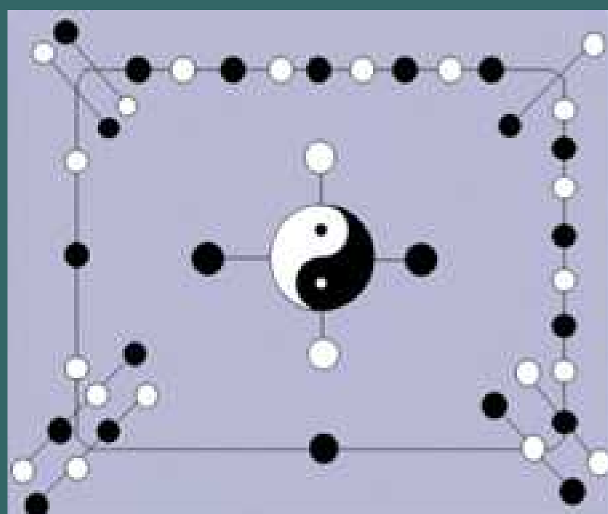
ISSN 1937 - 1055

VOLUME 2, 2016

INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

June, 2016

Vol.2, 2016 ISSN 1937-1055

International Journal of

Mathematical Combinatorics

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

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Aims and Scope: The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

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Famous Words:

If you would go up high, then use your own legs! Do not let yourselves carried aloft; do not seat yourselves on other people's backs and heads.

By F.W.Nietzsche, a German philosopher.

International J.Math. Combin. Vol.2(2016), 1-15

Smarandache Curves and Applications According to Type-2 Bishop Frame in Euclidean 3-Space

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Abstract: In this paper, we investigate Smarandache curves according to type-2 Bishop frame in Euclidean 3- space and we give some differential geometric properties of Smarandache curves. Also, some characterizations of Smarandache breadth curves in Euclidean 3-space are presented. Besides, we illustrate examples of our results.

Key Words: Smarandache curves, Bishop frame, curves of constant breadth.

AMS(2010): 53A05, 53B25, 53B30.

§1. Introduction

A regular curve in Euclidean 3-space, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve. M. Turgut and S. Yılmaz have defined a special case of such curves and call it Smarandache TB_2 curves in the space E_1^4 [10]. Moreover, special Smarandache curves have been investigated by some differential geometric [6]. A.T.Ali has introduced some special Smarandache curves in the Euclidean space [2]. Special Smarandache curves according to Sabban frame have been studied by [5]. Besides, It has been determined some special Smarandache curves E_1^3 by [12]. Curves of constant breadth were introduced by L.Euler [3].

We investigate position vector of curves and some characterizations case of constant breadth according to type-2 Bishop frame in E^3.

§2. Preliminaries

The Euclidean 3-space E^3 proved with the standard flat metric given by

$$<,>= dx_1^2 + dx_2^2 + dx_3^2$$

[1]Received November 26, 2015, Accepted May 6, 2016.

where (x_1, x_2, x_3) is rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $a \in E^3$ given by $\|a\| = \sqrt{< a, a >}$. φ is called a unit speed curve if velocity vector υ of φ satisfied $\|\upsilon\| = 1$

The Bishop frame or parallel transport frame is alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of orthonormal frame along a curve simply by parallel transporting each component of the frame [8]. The type-2 Bishop frame is expressed as

$$\begin{bmatrix} \xi_1' \\ \xi_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\varepsilon_1 \\ 0 & 0 & -\varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix} \tag{2.1}$$

In order to investigate type-2 Bishop frame relation with Serret-Frenet frame, first we

$$B' = -\tau N = \varepsilon_1 \xi_1 + \varepsilon_2 \xi_2 \tag{2.2}$$

Taking the norm of both sides, we have

$$\kappa(s) = \frac{d\theta(s)}{ds}, \qquad \tau(s) = \sqrt{\varepsilon_1^2 + \varepsilon_2^2} \tag{2.3}$$

Moreover, we may express

$$\varepsilon_1(s) = -\tau \cos\theta(s), \qquad \varepsilon_2(s) = -\tau \sin\theta(s) \tag{2.4}$$

By this way, we conclude $\theta(s) = Arc\tan\dfrac{\varepsilon_2}{\varepsilon_1}$. The frame $\{\xi_1, \xi_2, B\}$ is properly oriented, and τ and $\theta(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve $\alpha(s)$.

We write the tangent vector according to frame $\{\xi_1, \xi_2, B\}$ as

$$T = \sin\theta(s)\xi_1 - \cos\theta(s)\xi_2$$

and differentiate with respect to s

$$\begin{aligned} T' = \kappa N = \; & \theta'(s)(\cos\theta(s)\xi_1 + \sin\theta(s)\xi_2) \\ & + \sin\theta(s)\xi_1' - \cos\theta(s)\xi_2' \end{aligned} \tag{2.5}$$

Substituting $\xi_1' = -\varepsilon_1 B$ and $\xi_2' = -\varepsilon_2 B$ in equation (2.5) we have

$$\kappa N = \theta'(s)(\cos\theta(s)\xi_1 + \sin\theta(s)\xi_2)$$

In the above equation let us take $\theta'(s) = \kappa(s)$. So we immediately arrive at

$$N = \cos\theta(s)\xi_1 + \sin\theta(s)\xi_2$$

Considering the obtained equations, the relation matrix between Serret-Frenet and the type-2 Bishop frame can be expressed

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix} \tag{2.6}$$

§3. Smarandache Curves According to Type-2 Bishop Frame in $\mathbf{E}^3$

Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and denote by $\{\xi_1^\alpha, \xi_2^\alpha, B^\alpha\}$ the moving Bishop frame along the curve α. The following Bishop formulae is given by

$$\dot{\xi_1^\alpha} = -\varepsilon_1^\alpha B^\alpha, \quad \dot{\xi_2^\alpha} = -\varepsilon_2^\alpha B^\alpha, \quad \dot{B^\alpha} = \varepsilon_1^\alpha \xi_1^\alpha + \varepsilon_2^\alpha \xi_2^\alpha$$

3.1 $\xi_1\xi_2$-Smarandache Curves

Definition 3.1 *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{\xi_1^\alpha, \xi_2^\alpha, B^\alpha\}$ be its moving Bishop frame. $\xi_1\xi_2$-Smarandache curves can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\xi_1^\alpha + \xi_2^\alpha) \tag{3.1}$$

Now, we can investigate Bishop invariants of $\xi_1\xi_2$-Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.1.1) with respect to s, we get

$$\begin{aligned} \dot{\beta} = \frac{d\beta}{ds^*} \cdot \frac{ds^*}{ds} &= \frac{-1}{\sqrt{2}}(\varepsilon_1^\alpha + \varepsilon_2^\alpha)B^\alpha \\ T_\beta \cdot \frac{ds^*}{ds} &= \frac{-1}{\sqrt{2}}(\varepsilon_1^\alpha + \varepsilon_2^\alpha)B^\alpha \end{aligned} \tag{3.2}$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\varepsilon_1^\alpha + \varepsilon_2^\alpha) \tag{3.3}$$

The tangent vector of curve β can be written as follow;

$$T_\beta = -B^\alpha = -(\varepsilon_1^\alpha \xi_1^\alpha + \varepsilon_2^\alpha \xi_2^\alpha) \tag{3.4}$$

Differentiating (3.4) with respect to s, we obtain

$$\frac{dT_\beta}{ds^*} \cdot \frac{ds^*}{ds} = \varepsilon_1^\alpha \xi_1^\alpha + \varepsilon_2^\alpha \xi_2^\alpha \tag{3.5}$$

Substituting (3.3) in (3.5), we get

$$T^{'}_{\beta}=\frac{\sqrt{2}}{\varepsilon_1^{\alpha}+\varepsilon_2^{\alpha}}(\varepsilon_1^{\alpha}\xi_1^{\alpha}+\varepsilon_2^{\alpha}\xi_2^{\alpha})$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\left\|T^{'}_{\beta}\right\|=\kappa_{\beta}=\frac{\sqrt{2}}{\varepsilon_1^{\alpha}+\varepsilon_2^{\alpha}}\sqrt{(\varepsilon_1^{\alpha})^2+(\varepsilon_2^{\alpha})^2}$$

$$N_{\beta}=\frac{1}{\sqrt{(\varepsilon_1^{\alpha})^2+(\varepsilon_2^{\alpha})^2}}(\varepsilon_1^{\alpha}\xi_1^{\alpha}+\varepsilon_2^{\alpha}\xi_2^{\alpha})$$

On the other hand, we express

$$B_{\beta}=\frac{1}{\sqrt{(\varepsilon_1^{\alpha})^2+(\varepsilon_2^{\alpha})^2}}\det\begin{bmatrix}\xi_1^{\alpha} & \xi_2^{\alpha} & B^{\alpha}\\ 0 & 0 & -1\\ \varepsilon_1^{\alpha} & \varepsilon_2^{\alpha} & 0\end{bmatrix}.$$

So, the binormal vector of curve β is

$$B_{\beta}=\frac{1}{\sqrt{(\varepsilon_1^{\alpha})^2+(\varepsilon_2^{\alpha})^2}}(\varepsilon_2^{\alpha}\xi_1^{\alpha}-\varepsilon_1^{\alpha}\xi_2^{\alpha})$$

We differentiate $(3.2)_1$ with respect to s in order to calculate the torsion of curve β

$$\begin{aligned}\ddot{\beta}= & \tfrac{-1}{\sqrt{2}}\{[(\varepsilon_1^{\alpha})^2+\varepsilon_1^{\alpha}\varepsilon_2^{\alpha}]\xi_1^{\alpha}\\ & +[\varepsilon_1^{\alpha}\varepsilon_2^{\alpha}+(\varepsilon_2^{\alpha})^2]\xi_2^{\alpha}+[\dot{\varepsilon}_1^{\alpha}+\dot{\varepsilon}_2^{\alpha}]\}B^{\alpha}]\end{aligned}$$

and similarly

$$\dddot{\beta}=\frac{-1}{\sqrt{2}}(\delta_1\xi_1^{\alpha}+\delta_2\xi_2^{\alpha}+\delta_3B^{\alpha})$$

where

$$\begin{aligned}\delta_1= & 3\varepsilon_1^{\alpha}\dot{\varepsilon}_1^{\alpha}+\dot{\varepsilon}_1^{\alpha}\varepsilon_2^{\alpha}+2\varepsilon_1^{\alpha}\dot{\varepsilon}_2^{\alpha}-(\varepsilon_1^{\alpha})^3-(\varepsilon_1^{\alpha})^2\varepsilon_2^{\alpha}\\ \delta_2= & 2\dot{\varepsilon}_1^{\alpha}\varepsilon_2^{\alpha}+\varepsilon_1^{\alpha}\dot{\varepsilon}_2^{\alpha}+3\varepsilon_2^{\alpha}\dot{\varepsilon}_2^{\alpha}-\varepsilon_1^{\alpha}(\varepsilon_2^{\alpha})^2-(\varepsilon_2^{\alpha})^3\\ \delta_3= & \ddot{\varepsilon}_1^{\alpha}+\ddot{\varepsilon}_2^{\alpha}\end{aligned}$$

The torsion of curve β is

$$\tau_{\beta}=\frac{\varepsilon_1^{\alpha}+\varepsilon_2^{\alpha}}{4\sqrt{2}[(\varepsilon_1^{\alpha})^2+(\varepsilon_2^{\alpha})^2]}\{[(\varepsilon_1^{\alpha}+\varepsilon_2^{\alpha})(\varepsilon_1^{\alpha}\varepsilon_2^{\alpha}+(\varepsilon_2^{\alpha})^2]\delta_1-[(\varepsilon_1^{\alpha}+\varepsilon_2^{\alpha})((\varepsilon_1^{\alpha})^2+\varepsilon_1^{\alpha}\varepsilon_2^{\alpha})]\delta_2\}$$

3.2 ξ_1B-Smarandache Curves

Definition 3.2 *Let $\alpha=\alpha(s)$ be a unit speed regular curve in E^3 and $\{\xi_1^{\alpha},\xi_2^{\alpha},B^{\alpha}\}$ be its moving*

Bishop frame. $\xi_1 B$-Smarandache curves can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\xi_1^\alpha + B^\alpha) \tag{3.6}$$

Now, we can investigate Bishop invariants of $\xi_1 B$-Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.6) with respect to s, we get

$$\begin{aligned} \dot{\beta} = \frac{d\beta}{ds^*} \cdot \frac{ds^*}{ds} &= \frac{-1}{\sqrt{2}}(\varepsilon_1^\alpha B^\alpha + \varepsilon_1^\alpha \xi_1^\alpha + \varepsilon_2^\alpha \xi_2^\alpha) \\ T_\beta \cdot \frac{ds^*}{ds} &= \frac{-1}{\sqrt{2}}(-\varepsilon_1^\alpha B^\alpha + \varepsilon_1^\alpha \xi_1^\alpha + \varepsilon_2^\alpha \xi_2^\alpha) \end{aligned} \tag{3.7}$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2\left(\varepsilon_1^\alpha\right)^2 + \left(\varepsilon_2^\alpha\right)^2}{2}} \tag{3.8}$$

The tangent vector of curve β can be written as follow;

$$T_\beta = \frac{1}{\sqrt{2\left(\varepsilon_1^\alpha\right)^2 + \left(\varepsilon_2^\alpha\right)^2}}(\varepsilon_1^\alpha \xi_1^\alpha + \varepsilon_2^\alpha \xi_2^\alpha - \varepsilon_1^\alpha B^\alpha) \tag{3.9}$$

Differentiating (3.9) with respect to s, we obtain

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\left[2\left(\varepsilon_1^\alpha\right)^2 + \left(\varepsilon_2^\alpha\right)^2\right]^{\frac{3}{2}}}(\mu_1 \xi_1^\alpha + \mu_2 \xi_2^\alpha + \mu_3 B^\alpha) \tag{3.10}$$

where

$$\begin{aligned} \mu_1 &= \varepsilon_1^\alpha \varepsilon_2^\alpha \dot{\varepsilon}_2^\alpha + \dot{\varepsilon}_1^\alpha \left(\varepsilon_2^\alpha\right)^2 \\ \mu_2 &= 2\left(\varepsilon_2^\alpha\right)^2 \varepsilon_2^\alpha \text{-} 2\varepsilon_1^\alpha \dot{\varepsilon}_1^\alpha \varepsilon_2^\alpha + 2\left(\varepsilon_1^\alpha\right)^2 \varepsilon_2^\alpha \text{-} 2\left(\varepsilon_1^\alpha\right)^3 \varepsilon_2^\alpha \text{-} \varepsilon_1^\alpha \left(\varepsilon_2^\alpha\right)^3 \\ \mu_3 &= \varepsilon_1^\alpha \varepsilon_2^\alpha \dot{\varepsilon}_2^\alpha - 2\left(\varepsilon_1^\alpha\right)^4 + \left(\varepsilon_1^\alpha\right)^2 \left(\varepsilon_2^\alpha\right)^2 - \dot{\varepsilon}_1^\alpha \left(\varepsilon_2^\alpha\right)^2 \end{aligned}$$

Substituting (3.8) in (3.10), we have

$$T_\beta^{'} = \frac{\sqrt{2}}{\left[2\left(\varepsilon_1^\alpha\right)^2 + \left(\varepsilon_2^\alpha\right)^2\right]^2}(\mu_1 \xi_1^\alpha + \mu_2 \xi_2^\alpha + \mu_3 B^\alpha)$$

