPrecision} \to d \right] \}, \{n, 1, 21\} \]
digits = 50;
Rationalize[NLimit[NIntegrate[Exp[Pi*I*x]*x^(1/x), {x, 1, 2*a}],
   WorkingPrecision -> 50, PrecisionGoal -> 50, MaxRecursion -> 50], a -> Infinity,
   WorkingPrecision -> 50, Terms -> 15, Method -> SequenceLimit, WynnDegree -> 6], 0] -
   (NIntegrate[x^(1/x)*Cos[Pi*x], {x, 1, Infinity}, WorkingPrecision -> 2*digits] +
   I*NIntegrate[x^(1/x)*Sin[Pi*x], {x, 1, Infinity}, WorkingPrecision -> 2*digits] - I/Pi)

The following plots are hard to interpret, but here are what NIntegrate[x^(1/x)*Cos[Pi*x],{x,1,a}] and
NIntegrate[x^(1/x)*Cos[Pi*x],{x,1,a}] for a from 1 to 10 look like. At least we notice that the cosine plot is
shifted on both the x and f(x) axes, and they have the same, or about the same period of about 2.

In[30] := Plot[Integrate[Sin[Pi*x]*x^(1/x), {x, 1, a}], {a, 1, 10}]

Out[30] =

In[25] := Plot[Integrate[Cos[Pi*x]*x^(1/x), {x, 1, a}], {a, 1, 10}]

Out[25] =

For more precise measurements see Mathar’s work⁵, where Mathar also used many methods to
compute MKB. He called it M1, published many error-bounds of methods, and compared M1 to what he
called the MRB constant, M. R. Burns’ constant, and M.
Tools

We will use many integration and limit options available in Mathematica 11.0. We also use a Intel 6 core 3.5 GH extreme edition desktop for the computations.

Operations and Results

Realizing his mistake of confusing \( \int \sin(\pi x) x^{1/x}, \{x,1,a\} \) with \( \int \sin(\pi x^a) x^{1/x}, \{x,1,a\} \), through the following operations, on Monday, Aug 8, 2016 at 2:00PM, Marvin Ray Burns began to see that the limit of the ratio of a to a-1, as a goes to infinity, in \( \frac{\int \sin(\pi x^a) x^{1/x}, \{x,1,a\}}{\int \sin(\pi x) x^{1/x}, \{x,1,a-1\}} \) and \( \frac{\int -\cos(\pi x^a) x^{1/x}, \{x,1,a\}}{\int -\cos(\pi x) x^{1/x}, \{x,1,a-1\}} \) is Gelfond’s Constant, \((e^\pi)\).

\[
i_1 = \text{Table}\left[
\text{NIntegrate}\left[\sin(\pi x) x^{1/x}, \{x,1,a\}, \text{WorkingPrecision} \rightarrow 20\right], \{a, 9990, 10001\}\right]
\]

\[
i_2 = \text{Ratios}[i_1]
\]

\[
\text{N}\left[e^\pi, 20\right] = 23.140692632799006
\]

\[
\text{N}[i_2 - e^\pi, 20]
\]

Larger a: (Both sine and cosine in the this operation give Gelfond’s Constant.)
\textbf{Table \& NIntegrate} 
\begin{align*}
&\text{Cos[Pi*I*x] x^{(1/x)}, \{x, 1, a\}, \text{WorkingPrecision} \rightarrow 30]}, \\
&\{a, 999\,990, 1\,000\,000\} \\
&[8.16402704299180409525949988 \times 10^{1364}, \\
&1.889212445149992403409339252 \times 10^{1364}, \\
&4.3717683568857590343446623243 \times 10^{1364}, \\
&2.54184547494103980098507246551 \times 10^{1364}, \\
&5.417341377427506816005476577 \times 10^{1364}, \\
&1.25361031700280490130583526513 \times 10^{1364}, \\
&2.900941826671118226671546873 \times 10^{1364}, \\
&6.7129786401755096445850807654 \times 10^{1364}, \\
&1.55342975360723204911512762411 \times 10^{1364}].
\end{align*}