Then, the first curvature and principal normal vector field of curve β are respectively

$$\left\|T_\beta^{'}\right\| = \kappa_\beta = \frac{\sqrt{2}}{\left[2\left(\varepsilon_1^\alpha\right)^2 + \left(\varepsilon_2^\alpha\right)^2\right]^2}\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2}$$

$$N_\beta = \frac{1}{\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2}}(\mu_1 \xi_1^\alpha + \mu_2 \xi_2^\alpha + \mu_3 B^\alpha)$$

On the other hand, we get

$$B_\beta = \frac{1}{\sqrt{\mu_1^2+\mu_2^2+\mu_3^2}\sqrt{2\left(\varepsilon_1^\alpha\right)^2+\left(\varepsilon_2^\alpha\right)^2}}[(\mu_2\varepsilon_1^\alpha+\mu_3\varepsilon_2^\alpha)\xi_1^\alpha$$

$$-\left(\mu_1\xi_1^\alpha+\mu_3\xi_1^\alpha\right)\xi_2^\alpha+\left(\mu_2\varepsilon_1^\alpha\text{-}\mu_1\varepsilon_2^\alpha\right)B^\alpha]$$

We differentiate (3.7) with respect to s in order to calculate the torsion of curve β

$$\ddot{\beta} = \tfrac{-1}{\sqrt{2}}\{[\text{-}2\left(\varepsilon_1^\alpha\right)^2+\dot{\varepsilon}_1^\alpha]\xi_1^\alpha + [-\varepsilon_1^\alpha\varepsilon_2^\alpha+\dot{\varepsilon}_1^\alpha-(\varepsilon_2^\alpha)^2]\xi_2^\alpha-\dot{\varepsilon}_1^\alpha B^\alpha\}$$

and similarly

$$\dddot{\beta} = \frac{-1}{\sqrt{2}}(\Gamma_1\xi_1^\alpha+\Gamma_2\xi_2^\alpha+\Gamma_3 B^\alpha)$$

where

$$\Gamma_1 = \text{-}6\varepsilon_1^\alpha\dot{\varepsilon}_1^\alpha+\ddot{\varepsilon}_1^\alpha+2\left(\varepsilon_1^\alpha\right)^3$$

$$\Gamma_2 = \text{-}2\dot{\varepsilon}_1^\alpha\varepsilon_2^\alpha\text{-}\varepsilon_1^\alpha\dot{\varepsilon}_2^\alpha+\ddot{\varepsilon}_2^\alpha\text{-}2\varepsilon_2^\alpha\varepsilon_2^\alpha+\varepsilon_1^\alpha\left(\varepsilon_2^\alpha\right)^2\text{-}\dot{\varepsilon}_2^\alpha\varepsilon_2^\alpha+\left(\varepsilon_2^\alpha\right)^3$$

$$\Gamma_3 = \text{-}\ddot{\varepsilon}_1^\alpha$$

The torsion of curve β is

$$\tau_\beta = \text{-}\frac{[2\left(\varepsilon_1^\alpha\right)^2+\left(\varepsilon_2^\alpha\right)^2]^4}{4\sqrt{2}(\mu_1^2+\mu_2^2+\mu_3^2)}\{[(\text{-}\varepsilon_1^\alpha\varepsilon_2^\alpha\text{-}\dot{\varepsilon}_2^\alpha+\left(\varepsilon_2^\alpha\right)^2)\Gamma_1$$

$$\text{-}2(\left(\varepsilon_1^\alpha\right)^2-\dot{\varepsilon}_1^\alpha)\Gamma_2+(\text{-}\varepsilon_1^\alpha\varepsilon_2^\alpha\text{-}\dot{\varepsilon}_2^\alpha+\left(\varepsilon_2^\alpha\right)^2)\Gamma_3]\varepsilon_1^\alpha$$

$$\text{-}[(\dot{\varepsilon}_1^\alpha-2\left(\varepsilon_1^\alpha\right)^2)\Gamma_3+\dot{\varepsilon}_1^\alpha\Gamma_1]\varepsilon_2^\alpha\}$$

3.3 $\xi_2 B$-Smarandache Curves

Definition 3.3 *Let $\alpha=\alpha(s)$ be a unit speed regular curve in E^3 and $\{\xi_1^\alpha,\xi_2^\alpha,B^\alpha\}$ be its moving Bishop frame. $\xi_2 B$-Smarandache curves can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\xi_2^\alpha+B^\alpha) \tag{3.11}$$

Now, we can investigate Bishop invariants of $\xi_2 B$-Smarandache curves according to $\alpha=\alpha(s)$. Differentiating (3.11) with respect to s, we get

$$\begin{aligned}\dot{\beta} &= \frac{d\beta}{ds^*}\cdot\frac{ds^*}{ds} = (-\varepsilon_2^\alpha B^\alpha+\varepsilon_1^\alpha\xi_1^\alpha+\varepsilon_2^\alpha\xi_2^\alpha)\\ T_\beta\cdot\frac{ds^*}{ds} &= (\varepsilon_1^\alpha\xi_1^\alpha+\varepsilon_2^\alpha\xi_2^\alpha-\varepsilon_2^\alpha B^\alpha)\end{aligned} \tag{3.12}$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{\left(\varepsilon_1^\alpha\right)^2 + 2\left(\varepsilon_2^\alpha\right)^2}{2}} \tag{3.13}$$

The tangent vector of curve β can be written as follow;

$$T_\beta = \frac{\varepsilon_1^\alpha \xi_1^\alpha + \varepsilon_2^\alpha \xi_2^\alpha - \varepsilon_2^\alpha B^\alpha}{\sqrt{2\left(\varepsilon_1^\alpha\right)^2 + \left(\varepsilon_2^\alpha\right)^2}} \tag{3.14}$$

Differentiating (3.14) with respect to s, we obtain

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\left[\left(\varepsilon_1^\alpha\right)^2 + 2\left(\varepsilon_2^\alpha\right)^2\right]^{\frac{3}{2}}}\left(\eta_1 \xi_1^\alpha + \eta_2 \xi_2^\alpha + \eta_3 B^\alpha\right) \tag{3.15}$$

where

$$\begin{aligned}
\eta_1 &= 2\left(\dot{\varepsilon}_1^\alpha \left(\varepsilon_2^\alpha\right)^2 - \varepsilon_1^\alpha \varepsilon_2^\alpha\right) \\
\eta_2 &= \left(\varepsilon_2^\alpha\right)^2 \dot{\varepsilon}_2^\alpha + \left(\varepsilon_1^\alpha\right)^2 \dot{\varepsilon}_1^\alpha - \varepsilon_1^\alpha \dot{\varepsilon}_1^\alpha \varepsilon_2^\alpha \\
\eta_3 &= \left(\varepsilon_1^\alpha\right)^2 \varepsilon_2^\alpha + 2\left(\varepsilon_2^\alpha\right)^3 - \left(\varepsilon_1^\alpha\right)^4 - 2\left(\varepsilon_1^\alpha\right)^4 - 3\left(\varepsilon_1^\alpha\right)^2 \left(\varepsilon_2^\alpha\right)^2
\end{aligned}$$

Substituting (3.13) in (3.15), we have

$$T'_\beta = \frac{\sqrt{2}}{\left[2\left(\varepsilon_1^\alpha\right)^2 + \left(\varepsilon_2^\alpha\right)^2\right]^2}\left(\eta_1 \xi_1^\alpha + \eta_2 \xi_2^\alpha + \eta_3 B^\alpha\right)$$

Then, the first curvature and principal normal vector field of curve β are respectively

$$\left\|T'_\beta\right\| = \kappa_\beta = \frac{\sqrt{2}\sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2}}{\left[\left(\varepsilon_1^\alpha\right)^2 + 2\left(\varepsilon_2^\alpha\right)^2\right]^2}$$

$$N_\beta = \frac{1}{\sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2}}\left(\eta_1 \xi_1^\alpha + \eta_2 \xi_2^\alpha + \eta_3 B^\alpha\right)$$

On the other hand, we express

$$\begin{aligned}
B_\beta = & \frac{1}{\sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2}\sqrt{\left(\varepsilon_1^\alpha\right)^2 + 2\left(\varepsilon_2^\alpha\right)^2}}\left[\left(\eta_2 \varepsilon_2^\alpha + \eta_3 \varepsilon_2^\alpha\right)\xi_1^\alpha\right. \\
& \left. - \left(\eta_1 \xi_2^\alpha + \eta_3 \xi_1^\alpha\right)\xi_2^\alpha + \left(\eta_2 \varepsilon_1^\alpha - \eta_1 \varepsilon_2^\alpha\right) B^\alpha\right]
\end{aligned}$$

We differentiate $(3.12)_1$ with respect to s in order to calculate the torsion of curve β

$$\begin{aligned}
\ddot{\beta} = & \tfrac{1}{\sqrt{2}}\{[\varepsilon_1^\alpha \xi_1^\alpha + \dot{\varepsilon}_1^\alpha - \left(\varepsilon_1^\alpha\right)^2]\xi_1^\alpha \\
& + [\dot{\varepsilon}_2^\alpha - 2\left(\varepsilon_2^\alpha\right)^2]\xi_2^\alpha - \dot{\varepsilon}_2^\alpha B^\alpha\}
\end{aligned}$$

and similarly

$$\dddot{\beta} = \frac{1}{\sqrt{2}}(\eta_1\xi_1^\alpha + \eta_2\xi_2^\alpha + \eta_3 B^\alpha)$$

where

$$\eta_1 = -\dot{\varepsilon}_1^\alpha\varepsilon_2^\alpha - 5\dot{\varepsilon}_1^\alpha\varepsilon_1^\alpha + \ddot{\varepsilon}_1^\alpha + (\varepsilon_1^\alpha)^2\varepsilon_2^\alpha + (\varepsilon_1^\alpha)^3$$

$$\eta_2 = -4\dot{\varepsilon}_2^\alpha\varepsilon_2^\alpha + \ddot{\varepsilon}_2^\alpha + 2\varepsilon_2^\alpha$$

$$\eta_3 = -\ddot{\varepsilon}_2^\alpha$$

The torsion of curve β is

$$\tau_\beta = -\frac{[(\varepsilon_1^\alpha)^2 + 2(\varepsilon_2^\alpha)^2]^4}{4\sqrt{2}(\eta_1^2+\eta_2^2+\eta_3^2)}\{[\dot{\varepsilon}_2^\alpha\eta_2 + (\dot{\varepsilon}_2^\alpha - 2(\varepsilon_2^\alpha)^2)\eta_3]\varepsilon_1^\alpha$$
$$+[2(\varepsilon_2^\alpha)^2\eta_1 + (\varepsilon_1^\alpha\varepsilon_2^\alpha - \dot{\varepsilon}_1^\alpha + (\varepsilon_1^\alpha)^2)\eta_2$$
$$+(-\varepsilon_1^\alpha\varepsilon_2^\alpha + \dot{\varepsilon}_1^\alpha)\eta_3]\varepsilon_2^\alpha\}$$

3.4 $\xi_1\xi_2B$-Smarandache Curves

Definition 3.4 *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{\xi_1^\alpha, \xi_2^\alpha, B^\alpha\}$ be its moving Bishop frame. $\xi_1^\alpha\xi_2 B$-Smarandache curves can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{3}}(\xi_1^\alpha + \xi_2^\alpha + B^\alpha) \tag{3.16}$$

Now, we can investigate Bishop invariants of $\xi_1^\alpha\xi_2 B$-Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.16) with respect to s, we get

$$\begin{aligned}\dot{\beta} &= \frac{d\beta}{ds^*}\cdot\frac{ds^*}{ds} = \tfrac{1}{\sqrt{3}}[(\varepsilon_1^\alpha + \varepsilon_2^\alpha)B^\alpha - \varepsilon_1^\alpha\xi_1^\alpha - \varepsilon_2^\alpha\xi_2^\alpha] \\ T_\beta\cdot\frac{ds^*}{ds} &= \tfrac{1}{\sqrt{3}}[(\varepsilon_1^\alpha + \varepsilon_2^\alpha)B^\alpha - \varepsilon_1^\alpha\xi_1^\alpha - \varepsilon_2^\alpha\xi_2^\alpha)]\end{aligned} \tag{3.17}$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2[(\varepsilon_1^\alpha)^2 + \varepsilon_1^\alpha\varepsilon_2^\alpha + (\varepsilon_2^\alpha)^2]}{3}} \tag{3.18}$$

The tangent vector of curve β can be written as follow;

$$T_\beta = \frac{\varepsilon_1^\alpha\xi_1^\alpha + \varepsilon_2^\alpha\xi_2^\alpha - (\varepsilon_1^\alpha + \varepsilon_2^\alpha)B^\alpha}{\sqrt{2[(\varepsilon_1^\alpha)^2 + \varepsilon_1^\alpha\varepsilon_2^\alpha + (\varepsilon_2^\alpha)^2]}} \tag{3.19}$$

Differentiating (3.19) with respect to s, we get

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds}=\frac{(\lambda_1\xi_1^\alpha+\lambda_2\xi_2^\alpha+\lambda_3 B^\alpha)}{2\sqrt{2}\left[(\varepsilon_1^\alpha)^2+\varepsilon_1^\alpha\varepsilon_2^\alpha+(\varepsilon_2^\alpha)^2\right]^{\frac{3}{2}}} \tag{3.20}$$

where

$$\lambda_1= [\dot{\varepsilon}_1^\alpha\text{-}2\,(\varepsilon_1^\alpha)^2\text{-}\varepsilon_1^\alpha\varepsilon_2^\alpha]u(s)\text{-}\varepsilon_1^\alpha[2\varepsilon_1^\alpha\dot{\varepsilon}_1^\alpha+\dot{\varepsilon}_1^\alpha\varepsilon_2^\alpha+\varepsilon_1^\alpha\dot{\varepsilon}_2^\alpha+2\dot{\varepsilon}_2^\alpha\varepsilon_2^\alpha]$$

$$\lambda_2= [\dot{\varepsilon}_2^\alpha\text{-}2\,(\varepsilon_2^\alpha)^2\text{-}\varepsilon_1^\alpha\varepsilon_2^\alpha]u(s)\text{-}\varepsilon_2^\alpha[\dot{\varepsilon}_1^\alpha+\varepsilon_1^\alpha\dot{\varepsilon}_2^\alpha+2\varepsilon_2^\alpha\dot{\varepsilon}_2^\alpha]$$

$$\lambda_3= [\text{-}\dot{\varepsilon}_1^\alpha\text{-}\dot{\varepsilon}_2^\alpha]u(s)+\varepsilon_1^\alpha[2\varepsilon_1^\alpha\dot{\varepsilon}_1^\alpha+3\dot{\varepsilon}_1^\alpha\varepsilon_2^\alpha+\varepsilon_1^\alpha\dot{\varepsilon}_2^\alpha+2\dot{\varepsilon}_2^\alpha\varepsilon_2^\alpha]$$
$$+\dot{\varepsilon}_2^\alpha[\dot{\varepsilon}_1^\alpha\,(\varepsilon_2^\alpha)^2+2\,(\varepsilon_2^\alpha)^2]$$

Substituting (3.18) in (3.20), we have

$$T_\beta^{'}=\frac{\sqrt{3}(\lambda_1\xi_1^\alpha+\lambda_2\xi_2^\alpha+\lambda_3 B^\alpha)}{4\left[(\varepsilon_1^\alpha)^2+\varepsilon_1^\alpha\varepsilon_2^\alpha+(\varepsilon_2^\alpha)^2\right]^2}$$

Then, the first curvature and principal normal vector field of curve β are respectively

$$\left\|T_\beta^{'}\right\|=\kappa_\beta=\frac{\sqrt{3}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}{4\left[(\varepsilon_1^\alpha)^2+\varepsilon_1^\alpha\varepsilon_2^\alpha+(\varepsilon_2^\alpha)^2\right]^2}$$
$$N_\beta=\frac{1}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}(\lambda_1\xi_1^\alpha+\lambda_2\xi_2^\alpha+\lambda_3 B^\alpha) \tag{3.21}$$