\textbf{i12 = Ratios[i11]}
\begin{align*}
&[23.1406926324827036186672961841, 23.1406926324827041886616877912, \\
&23.1406926324827047586544156856, 23.140692632482705328645798737, \\
&23.1406926324827058986348803621, 23.1406926324827064686226171574, \\
&23.140692632482707386086902659, 23.140692632482708059396944, \\
&23.1406926324827081785758454491, 23.140692632482708748556927368].
\end{align*}

\textbf{N[E^Pi, 20]}
\[
23.140692632779269006
\]

\textbf{N[i12 - E^Pi, 20]}
\begin{align*}
&[-2.96565387067901839 \times 10^{-10}, -2.965648170673985768 \times 10^{-10}, \\
&-2.965642470746706824 \times 10^{-10}, -2.965636770836064943 \times 10^{-10}, \\
&-2.96562531064692106 \times 10^{-10}, -2.965619671203961020 \times 10^{-10}, \\
&-2.965608271532409188 \times 10^{-10}, -2.9656025712158312 \times 10^{-10}].
\end{align*}

\textbf{Even larger a, (about as big as Mathematica will tolerate)!}
\textbf{i21} = \textbf{Table}\{\textbf{NIntegrate}[\textbf{Cos}[\pi * x] x^\left(1/x\right), \{x, 1, a\}, \textbf{WorkingPrecision} \to 40], \{a, 9999990, 10000008\}\}

\begin{align*}
1.248758590784479767689088244974362158446 \times 10^{13} &\quad 4.3721204870011523727579796015963694 \times 10^{13} \\
2.88965356340081991397731740570292209 \times 10^{13} &\quad 7.52088716874064215569412546222015226236 \times 10^{13} \\
6.8695521195606949415827880642155765176 \times 10^{13} &\quad 1.5474077520922161463879355921643432162 \times 10^{13} \\
1.5474077520922161463879355921643432162 \times 10^{13} &\quad 3.58008716874064215569412546222015226236 \times 10^{13} \\
8.2623989394650135109671528590858985574115 \times 10^{13} &\quad 1.91749318791581179979519151425441918062 \times 10^{13} \\
9.1749318791581179979519151425441918062 \times 10^{13} &\quad 4.3721204870011523727579796015963694 \times 10^{13} \\
1.026801601654186402948901558249664519330 \times 10^{13} &\quad 2.3709002587213927292657934298398022129 \times 10^{13} \\
2.3709002587213927292657934298398022129 \times 10^{13} &\quad 5.498436956114034693155692172304668487 \times 10^{13} \\
5.498436956114034693155692172304668487 \times 10^{13} &\quad 1.272376381629768218138540861177504926 \times 10^{13} \\
1.272376381629768218138540861177504926 \times 10^{13} &\quad 2.94436707604978697140035525664234236912 \times 10^{13} \\
2.94436707604978697140035525664234236912 \times 10^{13} &\quad 6.8134693943284237431892945563446297757 \times 10^{13} \\
6.8134693943284237431892945563446297757 \times 10^{13} &\quad 1.5766840002603457160089104782265334778 \times 10^{13} \\
1.5766840002603457160089104782265334778 \times 10^{13} &\quad 3.648555982361789107489894116297179355389 \times 10^{13} \\
3.648555982361789107489894116297179355389 \times 10^{13} &\quad 8.44301126513984175383695730586199018495 \times 10^{13} \\
8.44301126513984175383695730586199018495 \times 10^{13} &\quad 1.95745374618293691296326908031257718779721784119022 \times 10^{13} \\
1.95745374618293691296326908031257718779721784119022 \times 10^{13} &\quad 4.52116207408743113708968531798126596848 \times 10^{13} \\
4.52116207408743113708968531798126596848 \times 10^{13} &\quad \text{\textbf{Closed forms, (including Gelfond's Constant), from MKB constant like integrals V 2.nb}}
\end{align*}
When we switch forms of the operations, Mathematica can give at least 42 digits of accuracy here. Again we will need the "NumericalCalculus" package.