On the other hand, we express

$$B_\beta=\frac{1}{\sqrt{2[(\varepsilon_1^\alpha)^2+\varepsilon_1^\alpha\varepsilon_2^\alpha+(\varepsilon_2^\alpha)^2]}\cdot\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\det\begin{bmatrix}\xi_1^\alpha & \xi_2^\alpha & B^\alpha\\ \varepsilon_1^\alpha & \varepsilon_2^\alpha & \text{-}(\varepsilon_1^\alpha+\varepsilon_2^\alpha)\\ \lambda_1 & \lambda_2 & \lambda_3\end{bmatrix}$$

So, the binormal vector field of curve β is

$$B_\beta=\frac{1}{\sqrt{2[(\varepsilon_1^\alpha)^2+\varepsilon_1^\alpha\varepsilon_2^\alpha+(\varepsilon_2^\alpha)^2]}\cdot\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\{[(\varepsilon_1^\alpha+\varepsilon_2^\alpha)\lambda_1$$
$$-\varepsilon_2^\alpha\lambda_3]\xi_1^\alpha+[-\varepsilon_1^\alpha\lambda_3\text{-}(\varepsilon_1^\alpha+\varepsilon_2^\alpha)]\xi_2^\alpha+[\varepsilon_1^\alpha\lambda_2\text{-}\varepsilon_2^\alpha\lambda_1]B^\alpha\}$$

We differentiate (3.20) with respect to s in order to calculate the torsion of curve β

$$\ddot{\beta}= \text{-}\frac{1}{\sqrt{3}}\{[2\,(\varepsilon_1^\alpha)^2+\varepsilon_1^\alpha\xi_1^\alpha\text{-}\dot{\varepsilon}_1^\alpha]\xi_1^\alpha$$
$$+[2\,(\varepsilon_2^\alpha)^2+\varepsilon_1^\alpha\varepsilon_2^\alpha\text{-}\dot{\varepsilon}_2^\alpha]\xi_2^\alpha+[\dot{\varepsilon}_1^\alpha+\dot{\varepsilon}_2^\alpha]B^\alpha\}$$

and similarly

$$\dddot{\beta} = -\frac{1}{\sqrt{3}}(\sigma_1\xi_1^\alpha + \sigma_2\xi_2^\alpha + \sigma_3 B^\alpha)$$

where

$$\eta_1 = 4\dot{\varepsilon}_1^\alpha\varepsilon_1^\alpha + 3\dot{\varepsilon}_1^\alpha\varepsilon_2^\alpha - \ddot{\varepsilon}_1^\alpha - 2\left(\varepsilon_1^\alpha\right)^3 - \left(\varepsilon_1^\alpha\right)^2\varepsilon_2^\alpha$$

$$\eta_2 = 5\dot{\varepsilon}_2^\alpha\varepsilon_2^\alpha + \dot{\varepsilon}_1^\alpha\varepsilon_2^\alpha + \varepsilon_1^\alpha\varepsilon_2^\alpha - \ddot{\varepsilon}_2^\alpha - 2\left(\varepsilon_2^\alpha\right)^3 - \varepsilon_1^\alpha\left(\varepsilon_2^\alpha\right)^2$$

$$\eta_3 = \ddot{\varepsilon}_2^\alpha + \ddot{\varepsilon}_2^\alpha$$

The torsion of curve β is

$$\begin{aligned}\tau_\beta = & -\tfrac{16[(\varepsilon_1^\alpha)^2+\varepsilon_1^\alpha\varepsilon_2^\alpha+(\varepsilon_2^\alpha)^2]^2}{9\sqrt{3}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\{[(2\left(\varepsilon_2^\alpha\right)^2 + \varepsilon_1^\alpha\varepsilon_2^\alpha - \dot{\varepsilon}_2^\alpha)\sigma_1 + (-\dot{\varepsilon}_2^\alpha - 2\left(\varepsilon_1^\alpha\right)^2 - \varepsilon_1^\alpha\varepsilon_2^\alpha)\sigma_2 \\ & + (2\left(\varepsilon_2^\alpha\right)^2 + \varepsilon_1^\alpha\varepsilon_2^\alpha - \dot{\varepsilon}_2^\alpha)\sigma_3]\varepsilon_1^\alpha + [-\dot{\varepsilon}_1^\alpha - 2\dot{\varepsilon}_2^\alpha + 2\left(\varepsilon_2^\alpha\right)^2 + \varepsilon_1^\alpha\varepsilon_2^\alpha)\sigma_1 \\ & + (-2\left(\varepsilon_1^\alpha\right)^2 - \varepsilon_1^\alpha\varepsilon_2^\alpha + \dot{\varepsilon}_1^\alpha)\sigma_2 + (2\left(\varepsilon_1^\alpha\right)^2 + \varepsilon_1^\alpha\varepsilon_2^\alpha - \dot{\varepsilon}_1^\alpha)\sigma_3]\varepsilon_2^\alpha\}.\end{aligned}$$

§4. Smarandache Breadth Curves According to Type-2 Bishop Frame in $\mathbf{E}^3$

A regular curve with more than 2 breadths in Euclidean 3-space is called Smarandache breadth curve.

Let $\alpha = \alpha(s)$ be a Smarandache breadth curve. Moreover, let us suppose $\alpha = \alpha(s)$ simple closed space-like curve in the space E^3. These curves will be denoted by (C). The normal plane at every point P on the curve meets the curve at a single point Q other than P.

We call the point Q the opposite point P. We consider a curve in the class Γ as in having parallel tangents ξ_1 and ξ_1^* opposite directions at opposite points α and $\alpha^*$of the curves.

A simple closed curve having parallel tangents in opposite directions at opposite points can be represented with respect to type-2 Bishop frame by the equation

$$\alpha^*(s) = \alpha(s) + \lambda\xi_1 + \varphi\xi_2 + \eta B \tag{4.1}$$

where $\lambda(s), \varphi(s)$ and $\eta(s)$ are arbitrary functions also α and α^* are opposite points.

Differentiating both sides of (4.1) and considering type-2 Bishop equations, we have

$$\begin{aligned}\frac{d\alpha^*}{ds} = \xi_1^*\frac{ds^*}{ds} = & \left(\frac{d\lambda}{ds} + \eta\varepsilon_1 + 1\right)\xi_1 + \left(\frac{d\varphi}{ds} + \eta\varepsilon_2\right)\xi_2 \\ & + \left(-\lambda\varepsilon_1 - \varphi\varepsilon_2 + \frac{d\eta}{ds}\right)B\end{aligned} \tag{4.2}$$

Since $\xi_1^* = -\xi_1$ rewriting (4.2) we have

$$\begin{aligned}\frac{d\lambda}{ds} &= -\eta\varepsilon_1 - 1 - \frac{ds^*}{ds}\\ \frac{d\varphi}{ds} &= -\varphi\varepsilon_2 \\ \frac{d\eta}{ds} &= \lambda\varepsilon_1 + \varphi\varepsilon_2\end{aligned} \tag{4.3}$$

If we call θ as the angle between the tangent of the curve (C) at point $\alpha(s)$ with a given direction and consider $\dfrac{d\theta}{ds} = \kappa$, we have (4.3) as follow:

$$\begin{aligned}\frac{d\lambda}{d\theta} &= -\eta\frac{\varepsilon_1}{\kappa} - f(\theta)\\ \frac{d\varphi}{d\theta} &= -\varphi\frac{\varepsilon_2}{\kappa} \\ \frac{d\eta}{d\theta} &= \lambda\frac{\varepsilon_1}{\kappa} + \varphi\frac{\varepsilon_2}{\kappa}\end{aligned} \tag{4.4}$$

where $f(\theta) = \delta + \delta^*$, $\delta = \dfrac{1}{\kappa}$, $\delta^* = \dfrac{1}{\kappa^*}$ denote the radius of curvature at α and α^* respectively. And using system (4.4), we have the following differential equation with respect to λ as

$$\begin{aligned}&\frac{d^3\lambda}{d\theta^3} - [\frac{\kappa}{\varepsilon_1}\frac{d}{d\theta}(\frac{\varepsilon_1}{\kappa})]\frac{d^2\lambda}{d\theta^2} + [\frac{\varepsilon_1^2}{\kappa^2} - \frac{\varepsilon_1}{\kappa} - \frac{d}{d\theta}(\frac{\kappa}{\varepsilon_1})\frac{d}{d\theta}(\frac{\varepsilon_1}{\kappa})\\ &-\frac{\kappa}{\varepsilon_1}\frac{d^2}{d\theta^2}(\frac{\varepsilon_1}{\kappa})]\frac{d\lambda}{d\theta} + [\frac{\varepsilon_1}{\kappa}\frac{d}{d\theta}(\frac{\varepsilon_1}{\kappa}) - \frac{\varepsilon_1^2}{\varepsilon_2\kappa}]\lambda + \\ &+[-\frac{\kappa}{\varepsilon_2} - \frac{\kappa}{\varepsilon_1}\frac{d}{d\theta}(\frac{\varepsilon_1}{\kappa})]\frac{d^2 f}{d\theta^2} - [\frac{\kappa}{\varepsilon_2} + 2\frac{\kappa}{\varepsilon_1}\frac{d}{d\theta}(\frac{\varepsilon_1}{\kappa})]\frac{df}{d\theta}\\ &-[\frac{\varepsilon_2^2}{\varepsilon_1\kappa} + \frac{\varepsilon_1}{\varepsilon_2} + 2\frac{d}{d\theta}(\frac{\kappa}{\varepsilon_1})\frac{d}{d\theta}(\frac{\varepsilon_1}{\kappa}) + \frac{\kappa}{\varepsilon_1}\frac{d^2}{d\theta^2}(\frac{\varepsilon_1}{\kappa})]f(\theta) = 0\end{aligned} \tag{4.5}$$

Equation (4.5) is characterization for α^*. If the distance between opposite points of (C) and (C^*) is constant, then we can write that

$$\|\alpha^* - \alpha\| = \lambda^2 + \varphi^2 + \eta^2 = l^2 = \text{constant} \tag{4.6}$$

Hence, we write

$$\lambda\frac{d\lambda}{d\theta} + \varphi\frac{d\varphi}{d\theta} + \eta\frac{d\eta}{d\theta} = 0 \tag{4.7}$$

Considering system (4.4) we obtain

$$\lambda \cdot f(\theta) = 0 \tag{4.8}$$

We write $\lambda = 0$ or $f(\theta) = 0$. Thus, we shall study in the following subcases.

Case 1. $\lambda = 0$. Then we obtain

$$\eta = -\int_0^\theta \frac{\kappa}{\varepsilon_1} f(\theta)d\theta, \qquad \varphi = \int_0^\theta (\int_0^\theta \eta \frac{\varepsilon_2}{\kappa} d\theta)\frac{\varepsilon_2}{\kappa} d\theta \tag{4.9}$$

and

$$\frac{d^2 f}{d\theta^2} - \frac{df}{d\theta} - [(\frac{\tau}{\kappa})^2 \frac{\sin^3\theta}{\cos\theta} - \frac{\tau}{\kappa}\cos\theta]f = 0 \tag{4.10}$$

General solution of (4.10) depends on character of $\frac{\tau}{\kappa}$. Due to this, we distinguish following subcases.

Subcase 1.1 $f(\theta) = 0$. then we obtain

$$\begin{aligned} \lambda &= \int_0^\theta \eta \frac{\varepsilon_1}{\kappa} d\theta \\ \varphi &= -\int_0^\theta \eta \frac{\varepsilon_2}{\kappa} d\theta \\ \eta &= \int_0^\theta \lambda \frac{\varepsilon_1}{\kappa} d\theta + \int_0^\theta \varphi \frac{\varepsilon_2}{\kappa} d\theta \end{aligned} \tag{4.11}$$

Case 2. Let us suppose that $\lambda \neq 0$, $\varphi \neq 0$,$\eta \neq 0$ and λ, φ, η constant. Thus the equation (4.4) we obtain $\frac{\varepsilon_1}{\kappa} = 0$ and $\frac{\varepsilon_2}{\kappa} = 0$.

Moreover, the equation (4.5) has the form $\frac{d^3\lambda}{d\theta^3} = 0$ The solution (4.12) is $\lambda = L_1\frac{\theta^2}{2} + L_2\theta + L_3$ where L_1, L_2 and L_3 real numbers. And therefore we write the position vector ant the curvature

$$\alpha^* = \alpha + A_1\xi_1 + A_2\xi_2 + A_3B$$

where $A_1 = \lambda$, $A_2 = \varphi$ and $A_3 = \eta$ real numbers. And the distance between the opposite points of (C) and (C^*) is

$$\|\alpha^* - \alpha\| = A_1^2 + A_2^2 + A_3^2 = \text{constant}$$

§5. Examples

In this section, we show two examples of Smarandache curves according to Bishop frame in E^3.

Example 5.1 First, let us consider a unit speed curve of E^3 by

$$\begin{aligned} \beta(s) = & \Big(\frac{25}{306}\sin(9s) - \frac{9}{850}\sin(25s), \\ & -\frac{25}{306}\cos(9s) + \frac{9}{850}\cos(25s), \frac{15}{136}\sin(8s)\Big) \end{aligned}$$

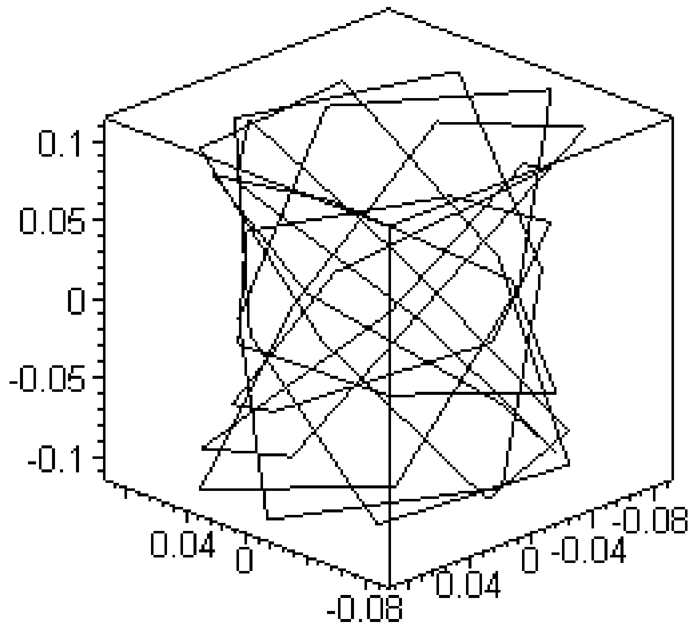

Fig.1 The curve $\beta = \beta(s)$

See the curve $\beta(s)$ in Fig.1. One can calculate its Serret-Frenet apparatus as the following

$$T = (\tfrac{25}{34}\cos 9s + \tfrac{9}{34}\cos 25s, \tfrac{25}{34}\sin 9s - \tfrac{9}{34}\sin 25s, \tfrac{15}{17}\cos 8s)$$

$$N = (\tfrac{15}{34}\csc 8s(\sin 9s - \sin 25s), -\tfrac{15}{34}\csc 8s(\cos 9s - \cos 25s), \tfrac{8}{17})$$

$$B = (\tfrac{1}{34}(25\sin 9s - 9\sin 25s), -\tfrac{1}{34}(25\cos 9s + 9\cos 25s), -\tfrac{15}{17}\sin 8s)$$

$\kappa = -15\sin 8s$ and $\tau = 15\cos 8s$

In order to compare our main results with Smarandache curves according to Serret-Frenet frame, we first plot classical Smarandache curve of β Fig.1.

Now we focus on the type-2 Bishop trihedral. In order to form the transformation matrix (2.6), let us express

$$\theta(s) = -\int_0^s 15\sin(8s)ds = \frac{15}{8}\cos(8s)$$

Since, we can write the transformation matrix

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin(\frac{15}{8}\cos 8s) & -\cos(\frac{15}{8}\cos 8s) & 0 \\ \cos(\frac{15}{8}\cos 8s) & \sin(\frac{15}{8}\cos 8s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix}$$

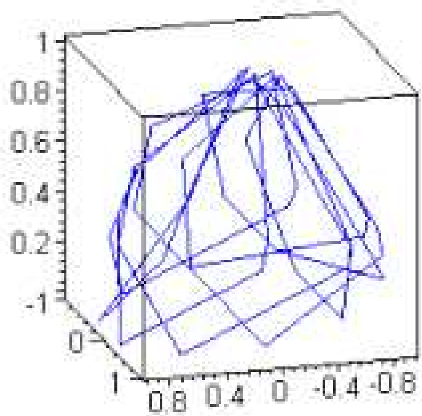

Fig.2 $\xi_1\xi_1$ Smarandache curve

By the method of Cramer, one can obtain type-2 Bishop frame of β as follows

$$\begin{aligned}\xi_1 = \quad & (\sin\theta(\tfrac{25}{34}\cos 9s - \tfrac{9}{34}\cos 25s) + \tfrac{15}{34}\cos\theta\csc 8s(\sin 9s - \sin 25s),\\ & \sin\theta(\tfrac{25}{34}\sin 9s - \tfrac{9}{34}\sin 25s) - \tfrac{15}{34}\cos\theta\csc 8s(\cos 9s - \cos 25s),\\ & \tfrac{15}{17}\sin\theta\cos 8s + \tfrac{8}{17}\cos\theta)\end{aligned}$$

$$\begin{aligned}\xi_2 = \quad & (-\cos\theta(\tfrac{25}{34}\cos 9s - \tfrac{9}{34}\cos 25s) + \tfrac{15}{34}\sin\theta\csc 8s(\sin 9s - \sin 25s),\\ & -\cos\theta(\tfrac{25}{34}\sin 9s - \tfrac{9}{34}\sin 25s) - \tfrac{15}{34}\sin\theta\csc 8s(\cos 9s - \cos 25s),\\ & -\tfrac{15}{17}\cos\theta\cos 8s + \tfrac{8}{17}\sin\theta)\end{aligned}$$

$$B = \quad (\tfrac{1}{34}(25\sin 9s - 9\sin 25s), -\tfrac{1}{34}(25\cos 9s + 9\cos 25s), -\tfrac{15}{17}\sin 8s)$$

where $\theta = \frac{15}{8}\cos(8s)$. So, we have Smarandache curves according to type-2 Bishop frame of the unit speed curve $\beta = \alpha(s)$, see Fig.2-4 and Fig.5.

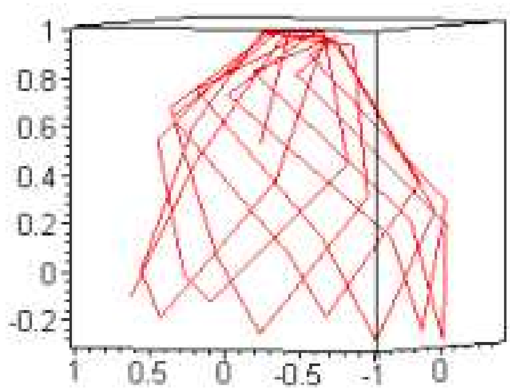

Fig.3 $\xi_1 B$ Smarandache curve

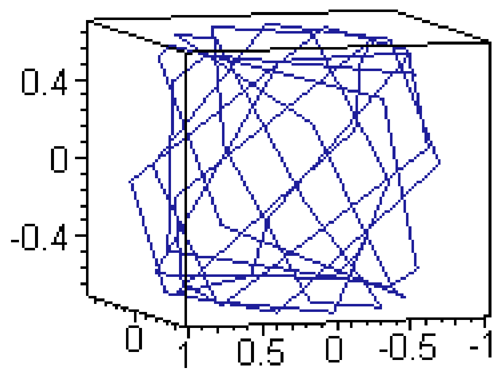

Fig.4 $\xi_2 B$ Smarandache curve

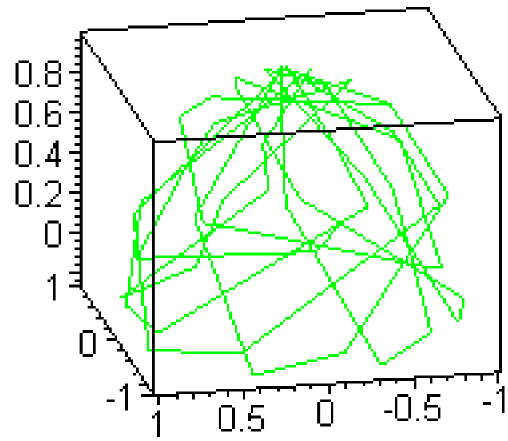

Fig.5 $\xi_1\xi_2 B$ Smarandache curve

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International J.Math. Combin. Vol.2(2016), 16-28

Ruled Surface Pair Generated by Darboux Vectors of a Curve and Its Natural Lift in $\mathbb{R}_1^3$

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Abstract: In this study,firstly, the darboux vector $\overline{W}$ of the natural lift $\overline{\alpha}$ of a curve α are calculated in terms of those of α in $\mathbb{R}_1^3$. Secondly, we obtained striction lines and distribution parameters of ruled surface pair generated by by Darboux vectors of the curve α and its natural lift $\overline{\alpha}$. Finally, for α and $\overline{\alpha}$ those notions are compared with each other.

Key Words: Lift, ruled surface, striction line, distribution parameter.

AMS(2010): 51B20,53A15,53A04.

§1. Introduction and Preliminary Notes

The concepts of the natural lift curve and geodesic sprays have first been given by Thorpe in [17]. Thorpe proved the natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray iff α is an geodesic on M. Çalışkan at al. studied the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and fixed centrode of a curve in [16]. They gave some interesting results about the original curve, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2\right)$. Some properties of $\overline{M}$-vector field Z defined on a hypersurface M of $\overline{M}$ were studied by Agashe in [1]. $\overline{M}$-integral curve of Z and $\overline{M}$-geodesic spray are defined by Çalışkan and Sivridağ. They gave the main theorem: The natural lift $\overline{\alpha}$ of the curve α (in $\overline{M}$) is an $\overline{M}$-integral curve of the geodesic spray Z iff α is an $\overline{M}$-geodesic in [8]. Bilici et al. have proposed the natural lift curves and the geodesic sprays for the spherical indicatrices of the the involute evolute curve couple in Euclidean 3-space. They gave some interesting results about the evolute curve, depending on the assumption that the natural lift curve of the spherical indicatrices of the involute should be the integral curve on the tangent bundle $T\left(S^2\right)$ in [6]. Then Bilici applied this problem to involutes of a timelike curve in Minkowski 3-space (see [7]). Ergün and Çalışkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space in [10]. The analogue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan in

[1]Received June 6, 2015, Accepted May 8, 2016.

[10]. Çalışkan and Ergün defined $\overline{M}$-vector field Z, $\overline{M}$-geodesic spray, $\overline{M}$-integral curve of Z, $\overline{M}$-geodesic in [9]. The analogue of the theorem of Sivridağ and Çalışkan was given in Minkowski 3-space by Ergün and Çalışkan in [10]. Walrave characterized the curve with constant curvature in Minkowski 3-space in [16]. In differential geometry, especially the theory of space curve, the Darboux vector is the areal velocity vector of the Frenet frame of a spacere curve. It is named after Gaston Darboux who discovered it. In term of the Frenet-Serret apparatus, the darboux vector W can be expressed as $W = \tau T + \kappa B$, details are given in Lambert et al. in [13].

Let Minkowski 3-space $\mathbb{R}^3_1$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ and lightlike (or null) if $g(X,X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(t)$ in $\mathbb{R}^3_1$ where t is a pseudo-arclength parameter, can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\dot{\alpha}(t)$ are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset \mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ $(resp. x_1 < 0)$ and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ $(resp.\ x_1 < 0)$.The norm of a vector X is defined by [14]

$$\|X\|_{IL} = \sqrt{|g(X,X)|}.$$

We denote by $\{T(t), N(t), B(t)\}$ the moving Frenet frame along the curve α. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively.

Let α be a unit speed timelike space curve with curvature κ and torsion τ. Let Frenet vector fields of α be $\{T, N, B\}$. In this trihedron, T is timelike vector field, N and B are spacelike vector fields.For this vectors, we can write

$$T \times N = B, \quad N \times B = -T, \quad B \times T = N,$$

where $\times$ is the Lorentzian cross product, [4]. in space $\mathbb{R}^3_1$. Then, Frenet formulas are given by

$$T' = \kappa N, N' = \kappa T + \tau B,\ B' = -\tau N, \text{ [16]}.$$

The Frenet instantaneous rotation vector for the timelike curve is given by $W = \tau T + \kappa B$.

Let α be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that T and B are spacelike vector fields and N is a timelike vector field. In this situation,

$$T \times N = B, \quad N \times B = T, \quad B \times T = -N,$$

Then, Frenet formulas are given by

$$T' = \kappa N, N' = \kappa T + \tau B,\ B' = \tau N, \text{ [16]}.$$

The Frenet instantaneous rotation vector for the spacelike space curve with a spacelike binormal is given by $W = \tau T - \kappa B$.

Let α be a unit speed spacelike space curve with a timelike binormal. In this trihedron, we assume that T and N are spacelike vector fields and B is a timelike vector field.In this situation,

$$T \times N = -B, \quad N \times B = T, \quad B \times T = N,$$

Then, Frenet formulas are given by,

$$T^{'} = \kappa N, N^{'} = -\kappa T + \tau B, \ B^{'} = \tau N, \ [16].$$

The Frenet instantaneous rotation vector for the spacelike space curve with a timelike binormal is given by $W = -\tau T + \kappa B$.

Lemma 1.1([15]) *Let X and Y be nonzero Lorentz orthogonal vectors in $\mathbb{R}^3_1$. If X is timelike, then Y is spacelike.*

Lemma 1.2([15]) *Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}^3_1$. Then*

$$g(X, Y) \leq \|X\| \|Y\|$$

whit equality if and only if X and Y are linearly dependent.

Lemma 1.3([15]) (1) *Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}^3_1$. By the Lemma 2, there is unique nonnegative real number $\varphi(X, Y)$ such that*

$$g(X, Y) = \|X\| \|Y\| \cosh \varphi(X, Y),$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X, Y)$.

(2) *Let X and Y be spacelike vektors in $\mathbb{R}^3_1$ that span a spacelike vector subspace. Then we have*

$$|g(X, Y)| \leq \|X\| \|Y\|.$$

Hence, there is a unique real number $\varphi(X, Y)$ between 0 and π such that

$$g(X, Y) = \|X\| \|Y\| \cos \varphi(X, Y),$$

the Lorentzian spacelike angle between X and Y is defined to be $\varphi(X, Y)$.

(3) *Let X and Y be spacelike vectors in $\mathbb{R}^3_1$ that span a timelike vector subspace. Then we have*

$$g(X, Y) > \|X\| \|Y\|.$$

Hence, there is a unique pozitive real number $\varphi(X, Y)$ between 0 and π such that

$$|g(X, Y)| = \|X\| \|Y\| \cosh \varphi(X, Y),$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X, Y)$.

(4) *Let X be a spacelike vector and Y be a pozitive timelike vector in $\mathbb{R}^3_1$. Then there is*

a unique nonnegative reel number $\varphi(X,Y)$ *such that*

$$|g(X,Y)| = \|X\| \|Y\| \sinh\varphi(X,Y),$$

the Lorentzian timelike angle between X *and* Y *is defined to be* $\varphi(X,Y)$.

For the curve α with a timelike tanget, let θ be a Lorentzian timelike angle between the spacelike binormal unit $-B$ and the Frenet instantaneous rotation vector W.

a) If $|\kappa| > |\tau|$, then W is a spacelike vector. In this situation, from Lemma 3 (3) we can write

$$\kappa = \|W\| \cosh\theta,\ \ \tau = \|W\| \sinh\theta$$

$\|W\|^2 = g(W,W) = \kappa^2 - \tau^2$ and $C = \frac{W}{\|W\|} = \sinh\theta T + \cosh\theta B$, where C is unit vector of direction W.

b) If $|\kappa| < |\tau|$, then W is a timelike vector. In this situation, from Lemma 3 (4) we can write

$$\kappa = \|W\| \sinh\theta,\ \ \tau = \|W\| \cosh\theta$$

$\|W\|^2 = -g(W,W) = -(\kappa^2 - \tau^2)$ and $C = \cosh\theta T + \sinh\theta B$.

For the curve α with a timelike principal normal, let θ be an angle between the B and the W, if B and W spacelike vectors that span a spacelike vektor subspace then by the Lemma 3 (2) we can write

$$\kappa = \|W\| \cos\theta, \quad \tau = \|W\| \sin\theta$$

$\|W\|^2 = g(W,W) = \kappa^2 + \tau^2$ and $C = \sin\theta T - \cos\theta B$.

For the curve α with a timelike binormal, let θ be a Lorentzian timelike angle between the $-B$ and the W.

a) If $|\kappa| < |\tau|$, then W is a spacelike vector. In this situation, from Lemma 3 (4) we can write

$$\kappa = \|W\| \sinh\theta,\ \ \tau = \|W\| \cosh\theta$$

$\|W\|^2 = g(W,W) = \tau^2 - \kappa^2$ and $C = -\cosh\theta T + \sinh\theta B$.

(*b*) If $|\kappa| > |\tau|$, then W is a timelike vector. In this situation, from Lemma 3 (1) we have

$$\kappa = \|W\| \cosh\theta,\ \ \tau = \|W\| \sinh\theta$$

$\|W\|^2 = -g(W,W) = -(\tau^2 - \kappa^2)$ and $C = -\sinh\theta T + \cosh\theta B$.

From [10], we know that if α be a unit speed timelike space curve, then the natural lift $\overline{\alpha}$ of α is a spacelike space curve; if α be a unit speed spacelike space curve with a spacelike binormal, then the natural lift $\overline{\alpha}$ of α is a timelike space curve; if α be a unit speed spacelike space curve with a timelike binormal, then the natural lift $\overline{\alpha}$ of α is a spacelike space curve. If α be a unit speed timelike space curve and $\overline{\alpha}$ be the natural lift of α, then from [12] we know that

$$\overline{T}(s) = N(s)\,,\ \overline{N}(s) = -\frac{\kappa(s)}{\|W\|}T(s) - \frac{\tau(s)}{\|W\|}B(s)\ \ ,\ \overline{B}(s) = -\frac{\tau(s)}{\|W\|}T(s) - \frac{\kappa(s)}{\|W\|}B(s)\,,$$

and if α be a unit speed spacelike space curve with a spacelike binormal and $\overline{\alpha}$ be the natural lift of α, then

$$\overline{T}(s) = N(s)\,,\ \overline{N}(s) = \frac{\kappa(s)}{\|W\|}T(s) + \frac{\tau(s)}{\|W\|}B(s)\ \ ,\ \overline{B}(s) = \frac{\tau(s)}{\|W\|}T(s) - \frac{\kappa(s)}{\|W\|}B(s)\,,$$

and if α be a unit speed spacelike space curve with a timelike binormal and $\overline{\alpha}$ be the natural lift of α, then

$$\overline{T}(s) = N(s)\,,\ \overline{N}(s) = -\frac{\kappa(s)}{\|W\|}T(s) - \frac{\tau(s)}{\|W\|}B(s)\ \ ,\ \overline{B}(s) = \frac{\tau(s)}{\|W\|}T(s) + \frac{\kappa(s)}{\|W\|}B(s)\,.$$

Definition 1.1([4]) *Let M be a hypersurface in $\mathbb{R}^3_1$ and let $\alpha : I \longrightarrow M$ be a parametrized curve. α is called an integral curve of X if*

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s))\ \text{(for all } s \in I)\,,$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup_{P\in M} T_P M = \chi(M)\,,$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields on M.