\[ \text{Needs["NumericalCalculus"]} \]

\[ e^\pi - \text{Rationalize[\text{NLimit[\text{NIntegrate[Cos[Pi*I*x] x^(1/x)], \{x, 1, a\}, WorkingPrecision \to 50, PrecisionGoal \to 50, MaxRecursion \to 50\} / \text{NIntegrate[Cos[Pi*I*x] x^(1/x)], \{x, 1, a - 1\}, WorkingPrecision \to 50, PrecisionGoal \to 50, MaxRecursion \to 50\}], a \to \text{Infinity}, WorkingPrecision \to 50, Terms \to 15, Method \to \text{SequenceLimit, WynnDegree \to 6}], 0], 50] \]
\[ -5.9702708245087317429519268920016913700154824404879 \times 10^{-42} \]

\[ e^\pi - \text{Rationalize[\text{NLimit[\text{NIntegrate[Sin[Pi*I*x] x^(1/x)], \{x, 1, a\}, WorkingPrecision \to 50, PrecisionGoal \to 50, MaxRecursion \to 50\} / \text{NIntegrate[Sin[Pi*I*x] x^(1/x)], \{x, 1, a - 1\}, WorkingPrecision \to 50, PrecisionGoal \to 50, MaxRecursion \to 50\}], a \to \text{Infinity}, WorkingPrecision \to 50, Terms \to 15, Method \to \text{SequenceLimit, WynnDegree \to 6}], 0], 50] \]
\[ -5.9702708245087317429519268920016913700154824404879 \times 10^{-42} \]

The following should help in a proof of the hypothesis: \( \cos[\pi i x] = \cosh[\pi x], \sin[\pi i x] = i \sinh[\pi x], \) and \( \text{Limit}[x^{1/x}, x \to \text{Infinity}] = 1. \)

Using L'Hospital's Rule, we have the following:

\[ e^\pi - \left( \cosh(\pi) - \lim_{a \to \infty} \left( \cos(\pi i) - \frac{x^{1/x} \cos(\pi i x) / \cdot x \to a}{x^{1/x} \cos(\pi i x) / \cdot x \to a - 1} \right) \right) \]
\[ 0 \]

\[ e^\pi - \left( i \sin(1) - \lim_{a \to \infty} \left( \frac{x^{1/x} \sinh(\pi x) / \cdot x \to a + \sinh(i)}{x^{1/x} \sinh(\pi x) / \cdot x \to a - 1} \right) \right) \]
\[ 0 \]

If we perform the same operations on only the integrals without the "I," (the later integrals), we get the following results, (proof not provided, since there are no known closed form solutions for two of the three ratios of the later integrals). Nonetheless, we will make a passing note to a few approximations.

\[ \text{Needs["NumericalCalculus"]} \]
That is close to the following:

\[
(137250424206313 \, e + 2085856139684320 \, e^2) / 1093465977254055
\]

and

\[
(70064463976981915 \, e^{\pi/2} + 2940546676813034 \, e^\pi) / 28060517609691197
\]

And for smaller coefficients it is also close the following:

\[
\frac{1}{28} (-70 \, e - 97 \, e^2 - 151 \, e^3 - 3 \, e^4 - 58 \, e^5 - 21 \, e^6 - 93 \, e^7 + 24 \, e^8 - 12 \, e^9 - 49 \, e^{10} - 37 \, e^{11} + 19 \, e^{12} - 48 \, e^{13} + 141 \, e^{14} - 87 \, e^{15} - 11 \, e^{16} - 58 \, e^{17} - 46 \, e^{18} - 31 \, e^{19} + 21 \, e^{20})
\]

and

\[
\frac{1}{38} (91 \, e^{\pi/20} - 168 \, e^{\pi/19} + 40 \, e^{\pi/18} + 80 \, e^{\pi/17} - 6 \, e^{\pi/16} + 8 \, e^{\pi/15} + 121 \, e^{\pi/14} + 12 \, e^{\pi/13} - 8 \, e^{\pi/12} + 21 \, e^{\pi/11} + 69 \, e^{\pi/10} - 11 \, e^{\pi/9} + 81 \, e^{\pi/8} + 72 \, e^{\pi/7} + 38 \, e^{\pi/6} + 56 \, e^{\pi/5} - 22 \, e^{\pi/4} - 38 \, e^{\pi/3} - 24 \, e^{\pi/2} - 5 \, e^\pi)
\]

We arrive at about 1. The random digits could be some hard to avoid round of error, but we make no claim of certianty here.