Definition 1.2([5]) *A parameterized curve $\alpha : I \longrightarrow M$, $\overline{\alpha} : I \longrightarrow TM$ given by*

$$\overline{\alpha}(s) = \left(\alpha(s)\,, \alpha'(s)\right) = \alpha'(s)\,|_{\alpha(s)}$$

is called the natural lift of α on TM. Thus, we can write

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}\left(\alpha'(s)\,|_{\alpha(s)}\right) = D_{\alpha'(s)}\alpha'(s)$$

where D is the Levi-Civita connection on $\mathbb{R}^3_1$.

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation

$$X(s,v) = \alpha(s) + ve(s)\,,$$

where $\alpha(s)$ represents a space curve which is called the base curve and e is a unit vector representing the direction of a straight line.

The striction point on a ruled surface X is the foot of the common normal between two consecutive generators (or ruling). The set of striction points defines the striction curve given as

$$\beta(s) = \alpha(s) - \frac{g\left(\alpha', e'\right)}{g\left(e', e'\right)} e(s) \ \ [2].$$

The distribution parameter of the ruled surface X is defined by ([2])

$$P_e = \frac{\det\left(\alpha', e, e'\right)}{\|e'\|^2}$$

and the ruled surface is developable if and only if $P_e = 0$.

§2. Ruled Surface Pair Generated by Darboux Vectors of a Curve and Its Natural Lift in $\mathbb{R}_1^3$

In this section the darboux vector $\overline{W}$ of the natural lift $\overline{\alpha}$ of a curve α are calculated in terms of those of α in $\mathbb{R}_1^3$. We obtained striction lines and distribution parameters of ruled surface pair generated by Darboux vectors of the curve α and its natural lift $\overline{\alpha}$. Let α be a unit speed timelike space curve. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve.

Proposition 2.1 *Let α be a unit speed timelike space curve and the natural lift $\overline{\alpha}$ of the curve α be a space curve with curvature $\overline{\kappa}$ and torsion $\overline{\tau}$. Then*

$$\overline{\kappa}(s) = \frac{\|W\|}{\kappa(s)}, \overline{\tau}(s) = -\frac{\kappa'(s)\tau(s) + \kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}.$$

Proposition 2.2 *Let α be a unit speed timelike space curve and the natural lift $\overline{\alpha}$ of the curve α be a space curve.*

(i) If the natural lift $\overline{\alpha}$ is a unit speed spacelike space curve with a spacelike binormal, then

$$\bar{W} = \frac{\overline{\tau}(s)}{\kappa(s)} T - \left(\frac{\kappa'(s)\tau(s) + \kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}\right) N + B$$

(ii) If the natural lift $\overline{\alpha}$ is a unit speed spacelike space curve with a timelike binormal, then

$$\bar{W} = \frac{\tau(s)}{\kappa(s)} T + \left(\frac{\kappa'(s)\tau(s) + \kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}\right) N + B$$

Let X and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v) = \alpha(s) + vC(s), \ \overline{X}(s,v) = \overline{\alpha}(s) + v\overline{C}(s)$$

The striction curves of X and $\overline{X}$ are given by$\beta(s) = \alpha(s) - \lambda C(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) -$

$\mu\overline{C}(s)$,respectively. The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by

$$P_C = \frac{\det\left(\alpha', C, C'\right)}{\|C'\|^2} \text{ and } \overline{P}_{\overline{C}} = \frac{\det\left(\overline{\alpha}', \overline{C}, \overline{C}'\right)}{\left\|\overline{C}'\right\|^2}.$$

Proposition 2.3 *If the natural lift $\overline{\alpha}$ is a unit speed spacelike space curve with a timelike binormal, then*

$$\lambda = \frac{\tau'(s)}{[\kappa'(s)]^2 - [\tau'(s)]^2}\|W\|,$$

$$\mu = -\frac{\|\bar{W}\|\kappa(s)\sigma'(s)}{-\left[\frac{-\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)^2} + \kappa(s)\sigma(s)\right]^2 + [\sigma'(s)]^2 + [\sigma(s)\tau(s)]^2},$$

$$P_C = 0,\ \overline{P}_{\overline{C}} = \frac{\sigma(s)\tau(s)^2 - \kappa(s)\left(\frac{-\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)^2} + \kappa(s)\sigma(s)\right)}{\left|-\left[\frac{-\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)^2} + \kappa(s)\sigma(s)\right]^2 + [\sigma'(s)]^2 + [\sigma(s)\tau(s)]^2\right|}$$

where $\sigma(s) = \frac{\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}$.

Proposition 2.4 *If the natural lift $\overline{\alpha}$ is a unit speed spacelike space curve with a spacelike binormal, then*

$$\lambda = \frac{\tau'(s)}{[\kappa'(s)]^2 - [\tau'(s)]^2}\|W\|,$$

$$\mu = \frac{\|\bar{W}\|\kappa(s)\sigma'(s)}{-\left[\frac{-\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)^2} - \kappa(s)\sigma(s)\right]^2 + [-\sigma'(s)]^2 + [-\sigma(s)\tau(s)]^2},$$

$$P_C = 0,\ \overline{P}_{\overline{C}} = \frac{-\sigma(s)\tau(s)^2 - \kappa(s)\left(\frac{-\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)^2} - \kappa(s)\sigma(s)\right)}{\left|-\left[\frac{-\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)^2} - \kappa(s)\sigma(s)\right]^2 + [-\sigma'(s)]^2 + [-\sigma(s)\tau(s)]^2\right|}$$

where $\sigma(s) = \frac{\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}$.

Let α be a unit speed spacelike space curve with a spacelike binormal. Then the natural lift $\overline{\alpha}$ of α is a timelike space curve.

Proposition 2.5 *Let α be a unit speed spacelike space curve with a spacelike binormal and the natural lift $\overline{\alpha}$ of the curve α be a space curve with curvature $\overline{\kappa}$ and torsion $\overline{\tau}$. Then*

$$\overline{\kappa}(s) = \frac{\|W\|}{\kappa(s)}, \overline{\tau}(s) = \frac{-\kappa'(s)\tau(s) + \kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}.$$

.

Proposition 2.6 *Let α be a unit speed spacelike space curve with a spacelike binormal and the natural lift $\overline{\alpha}$ of the curve α be a space curve, then*

$$\bar{W} = \frac{\tau(s)}{\kappa(s)}T + \left(\frac{-\kappa'(s)\tau(s) + \kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}\right)N + B.$$

.

Let X and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v) = \alpha(s) + vC(s),\ \overline{X}(s,v) = \overline{\alpha}(s) + v\overline{C}(s)$$

Proposition 2.7 *The striction curves of X and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda C(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu\overline{C}(s)$,respectively. The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by $P_C = \frac{\det\left(\alpha',C,C'\right)}{\|C'\|^2}$ and $\overline{P}_{\overline{C}} = \frac{\det\left(\overline{\alpha}',\overline{C},\overline{C}'\right)}{\|\overline{C}'\|^2}$. Then we have*

$$\lambda = -\frac{\tau'(s)}{[\kappa'(s)]^2 + [\tau'(s)]^2}\|W\|,$$

$$\mu = -\frac{\left(2\tau(s) + \sigma'(s)\right)\kappa(s)\|\bar{W}\|}{\left[\frac{-\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)^2}\right]^2 - [2\tau(s) + \sigma'(s)]^2 + [\sigma(s)\tau(s)]^2},$$

$$P_C = 0,\quad \overline{P}_{\overline{C}} = \frac{-\kappa(s)\left(\frac{\kappa(s)\tau'(s)-\kappa'(s)\tau(s)}{\kappa(s)^2} + \kappa(s)\sigma(s)\right) + \sigma(s)\tau(s)^2}{\left|\left[\frac{-\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)^2} + \kappa(s)\sigma(s)\right]^2 - [2\tau(s) + \sigma'(s)]^2 + [\sigma(s)\tau(s)]^2\right|}$$

where $\sigma(s) = \frac{\kappa'(s)\tau(s)+\kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}$.

Let α be a unit speed spacelike space curve with a timelike binormal. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve.

Proposition 2.8 *Let α be a unit speed spacelike space curve with a timelike binormal and the natural lift $\overline{\alpha}$ of the curve α be a space curve with curvature $\overline{\kappa}$ and torsion $\overline{\tau}$.Then*

$$\overline{\kappa}(s) = \frac{\|W\|}{\kappa(s)}, \overline{\tau}(s) = \frac{-\kappa'(s)\tau(s) - \kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}.$$

Proposition 2.9 *Let α be a unit speed spacelike space curve with a timelike binormal and the natural lift $\overline{\alpha}$ of the curve α be a space curve.*

(i) If the natural lift $\overline{\alpha}$ is a unit speed spacelike space curve with a spacelike binormal, then

$$\bar{W} = -\frac{\tau(s)}{\kappa(s)}T + \left(\frac{-\kappa'(s)\tau(s) - \kappa(s)\tau'(s)}{\kappa(s)\|W\|^2}\right)N - B.$$

(ii) If the natural lift $\overline{\alpha}$ is a unit speed spacelike space curve with a timelike binormal, then

$$\bar{W}=\frac{\tau(s)}{\kappa(s)}T+\left(\frac{\kappa^{'}(s)\tau(s)+\kappa(s)\tau^{'}(s)}{\kappa(s)\|W\|^{2}}\right)N+B.$$

Let X and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v)=\alpha(s)+vC(s),\ \overline{X}(s,v)=\overline{\alpha}(s)+v\overline{C}(s)$$

The striction curves of X and $\overline{X}$ are given by $\beta(s)=\alpha(s)-\lambda C(s)$ and $\overline{\beta}(s)=\overline{\alpha}(s)-\mu\overline{C}(s)$,respectively. The distribution parameters of the ruled surfaces X and $\overline{X}$ are defined by

$$P_{C}=\frac{\det\left(\alpha^{'},C,C^{'}\right)}{\|C^{'}\|^{2}}\text{ and }\overline{P}_{\overline{C}}=\frac{\det\left(\overline{\alpha}^{'},\overline{C},\overline{C}^{'}\right)}{\left\|\overline{C}^{'}\right\|^{2}}.$$

Proposition 2.10 *If the natural lift $\overline{\alpha}$ is a unit speed spacelike space curve with a timelike binormal, then*

$$\lambda=-\frac{\tau^{'}(s)}{\left[\tau^{'}(s)\right]^{2}-\left[\kappa^{'}(s)\right]^{2}}\|W\|,$$

$$\mu=-\frac{\left(2\tau(s)+\sigma^{'}(s)\right)\kappa(s)\left\|\bar{W}\right\|}{\left[\frac{-\kappa^{'}(s)\tau(s)+\kappa(s)\tau^{'}(s)}{\kappa(s)^{2}}\right]^{2}+\left[2\tau(s)+\sigma^{'}(s)\right]^{2}-\left[\sigma(s)\tau(s)\right]^{2}},$$

$$P_{C}=0,\quad \overline{P}_{\overline{C}}=\frac{\kappa(s)\left(\frac{\kappa(s)\tau^{'}(s)-\kappa^{'}(s)\tau(s)}{\kappa(s)^{2}}-\kappa(s)\sigma(s)\right)+\sigma(s)\tau(s)^{2}}{\left|\left[\frac{-\kappa^{'}(s)\tau(s)+\kappa(s)\tau^{'}(s)}{\kappa(s)^{2}}-\kappa(s)\sigma(s)\right]^{2}+\left[2\tau(s)+\sigma^{'}(s)\right]^{2}-\left[\sigma(s)\tau(s)\right]^{2}\right|}$$

where $\sigma(s)=\frac{\kappa^{'}(s)\tau(s)+\kappa(s)\tau^{'}(s)}{\kappa(s)\|W\|^{2}}$.

Proposition 2.11 *If the natural lift $\overline{\alpha}$ is a unit speed spacelike space curve with a spacelike binormal, then*

$$\lambda=-\frac{\tau^{'}(s)}{\left[\tau^{'}(s)\right]^{2}-\left[\kappa^{'}(s)\right]^{2}}\|W\|,$$

$$\mu=-\frac{\left(-2\tau(s)+\sigma^{'}(s)\right)\kappa(s)\left\|\bar{W}\right\|}{\left[-\frac{-\kappa^{'}(s)\tau(s)+\kappa(s)\tau^{'}(s)}{\kappa(s)^{2}}\right]^{2}+\left[-2\tau(s)+\sigma^{'}(s)\right]^{2}-\left[\sigma(s)\tau(s)\right]^{2}},$$

$$P_{C}=0,\quad \overline{P}_{\overline{C}}=\frac{\kappa(s)\left(-\frac{\kappa(s)\tau^{'}(s)-\kappa^{'}(s)\tau(s)}{\kappa(s)^{2}}+\kappa(s)\sigma(s)\right)-\sigma(s)\tau(s)^{2}}{\left|\left[-\frac{-\kappa^{'}(s)\tau(s)+\kappa(s)\tau^{'}(s)}{\kappa(s)^{2}}+\kappa(s)\sigma(s)\right]^{2}+\left[-2\tau(s)-\sigma^{'}(s)\right]^{2}-\left[-\sigma(s)\tau(s)\right]^{2}\right|}$$

where $\sigma(s)=\frac{\kappa^{'}(s)\tau(s)+\kappa(s)\tau^{'}(s)}{\kappa(s)\|W\|^{2}}$.