Next, we will perform our operations on the MKB constant, mentioned at the top of this paper. For reference we will call its integrand "e."
Needs["NumericalCalculus`"]


5.15808219431339787028600589211 - 6.211244321438292793271496484947123567

That is close to the following:

\[
\frac{\left(3635\,966\,283\,803\,821\,e + 902\,615\,612\,958\,241\,e^2\right)}{\left(1\,211\,008\,478\,422\,557\,e + 27\,332\,794\,326\,472\,e^2\right)} / 562\,500\,154\,689\,724
\]

and

\[
\frac{\left(-1\,515\,494\,137\,444\,387\,e^{\pi/2} + 389\,348\,082\,242\,624\,e^\pi\right)}{\left(437\,182\,811\,555\,465\,e^{\pi/2} - 2\,053\,701\,405\,670\,e^\pi\right)} / 330\,937\,546\,000\,493
\]

And for smaller coefficients it is also close the following:

\[
\frac{1}{2} \left(41\,e^{\pi/24} + 16\,e^{\pi/23} + 70\,e^{\pi/22} - 30\,e^{\pi/21} - 55\,e^{\pi/20} - 12\,e^{\pi/19} + 42\,e^{\pi/18}\right)
\]

\[
\frac{1}{6} \left(-5\,e^{\pi/27} - 30\,e^{\pi/26} + 11\,e^{\pi/25} - 6\,e^{\pi/24} - 34\,e^{\pi/23} - 25\,e^{\pi/22} + 20\,e^{\pi/21} + 50\,e^{\pi/20}\right)
\]

And

\[
\frac{1}{3} \left(-11\,e - 12\,e^2 - 26\,e^3 - 34\,e^4 + 19\,e^5 + 13\,e^6 - 13\,e^7 + 19\,e^8 - 52\,e^9 + 26\,e^{10} + 36\,e^{11} - 30\,e^{12} + 14\,e^{13} + 6\,e^{14} - 16\,e^{15} - 45\,e^{16} - 8\,e^{17} + 28\,e^{18} - 28\,e^{20} + 18\,e^{21} + 8\,e^{22} + 12\,e^{23} + 34\,e^{24} - 12\,e^{25} - 26\right)
\]

\[
\frac{1}{4} \left(-18\,e + 49\,e^2 - 23\,e^3 + 31\,e^4 + 4\,e^5 + 78\,e^6 + 27\,e^7 - 23\,e^8 + 43\,e^9 - 22\,e^{10} - 6\,e^{11} - 19\,e^{12} - 51\,e^{13} + 2\,e^{14} + 22\,e^{15} - 6\,e^{16} - 44\,e^{17} + 21\,e^{18} - 58\,e^{19} + 7\,e^{20} - 15\,e^{21} - 41\,e^{22} + 5\,e^{23} + 28\,e^{24} + 26\,e^{25} + 14\,e^{26}\right)
\]

Using Mathematica, the limit of the differences of the later integrals appear to go by the following rule: cosine->0; sine-> -2/Pi, and e-> -2/Pi * I; (reaching as far as 20 to 28 digits of precision).

Needs["NumericalCalculus`"]
Open Question

Do there exist closed form solutions for the ratios of the later integrals, specifically those with the integrands sine and e? And maybe we might ask about cosine too.

Sumary

The limits of ratios of the hyperbolic integrals and the limits of limits of the differences of the later integrals had exact solutions that were easy to recognize, while we have not found exact values for the limits of ratios of the later integrals, except, perhaps cosine which seems to give 1. The limits of differences of the later integrals have known closed forms.

References

1 S. R. Finch, Mathematical Constants, Cambridge, 2003, pp 450, 452
3 http://www.plouffe.fr/simon/articles/Tableofconstants.pdf
4 www.people.fas.harvard.edu/~sfinch/csolve/erradd.pdf p. 64.
5 http://arxiv.org/abs/0912.3844