Example 2.1 Let $\alpha(s) = \left(\frac{2\sqrt{3}}{3}s, \frac{1}{3}\cos\left(\sqrt{3}s\right), \frac{1}{3}\sin\left(\sqrt{3}s\right)\right)$ be a unit speed (timelike curve) timelike circular helix with

$$\begin{aligned} T(s) &= \left(\frac{2\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), \frac{\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right), \\ N(s) &= \left(0, -\cos\left(\sqrt{3}s\right), -\sin\left(\sqrt{3}s\right)\right), \\ B(s) &= \left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), -\frac{2\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right) \text{ and } \kappa = 1, \tau = 2, \\ C(s) &= (1, 0, 0). \end{aligned}$$

$$X(s,t) = \left(\frac{2\sqrt{3}}{3}s + t, \frac{1}{3}\cos\left(\sqrt{3}s\right), \frac{1}{3}\sin\left(\sqrt{3}s\right)\right)$$

and

$$\bar{X}(s,t) = \left(\frac{2\sqrt{3}}{3} + \frac{2t}{\sqrt{3}}, -\frac{\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), \frac{\sqrt{3}}{3}\cos\left(\sqrt{3}s\right) + \frac{t}{\sqrt{3}}\right).$$

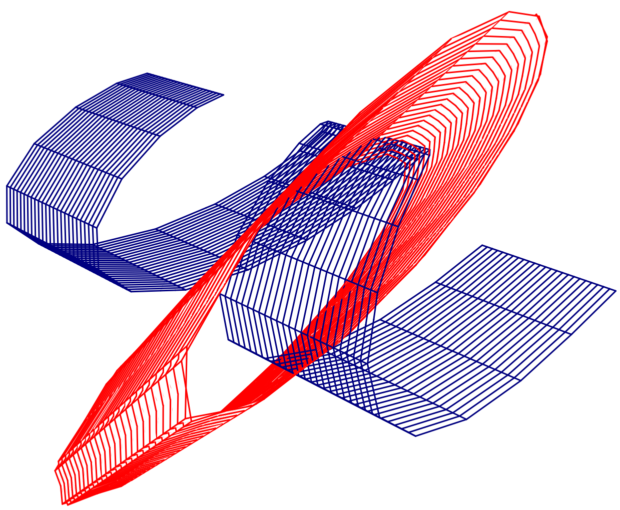

Figure 1

Example 2.2 Let $\alpha(s) = \left(\cosh\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, \sinh\left(\frac{s}{\sqrt{2}}\right)\right)$ be a unit speed spacelike hyperbolic helix with

$$\begin{aligned} T(s) &= \left(\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right) \\ N(s) &= \left(\cosh\left(\frac{s}{\sqrt{2}}\right), 0, \sinh\left(\frac{s}{\sqrt{2}}\right)\right), \end{aligned}$$

$$
\begin{aligned}
B(s) &= \left(-\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right), \text{ and } \kappa = \frac{1}{2}, \tau = \frac{1}{2} \\
C(s) &= \left(\sinh\left(\frac{s}{\sqrt{2}}\right), 0, \cosh\left(\frac{s}{\sqrt{2}}\right)\right),
\end{aligned}
$$

$$
X(s,t) = \left(\cosh\left(\frac{s}{\sqrt{2}}\right) + t\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, \sinh\left(\frac{s}{\sqrt{2}}\right) + t\cosh\left(\frac{s}{\sqrt{2}}\right)\right)
$$

and

$$
\bar{X}(s,t) = \left(\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}t, \frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right)
$$

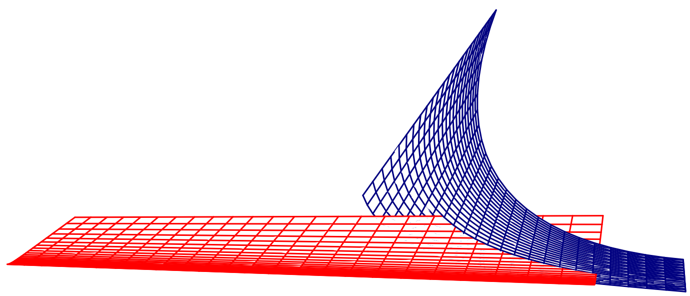

Figure 2

Example 2.3 Let $\alpha(s) = \left(\frac{\sqrt{3}}{3}s, \frac{2}{3}\cos\left(\sqrt{3}s\right), \frac{2}{3}\sin\left(\sqrt{3}s\right)\right)$ be a unit speed (spacelike curve with timelike binormal) spacelike circular helix with

$$
\begin{aligned}
T(s) &= \left(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), \frac{2\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right), \\
N(s) &= \left(0, -\cos\left(\sqrt{3}s\right), -\sin\left(\sqrt{3}s\right)\right), \\
B(s) &= \left(\frac{2\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), \frac{\sqrt{3}}{3}\cos\left(\sqrt{3}s\right)\right) \text{ and } \kappa = 2, \tau = 1 \\
C(s) &= (1,0,0)
\end{aligned}
$$

$$
X(s,t) = \left(\frac{\sqrt{3}}{3}s + t, -\frac{2}{3}\cos\left(\sqrt{3}s\right), \frac{2}{3}\sin\left(\sqrt{3}s\right)\right)
$$

and

$$
\bar{X}(s,t) = \left(\frac{\sqrt{3}}{3}s - \frac{1}{\sqrt{3}}t, -\frac{2\sqrt{3}}{3}\sin\left(\sqrt{3}s\right), \frac{2\sqrt{3}}{3}\cos\left(\sqrt{3}s\right) - \frac{2}{\sqrt{3}}t\right)
$$

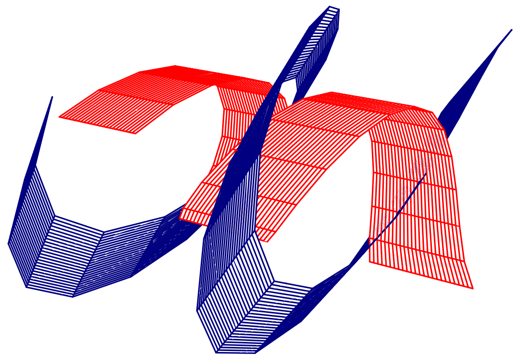

Figure 3

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International J.Math. Combin. Vol.2(2016), 29-42

Tutte Polynomial of Generalized Flower Graphs

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Abstract: The book graph denoted by $B_{n,2}$ is the Cartesian Product $S_{n+1} \times P_2$ where S_{n+1} is a star graph with n vertices of degree 1 and one vertex of degree n and P_2 is the path graph of 2 vertices. Let $X_{n,p}$ denote the generalized form of Book graph where a family of p cycles which are n in number, is merged at a common edge. The generalized flower graph is obtained by merging t copies of $X_{n,p}$ with a base cycle C_t of length t at the common edges. The resultant structure looks like flower with petals. In this paper we discuss some properties satisfied by Tutte polynomial of this special graph and the related graphs.

Key Words: Tutte polynomial, recurrence relation, flower graph.

AMS(2010): 05C30, 05C99, 68R05.

§1. Introduction and Preliminaries

Tutte polynomial is a polynomial in two variables x, y with remarkable properties and it can be defined for a graph, matrix and more generally for matroids. Tutte polynomial is closely associated with many graphical invariants and in fact the following are the special cases of Tutte polynomial along particular curves of (x, y) plane.

(1) The chromatic and flow polynomial of a graph;

(2) The partition function of a $\mathbb{Q}$-state Pott's model;

(3) The Jone's polynomial of an alternating knot;

(4) The weight enumerator of a linear code over $GF(q)$;

(5) The all terminal reliability probability of a network;

(6) The number of spanning trees,number of forests, number of connected spanning subgraphs, the dimension of bicycle space and so on.

Tutte polynomial is widely studied for the reason that it provides structural information about the graph.

Definition 1.1 (*i*) *Let $G = (V, E)$ be an undirected connected multi-graph. The Tutte polyno-*

[1]Received November 3, 2015, Accepted May 12, 2016.

mial of the graph G is given by

$$\begin{aligned} T(G,x,y) &= 1 \ if \ E(G)=\phi; \\ &= xT(G.e,x,y) \ if \ e \in E \ and \ e \ is \ a \ cut \ edge; \\ &= yT(G-e,x,y) \ if \ e \in E \ and \ e \ is \ a \ loop; \\ &= T(G-e,x,y)+T(G.e,x,y) \ if \ e \in E \ and \ e \ is \ neither \ a \ loop \ nor \ a \ cut \ edge. \end{aligned}$$

(ii) If G is a disconnected graph with connected components $G_1, G_2, \cdots, G_t$ with $t \geq 2$, then the Tutte Polynomial of G denoted by $T(G,x,y)$ is defined as $T(G,x,y) = \prod_{i=1}^{t} T(G_i,x,y)$.

Tutte polynomial of some of standard graphs are given below.

Theorem 1.2 *Let T_n be a tree on n vertices and let C_n be a cycle on n vertices then*

(1) $T(T_n,x,y) = x^{n-1}$;

(2) $T(C_n,x,y) = y + \sum_{i=1}^{n-1} x^i$.

Theorem 1.3 *Let G be a bi connected graph. Let u,v be two vertices in G such that u,v are joined by a path P^s of length s where degree of each vertex in P^s is two except possibly for u,v then*

$$T(G) = (1+x+x^2+\cdots x^{s-1})T(G-P^s) + T(G.P^s).$$

Proof Let $e_1, e_2, \cdots e_s$ be the s edges in the path P^s, then

$$\begin{aligned} T(G) &= T(G-e_1)+T(G.e_1) \\ &= x^{s-1}T(G-P^s)+x^{s-2}T(G-P^s)+\cdots x^{s-s}T(G-P^s)+T(G.P^s) \\ &\quad (G \text{ is bi}-\text{connected}, e^t \text{ is not a bridge in } G-G_1-G_2\cdots-G_{t-1}) \\ &= (1+x+x^2+\cdots x^{s-1})T(G-P^s)+T(G.P^s). \end{aligned}$$ □

We study the Tutte polynomial of generalized Book graph. Cartesian product of two graphs G_1, G_2 denoted by $G_1 \times G_2$ is a graph with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices $(u_1,v_1),(u_2,v_2)$ of $G_1 \times G_2$ are adjacent if and only if either $u_1 = u_2$ and (v_1,v_2) is an edge in G_2 or $v_1 = v_2$ and (u_1,u_2) is an edge of G_1. The book graph denoted by $B_{n,2}$ or simplyB_n is the Cartesian Product $S_{n+1} \times P_2$ where S_{n+1} is a star graph with n vertices of degree 1 and one vertex of degree n and P_2 is the path graph of 2 vertices. It can be observed that book graphs are planar. Some book graphs and their planar representation are given below.

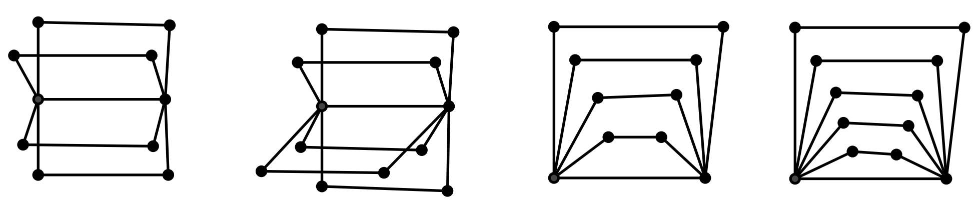

Figure 1 Book graph B_4 , B_5 and their Planar representation

We make a generalization of this graph. Through out this section $T(G,x,y) = T(G)$ denotes the Tutte polynomial for the graph G. We make use of the following notation. $X_{n,p}$ denote a graph with n number of p-cycles with a common edge $e = xy$ and let $Y_{n,p} = X_{n,p} - e$ and $Z_{n,p} = X_{n,p}.e$. Note that $Z_{n,p}$ is actually a graph with n number of $p-1$ cycles with a common vertex.

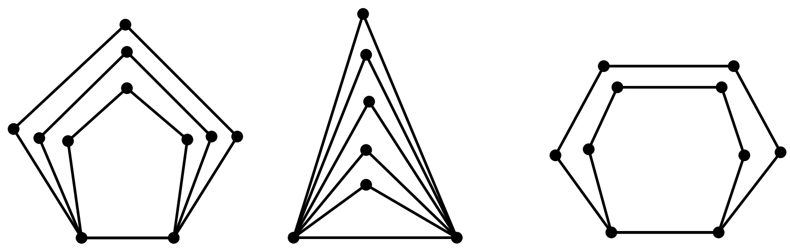

Figure 2 $X_{3,5}$, $X_{5,3}$,$X_{2,6}$

Thus $B_n = X_{n,4}$ is a particular case of the graph we have defined which we call as generalized book graph. We first arrive at some recurrence relation satisfied by these graphs. Before we prove the relations satisfied by these graphs we will prove some preliminary results.

Notations and Conventions 1.4

(1) Let G_1 and G_2 be two disjoint graphs each of them having a unique identified vertex. The graph obtained by merging an identified vertex of G_1 to an identified vertex of G_2 is denoted by $G_1 \times G_2$.

(2) Let G_1, G_2 be two disjoint graphs each of them having two designated vertices namely x, y and x', y'. The graph obtained by merging the identified vertex x with x' and the vertex y with y' is denoted by $G_1 * G_2$.

(3) Let G be a graph with an identified vertex v. The graph obtained by taking n copies of G and joining all the copies at the identified vertex v is denoted by $G^{(n)}$.

(4) Let G_1, G_2 be two disjoint graphs each with two identified vertices x, y and x', y' respectively. Let $xy \in E(G)$ and $x'y' \in E(G')$. The graph obtained by merging the two vertices x, x' and y, y' and the edges xy and $x'y'$ to a single edge is denoted by $G_1 \odot G_2$.

Proposition 1.5 *G be a graph which can be expressed as $G = H \times P_l$ where P_l is a tree of order $l+1$, then $T(G) = x^l T(H)$.*

Proposition 1.6 $T(C_p^{(n)}) = T(C_p)^n = \left[y + \sum\limits_{k=1}^{p-1} x^k\right]^n$.

Proof Induction on n. For $n = 1, T(C_p) = y + \sum\limits_{k=1}^{p-1} x^k$ by Theorem 1.2. Let v be the identified vertex. Assume that the result is true for $n-1$. Let $G = C_p^{(n)}$. Note that $C_p^{(n)} = C_p^{(n-1)} \times C_p$.

Let e be any edge adjacent to v. By recurrence relation

$$\begin{aligned}
T(G) &= T(G-e)+T(G.e)\\
&= T(C_p^{(n-1)}\times P_{p-1})+T(C_p^{(n-1)}\times C_{p-1})\\
&= x^{p-1}T(C_p^{(n-1)})+T(C_p^{(n-1)}\times C_{p-1})\\
&= x^{p-1}T(C_p^{(n-1)})+T(C_p^{(n-1)}\times P_{p-2})+T(C_p^{(n-1)}\times C_{p-2})\\
&= x^{p-1}T(C_p^{(n-1)})+x^{p-2}T(C_p^{(n-1)})+T(C_p^{(n-1)}\times C_{p-2})\\
&\qquad \text{by using Proposition 1.5}\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&= (x^{p-1}+x^{p-2}+\cdots+x)T(C_p^{(n-1)})+T(C_p^{(n-1)}\times C_1)\\
&= (x^{p-1}+x^{p-2}+\cdots+x)T(C_p^{(n-1)})+yT(C_p^{(n-1)}) \text{ as } C_1 \text{ is a loop}\\
&= (x^{p-1}+x^{p-2}+\cdots+x+y)T(C_p^{(n-1)})\\
&= \{y+\sum_{k=1}^{p-1}x^k\}\left[y+\sum_{k=1}^{p-1}x^k\right]^{n-1} = \left[y+\sum_{k=1}^{p-1}x^k\right]^{n}.
\end{aligned}$$

□

§2. Tutte Polynomial of Generalized Book Graph

Theorem 2.1 *Let $X_{n,p}$ denote a graph with n number of p-cycles with a common edge $e=xy$ and let $Y_{n,p}=X_{n,p}-e$ then,*
$X_{n,p}$ and $Y_{n,p}$ satisfy the following recurrence relations

(i) $T(X_{n,p})=T(Y_{n,p})+T(C_{p-1})^n$;

(ii) $T(Y_{n,p})=\left[\sum_{k=1}^{p-2}x^k\right]T(Y_{n-1,p})+T(X_{n-1,p})$ *for* $n\geq 2$ *with* $T(Y_{1,p})=x^{p-1}$.

Proof *(i)* e is neither a loop nor a cut edge and hence using recurrence relation of Tutte polynomial

$$\begin{aligned}
T(X_{n,p}) &= T(X_{n,p}-e)+T(X_{n,p}.e)\\
&= T(Y_{n,p})+T(C_{p-1}^{(n)})=T(Y_{n,p})+T(C_{p-1})^n
\end{aligned}$$

using Proposition 1.6. This proves (1).

(ii) Clearly, $Y_{1,p}=$ a path of length $p-1$ and hence $T(Y_{1,p})=x^{p-1}$. We prove this result by induction on n.

For $n=2$, $T(Y_{2,p})=T(Y_{2,p}-e')+T(Y_{2,p}.e')$ where e' is any edge of $Y_{2,p}$ adjacent to x

other than e.

$$\begin{aligned} T(Y_{2,p}) &= x^{p-2}T(Y_{1,p}) + \mathbf{T}(\mathbf{Y_{1,p}} * \mathbf{P_{p-2}}) \\ &= x^{p-2}T(Y_{1,p}) + x^{p-3}T(Y_{1,p}) + \mathbf{T}(\mathbf{Y_{1,p}} * \mathbf{P_{p-3}}) \\ &= x^{p-2}T(Y_{1,p}) + x^{p-3}T(Y_{1,p}) + \cdots + xT(Y_{1,p}) + \mathbf{T}(\mathbf{Y_{1,p}} * \mathbf{P_{1}}) \\ &= \left[\sum_{k=1}^{p-2} x^k\right] T(Y_{1,p}) + T(X_{1,p}), \end{aligned}$$

which proves the result for $n = 2$.

Assume that the result is true for a graph $Y_{n-1,p}$. Consider $Y_{n,p}$ and let e' is any edge of $Y_{n-1,p}$ adjacent to x other than e. Then,

$$\begin{aligned} T(Y_{n,p}) &= T(Y_{n-1,p} - e') + T(Y_{n-1,p}.e') \\ &= x^{p-2}T(Y_{n-1,p}) + \mathbf{T}(\mathbf{Y_{n-1,p}} * \mathbf{P_{p-2}}) \\ &= x^{p-2}T(Y_{n-1,p}) + x^{p-3}T(Y_{n-1,p}) + \mathbf{T}(\mathbf{Y_{n-1,p}} * \mathbf{P_{p-3}}) \\ &= x^{p-2}T(Y_{n-1,p}) + x^{p-3}T(Y_{n-1,p}) + \cdots + xT(Y_{n-1,p}) + \mathbf{T}(\mathbf{Y_{n-1,p}} * \mathbf{P_{1}}) \\ &= \left[\sum_{k=1}^{p-2} x^k\right] T(Y_{n-1,p}) + T(X_{n-1,p}), \end{aligned}$$

which proves the result for $n \geq 2$. □

Theorem 2.2 *(i)* $T(Y_{n,p}) = b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k(b+y-1)^{n-1-k}\right]$ *for* $n \geq 2$*;*

(ii) $T(X_{n,p}) = b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k(b+y-1)^{n-1-k}\right] + (b+y-1)^n$*, where* $b = \sum_{k=0}^{p-2} x^k$ *for* $n \geq 2$.

Proof By Theorem 2.1,

$$\begin{aligned} T(Y_{n,p}) &= \left[\sum_{k=1}^{p-2} x^k\right] T(Y_{n-1,p}) + T(X_{n-1,p}) \\ &= \left[\sum_{k=1}^{p-2} x^k\right] T(Y_{n-1,p}) + T(Y_{n-1,p}) + T(C_{p-1})^{n-1} \\ &= \left[\sum_{k=0}^{p-2} x^k\right] T(Y_{n-1,p}) + T(C_{p-1})^{n-1} \\ &= bT(Y_{n-1,p}) + T(C_{p-1})^{n-1}. \end{aligned}$$

Note that $T(C_{p-1}) = x^{p-2} + x^{p-3} + \cdots + x + y = b + y - 1$. We solve the recurrence relation,

$$\begin{aligned}
T(Y_{1,p}) &= x^{p-1}, \\
T(Y_{2,p}) &= bT(Y_{1,p}) + T(C_{p-1}) \\
&= bx^{p-1} + (y+b-1) = b^1x^{p-1} + \left[\sum_{k=0}^{2-2} b^k(b+y-1)^{2-1-k}\right], \\
T(Y_{3,p}) &= bT(Y_{2,p}) + T(C_{p-1})^2 \\
&= b^2x^{p-1} + bT(C_{p-1}) + (y+b-1)^2 \\
&= b^2x^{p-1} + b(y+b-1) + (y+b-1)^2 \\
&= b^{3-1}x^{p-1} + \left[\sum_{k=0}^{3-2} b^k(b+y-1)^{3-1-k}\right], \\
T(Y_{4,p}) &= bT(Y_{3,p}) + T(C_{p-1})^3 \\
&= b^3x^{p-1} + b^2T(C_{p-1}) + bT(C_{p-1})^2 + T(C_{p-1})^3 \\
&= b^3x^{p-1} + b^2(b+y-1) + b(b+y-1)^2 + (b+y-1)^3 \\
&= b^{4-1}x^{p-1} + \left[\sum_{k=0}^{4-2} b^k(b+y-1)^{4-1-k}\right].
\end{aligned}$$

Assume that by induction

$$\begin{aligned}
T(Y_{n-1,p}) &= b^{n-2}x^{p-1} + \left[\sum_{k=0}^{n-3} b^k(b+y-1)^{n-2-k}\right], \\
T(Y_{n,p}) &= bT(Y_{n-1,p}) + T(C_{p-1})^{n-1} \\
&= b\left\{b^{n-2}x^{p-1} + \left[\sum_{k=0}^{n-3} b^k(b+y-1)^{n-2-k}\right]\right\} + (b+y-1)^{n-1} \\
&= b^{n-1}x^{p-1} + b[(b+y-1)^{n-2} + b(b+y-1)^{n-3} + \cdots + b^{n-3}] + (b+y-1)^{n-1} \\
&= b^{n-1}x^{p-1} + (b+y-1)^{n-1} + b(b+y-1)^{n-2} + \cdots + b^{n-2}(b+y-1) \\
&= b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k(b+y-1)^{n-1-k}\right].
\end{aligned}$$

This completes the proof of (i).

$$\begin{aligned}
T(X_{n,p}) &= T(Y_{n,p}) + (b+y-1)^n \\
&= b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k(b+y-1)^{n-1-k}\right] + (b+y-1)^n,
\end{aligned}$$

which completes proof of (ii). □

Remark 2.3 For $n = 1$,

$$\begin{aligned} T(X_{n,p}) &= T(C_p) \\ &= x^{p-1} + x^{p-2} + \cdots + x + yx^{p-1} + (b + y - 1), \end{aligned}$$

which matches with the Theorem 2.2

An equivalent representation of Tutte polynomial for generalized book graph is the following.

Theorem 2.4 $T(X_{n,p}) = xb^n + \left[\sum_{k=0}^{n-1} b^k(y + b - 1)^{n-1-k}\right] y$

Proof By Theorem 2.2,

$$T(X_{n,p}) = b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k(b + y - 1)^{n-1-k}\right] + (b + y - 1)^n <$$

$$\begin{aligned} b &= x^{p-2} + x^{p-3} + \cdots + x + 1 = \frac{x^{p-1} - 1}{x - 1} \\ &\Rightarrow x^{p-1} = b(x - 1) + 1 = bx - b + 1 \\ &\Rightarrow x^{p-1}b^{n-1} = (bx - b + 1)b^{n-1} = xb^n - b^n + b^{n-1}. \end{aligned}$$

Hence,

$$\begin{aligned} T(X_{n,p}) &= xb^n - b^n + b^{n-1} + \left[\sum_{k=0}^{n-2} b^k(b + y - 1)^{n-1-k}\right] + (b + y - 1)^n \\ &= xb^n - b^n + b^{n-1} + (b + y - 1)^{n-1} + b(b + y - 1)^{n-2} + \cdots \\ &\quad + b^{n-2}(b + y - 1) + (b + y - 1)^n. \qquad (*) \end{aligned}$$

Now consider

$$\begin{aligned} & xb^n + \left[\sum_{k=0}^{n-1} b^k(y + b - 1)^{n-1-k}\right] y \\ &= xb^n + y\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &= xb^n + (y + b - 1 - (b - 1))\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &= xb^n + (y + b - 1)\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &\quad -(b - 1)\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &= xb^n + \left[(y - b - 1)^n + b(y - b + 1)^{n-1} + \cdots + b^{n-1}(y + b - 1)\right] \\ &\quad -b(\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &\quad + \left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \end{aligned}$$

$$\begin{aligned} &= xb^n + (y+b-1)^n + b(y+b-1)^{n-1} + \cdots + (y+b-1)b^{n-1} \\ &\quad -b(y+b-1)^{n-1} - b^2(y+b-1)^{n-2} - \cdots - b^{n-1}(b+y-1) - b^n \\ &\quad +(y+b-1)^{n-1} + b(y+b-1)^{n-2} + \cdots + b^{n-2}(y+b-1) + b^{n-1} \\ &= xb^n + (y+b-1)^n - b^n + (y+b-1)^{n-1} + b(y+b-1)^{n-2} + \cdots \\ &\quad +b^{n-2}(y+b-1) + b^{n-1}. \qquad (**) \end{aligned}$$

From (*) and (**) we get

$$T(X_{n,p}) = xb^n + \left[\sum_{k=0}^{n-1} b^k (y+b-1)^{n-1-k}\right] y. \qquad \square$$

§3. The Generalized Flower Graph

The generalized flower graph is obtained by merging $X_{n,p}$ at each of the edge of a basic cycle of length t. We define the generalized complete flower graph and generalized Flower graph with k petals.

Definition 3.1 (*i*) *A graph in which i copies of $X_{n,p}$ is taken and is merged with any of the i out of t edges of the base cycle C_t of length t where, $1 \leq i \leq t-1$ is called a generalized flower graph with i petals and is denoted by $G^{(i)}_{n,p,t}$.*

(*ii*) *A graph obtained by taking a base cycle C_t of length t and t copies of $X_{n,p}$ and merging the two graphs at the common edge of $X_{n,p}$ with each of the edge of the basic cycle C_t is referred to as Generalized Flower Graph or Generalized Complete Flower graph and is denoted by $G_{n,p,t}$. In fact $G_{n,p,t} = G^{(t)}_{n,p,t}$.*

(*iii*) *The graph obtained by taking i copies of $X_{n,p}$ with each of the cycle containing a designated edge and joining the i copies at the end vertices of the designated edges is denoted by $H^{(i)}_{n,p}$.*

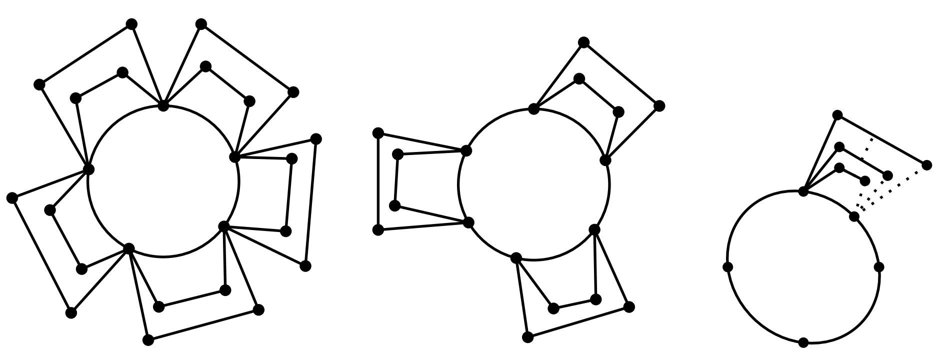

Figure 3 Generalized complete flower graph $G_{2,4,5}$ and flower graph $G^{(3)}_{2,4,6}$ with 3 petals and generalized flower graph with one petal

Theorem 3.2 *Let $X_{n,p}$ has the common edge e. Let $G^{(1)}_{n,p,t}$ be the generalized flower graph with* *one petal. Then,*

$$T(G^{(1)}_{n,p,t}) = (1 + x + x^2 + \cdots + x^{t-2})T(X_{n,p}) + y(y + b - 1)^n.$$

Proof Let e_1 be any edge on C_t other than e. By deletion contraction formula we get

$$\begin{aligned} T(G_{n,p,t}) &= x^{t-2}T(X_{n,p}) + T(G_{n,p,t-1}) \\ &= x^{t-2}T(X_{n,p}) + x^{t-3}T(X_{n,p}) + \cdots + xT(X_{n,p}) + T(G_{n,p,2}) \\ &= (x + x^2 + \cdots + x^{t-2})T(X_{n,p}) + T(X_{n,p}) + yT(C^n_{p-1}) \\ &= (1 + x + x^2 + \cdots + x^{t-2})T(X_{n,p}) + y(y + b - 1)^n, \end{aligned}$$

which completes the proof. □

Corollary 3.3 $T(G_{n,p,p}) = b^n x^{n-1} + \sum\limits_{k=1}^{n} b^k (b + y - 1)^{n-k} + (y + b - 1)^{n+1}$

Proof By Theorem 2.5 taking $p = t$ we get

$$\begin{aligned} T(G_{n,p,p}) &= (1 + x + x^2 + \cdots + x^{p-2})T(X_{n,p}) + y(y + b - 1)^n \\ &= b\left\{ b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k (b + y - 1)^{n-1-k}\right] + (b + y - 1)^n \right\} + y(y + b - 1)^n \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n-2} b^{k+1} (b + y - 1)^{n-1-k}\right] + b(b + y - 1)^n + y(y + b - 1)^n \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n-2} b^{k+1} (b + y - 1)^{n-1-k}\right] + (b + y)(b + y - 1)^n \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n-2} b^{k+1} (b + y - 1)^{n-1-k}\right] + (b + y - 1 + 1)(y + b - 1)^n \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n-2} b^{k+1} (b + y - 1)^{n-1-k}\right] + (b + y - 1)^n + (y + b - 1)^{n+1} \\ &= b^n x^{p-1} + b(b + y - 1)^{n-1} + b^2(b + y - 1)^{n-2} + \cdots + b^{n-1}(b + y - 1) \\ &\quad + (b + y - 1)^n + (b + y - 1)^{n+1} \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n} b^{k+1} (b + y - 1)^{n-k}\right] + (y + b - 1)^{n+1}. \end{aligned}$$

Lemma 3.4 *Let u, v be two vertices of a graph G which are joined by n disjoint paths of length $p - 1$, namely $P_1, P_2, \cdots P_n$ such that degree of each of vertices in $P_1, P_2, \cdots P_n$ other than u, v is 2 in G and removal of these n paths does not disconnect u and v, then*

$$T(G) = b^n T(G'') + \left[\sum_{k=0}^{n-1} b^k (y + b - 1)^{n-1-k}\right] T(G'),$$

where G' is obtained from G by removing the n disjoint paths of length $p-1$ between u, v and identifying u, v and G'' is the graph is obtained by removing the n disjoint paths of length $p-1$ from G.

Proof Using Theorem 1.3, $T(G) = bT(G - P_1) + T(G.P_1)$. But

$$T(G.P_1) = T(C_{p-1}^{n-1} \times G') = (y + b - 1)^{n-1} T(G').$$

Hence

$$\begin{aligned}
T(G) &= bT(G - P_1) + (y + b - 1)^{n-1} T(G') \\
&= b\left[bT(G - P_1 - P_2) + (y + b - 1)^{n-2} T(G')\right] + (y + b - 1)^{n-1} T(G') \\
&= b^2 T(G - P_1 - P_2) + \left[b(y + b - 1)^{n-2} + (y + b - 1)^{n-1}\right] T(G') \\
&= b^2 T(G - P_1 - P_2 - P_3) + \left[b^2(y + b - 1)^{n-3} + b(y + b - 1)^{n-2}\right. \\
&\quad \left. + (y + b - 1)^{n-1}\right] T(G') \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&= b^n T(G - P_1 - P_2 \cdots P_n) + \left[b^{n-1}(y + b - 1)^0 + b^{n-2}(y + b - 1)^1 + \cdots\right. \\
&\quad \left. + b(y + b - 1)^{n-2} + (y + b - 1)^{n-1}\right] T(G') \\
&= b^n T(G'') + \left[\sum_{k=0}^{n-1} b^k (y + b - 1)^{n-1-k}\right] T(G'),
\end{aligned}$$

which completes the proof. □

From the above theorem we get another method of proving Theorem 2.4.

Corollary 3.5 $T(X_{n,p}) = xb^n + \left[\sum\limits_{k=0}^{n-1} b^k (y + b - 1)^{n-1-k}\right] y.$

Proof Applying Lemma 3.5 to $X_{n,p}$we get $G'' = K_2$ and $G' =$ a single loop so that $T(G'') = x$, $T(G') = y$ and we obtain the result. □

Theorem 3.6 *Let $H_{n,p}^{(i)}$ denote a graph obtained by taking i copies of $X_{n,p}$ and joining it at a common vertex in succession. Then, $H_{n,p}^{(i)} = [T(X_{n,p})]^i$.*

Proof Let $e = uv$ be the common edge of the i^{th} copy of $X_{n,p}$. If G'' is the graph obtained by removing the n distinct paths of length $p-1$, then the resultant graph is a graph obtained by joining the $i-1$ copies of $X_{n,p}$ with edge $e = uv$ at the vertex u and hence $T(G'') = xT(H_{n,p}^{(i-1)})$. If G' is obtained by removing n disjoint paths of length $p-1$ between u, v and identifying u and v then, $T(G') = yT(H_{n,p}^{(i-1)})$. Thus using Lemma 3.4

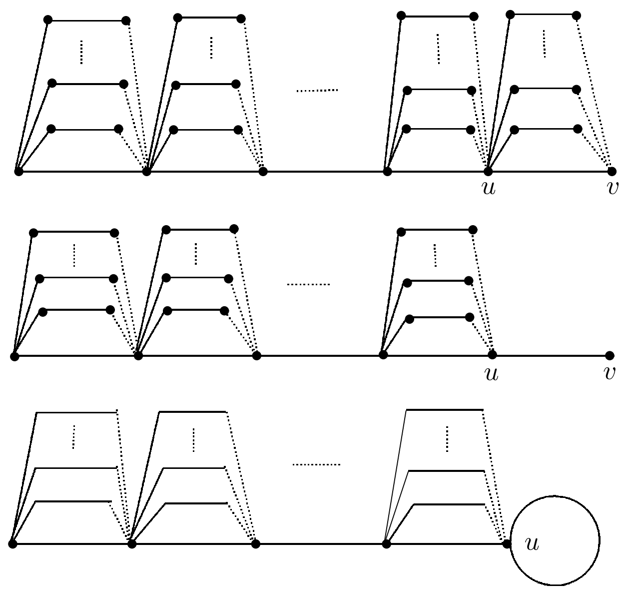

Figure 4 $H_{n,p}^{(i)}$ graph G'' and G' segregation

$$\begin{aligned}
T(H_{n,p}^{(i)}) &= xb^nT(H_{n,p}^{(i-1)}) + y\left[\sum_{k=0}^{n-1} b^k(y+b-1)^{n-1-k}\right]T(H_{n,p}^{(i-1)})\\
&= \left\{xb^n + y\left[\sum_{k=0}^{n-1} b^k(y+b-1)^{n-1-k}\right]\right\}T(H_{n,p}^{(i-1)})\\
&= [T(X_{n,p})]\,T(H_{n,p}^{(i-1)})\\
&= [T(X_{n,p})]^2\,T(H_{n,p}^{(i-2)}),\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$[T(X_{n,p})]^{i-1}\,T(H_{n,p}^{(1)}) = [T(X_{n,p})]^{i-1}\,T(X_{n,p}) = \left[T(X_{n,p})^i\right]. \qquad \square$$

Corollary 3.7 *Let G denote a graph obtained by taking i copies of $X_{n,p}$ and t copies of K_2 and joining it in succession in any order then,*

$$T(G) = x^tT(H_{n,p}^{(i)}) = x^t\,[T(X_{n,p})]^i\,.$$

Theorem 3.8 *Let $G_{n,p,t}$ denote a graph obtained by taking t copies of $X_{n,p}$ and taking $\odot$ product with C_t in succession, then*

$$\begin{aligned}
T(G_{n,p,t}) &= b^n\sum_{k=0}^{t-2}(b^n+\alpha y)^k(b^nx+\alpha y)^{t-1-k}\\
&\quad +(b^n+\alpha y)^{t-2}\alpha y(b+y-1)^n + (b^n+\alpha y)^{t-2}y(b+y-1)^{2n},
\end{aligned}$$

where $\alpha = \left[\sum\limits_{k=0}^{n-1} b^k(y+b-1)^{n-1-k}\right]$.

Proof Let G'' be the graph obtained by removing n distinct paths of length $p-1$ between the two vertices which are end points of common edge e of any copy of $X_{n,p}$ on the cycle C_t and let G' be the graph obtained by removing n distinct paths of length $p-1$ as described for G'' and identifying the two end vertices of e in C_t. Then by Lemma 3.4.

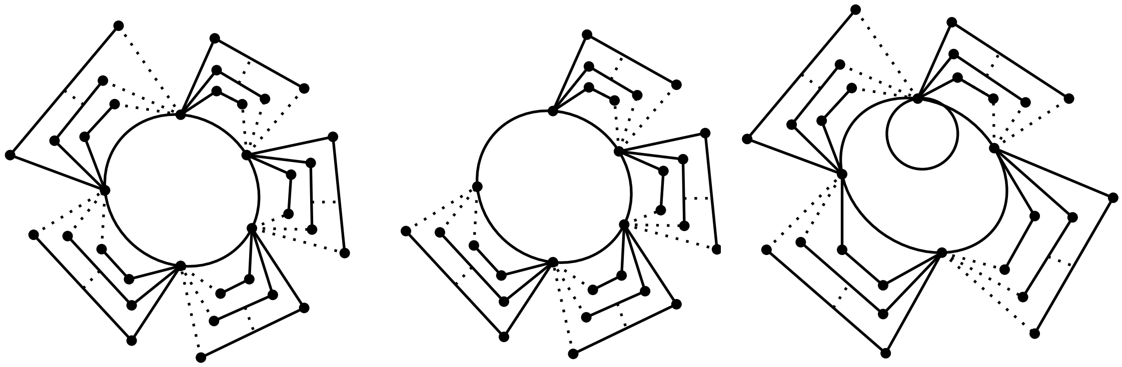

Figure 5 $G_{n,p,t}$ graph G'' and G' segregation

$$T(G_{n,p,t}) = b^n T(G'') + \alpha T(G'),$$

where $\alpha = \left[\sum\limits_{k=0}^{n-1} b^k (y+b-1)^{n-1-k}\right]$. Also using deletion contraction formula of Tutte polynomial $T(G') = yT(G_{n,p,t-1})$ and

$$T(G'') = T(H_{n,p}^{(t-1)}) + T(G_{n,p,t-1}) = T(X_{n,p})^{t-1} + T(G_{n,p,t-1})$$

from Theorem 3.6. But by Corollary 3.5,

$$T(X_{n,p}) = xb^n + \left[\sum_{k=0}^{n-1} b^k (y+b-1)^{n-1-k}\right] y = b^n x + \alpha y$$

Thus, $T(G'') = (b^n x + \alpha y)^{t-1} + T(G_{n,p,t-1})$ and

$$\begin{aligned}
T(G_{n,p,t}) &= b^n(b^n x + \alpha y)^{t-1} + b^n T(G_{n,p,t-1}) + \alpha y T(G_{n,p,t-1}) \\
&= b^n(b^n x + \alpha y)^{t-1} + (b^n + \alpha y) T(G_{n,p,t-1}) \\
&= b^n(b^n x + \alpha y)^{t-1} + (b^n + \alpha y)\left[b^n(b^n x + \alpha y)^{t-2} + (b^n + \alpha y) T(G_{n,p,t-2})\right] \\
&= b^n(b^n x + \alpha y)^{t-1} + b^n(b^n x + \alpha y)^{t-2}(b^n + \alpha y) + (b^n + \alpha y)^2 T(G_{n,p,t-2}) \\
&= b^n(b^n x + \alpha y)^{t-1} + b^n(b^n x + \alpha y)^{t-2}(b^n + \alpha y) \\
&\quad + (b^n + \alpha y)^2\left[b^n(b^n x + \alpha y)^{t-3} + (b^n + \alpha y) T(G_{n,p,t-3})\right] \\
&= b^n(b^n x + \alpha y)^{t-1} + b^n(b^n x + \alpha y)^{t-2}(b^n + \alpha y) \\
&\quad + b^n(b^n x + \alpha y)^{t-3}(b^n + \alpha y)^2 + (b^n + \alpha y)^3 T(G_{n,p,t-3}) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$\begin{aligned} &= b^n(b^nx+\alpha y)^{t-1}+b^n(b^nx+\alpha y)^{t-2}(b^n+\alpha y) \\ &\quad +b^n(b^nx+\alpha y)^{t-3}(b^n+\alpha y)^2+\cdots+(b^n+\alpha y)^{t-2}T(G_{n,p,t-(t-2)}) \\ &= b^n(b^nx+\alpha y)^{t-1}+b^n(b^nx+\alpha y)^{t-2}(b^n+\alpha y)+b^n(b^nx+\alpha y)^{t-3}(b^n+\alpha y)^2 \\ &\quad +\cdots+(b^n+\alpha y)^{t-2}T(G_{n,p,2})). \end{aligned}$$

But

$$\begin{aligned} T(G_{n,p,2}) &= b^nT(X_{n,p})+\alpha y(b+y-1)^n+y(b+y-1)^{2n} \\ &= b^n(b^nx+\alpha y)+\alpha y(b+y-1)^n+y(b+y-1)^{2n}. \end{aligned}$$

Thus

$$\begin{aligned} T(G_{n,p,t}) &= b^n(b^nx+\alpha y)^{t-1}+b^n(b^nx+\alpha y)^{t-2}(b^n+\alpha y) \\ &\quad +b^n(b^nx+\alpha y)^{t-3}(b^n+\alpha y)^2+\cdots+b^n(b^nx+\alpha y)^2(b^n+\alpha y)^{t-3} \\ &\quad +(b^n+\alpha y)^{t-2}\left[b^n(b^nx+\alpha y)+\alpha y(b+y-1)^n+y(b+y-1)^{2n}\right] \\ &= b^n(b^nx+\alpha y)^{t-1}+b^n(b^nx+\alpha y)^{t-2}(b^n+\alpha y) \\ &\quad +b^n(b^nx+\alpha y)^{t-3}(b^n+\alpha y)^2+\cdots \\ &\quad +b^n(b^nx+\alpha y)^2(b^n+\alpha y)^{t-3}+b^n(b^nx+\alpha y)(b^n+\alpha y)^{t-2} \\ &\quad +(b^n+\alpha y)^{t-2}\alpha y(b+y-1)^n+(b^n+\alpha y)^{t-2}y(b+y-1)^{2n} \\ &= b^n\sum_{k=0}^{t-2}(b^n+\alpha y)^k(b^nx+\alpha y)^{t-1-k} \\ &\quad +(b^n+\alpha y)^{t-2}\alpha y(b+y-1)^n+(b^n+\alpha y)^{t-2}y(b+y-1)^{2n}, \end{aligned}$$

which completes the proof. □

Theorem 3.9 $T(G^{(i)}_{n,p,t})=(1+x+\cdots+x^{t-i-1})(b^nx+\alpha y)^i+b^n\sum\limits_{k=0}^{i-2}(b^n+\alpha y)^k(b^nx+\alpha y)^{i-1-k}+(b^n+\alpha y)^{i-2}\alpha y(b+y-1)^n+(b^n+\alpha y)^{i-2}y(b+y-1)^{2n}$.

Proof In $G^{(i)}_{n,p,t}$ there are $t-i$ sides of C_t without petals. Let e be any side of $G^{(i)}_{n,p,t}$ without petal. Clearly e is neither a loop nor a bridge. Applying deletion and contraction formula

$$\begin{aligned} G^{(i)}_{n,p,t} &= x^{t-i-1}T(H^{(i)}_{n,p})+T(G^{(i)}_{n,p,t-1}) \\ &= x^{t-i-1}T(H^{(i)}_{n,p})+x^{t-i-2}T(H^{(i)}_{n,p})+T(G^{(i)}_{n,p,t-2}) \\ &= (x^{t-i-1}+x^{t-i-2}+\cdots+x^{t-i-(t-i)})T(H^{(i)}_{n,p})+T(G^{(i)}_{n,p,t-(t-i)}) \\ &= (1+x+\cdots+x^{t-i-1})T(H^{(i)}_{n,p}+T(G^{(i)}_{n,p,i}) \\ &= (1+x+\cdots+x^{t-i-1})T(X^{i}_{n,p})+T(G_{n,p,i}) \\ &= (1+x+\cdots+x^{t-i-1})(b^nx+\alpha y)^i+b^n\sum_{k=0}^{i-2}(b^n+\alpha y)^k(b^nx+\alpha y)^{i-1-k} \\ &\quad +(b^n+\alpha y)^{i-2}\alpha y(b+y-1)^n+(b^n+\alpha y)^{i-2}y(b+y-1)^{2n}. \end{aligned}$$

§4. Conclusion

Tutte Polynomial has been an open topic for research for mathematicians for the last 30 years. It is a two variable polynomial which reduces to many graph polynomials associated with the graph. It gives various information about the graph like the number of spanning trees, number of cyclic orientations not resulting in oriented cycles and colorability of graphs.

In this research paper, Tutte polynomial of many specialized graphs have been studied in detail. The properties and Tutte polynomials of , generalized Book graph, Generalized Book graph with petals, Complete Generalized Book graph have been arrived at which in turn reveal various other related information by substituting appropriate values for the two variables.

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International J.Math. Combin. Vol.2(2016), 43-50

Entire Equitable Dominating Graph

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Abstract: The entire equitable dominating graph $EE_qD(G)$ of a graph G with vertex set $V \cup S$, where S is the collection of all minimal equitable dominating sets of G and two vertices $u, v \in V \cup S$ are adjacent if u, v are not disjoint minimal equitable dominating sets in S or $u, v \in D$, where D is the minimal equitable dominating set in S or $u \in V$ and v is a minimal equitable dominating set in S containing u. In this paper, we initiate a study of this new graph valued function and also established necessary and sufficient conditions for $EE_qD(G)$ to be connected and complete. Other properties of $EE_qD(G)$ are also obtained.

Key Words: Dominating set, equitable dominating set, entire equitable dominating graph, Smarandachely dominating set.

AMS(2010): 05C69(70).

§1. Introduction

All graphs considered here are finite, undirected with no loops and multiple edges. We denote by p the order(i.e number of vertices) and by q the size (i.e number of edges) of such a graph G. Any undefined term and notation in this paper may be found in Harary [5].

A set of vertices which covers all the edges of a graph G is called *vertex cover* for G. The smallest number of vertices in any vertex cover for G is called its *vertex covering number* and is denoted by $\alpha_0(G)$ or α_0. A set of vertices in G is *independent* if no two of them are adjacent. The largest number of vertices in such a set is called the *vertex independence number* of G and is denoted by $\beta_0(G)$ or β_0. The *connectivity* $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results a disconnected or trivial graph. Analogously the *edge-connectivity* $\lambda = \lambda(G)$ is the minimum number of edges whose removal results a disconnected or trivial graph. The *diameter* of a connected graph is the maximum distance between two vertices in G and is denoted by $diam(G)$. If G and H are graphs with the property that the identification of any vertex of G with an arbitrary vertex of H results in a unique graph (up to

[1]Supported by UGC-SAP DRS-II New Delhi, India: for 2010-2015 and the University Grants Commission, New Delhi, India. No.F.4-1/2006(BSR)/7-101/2007(BSR) dated: 20th June, 2012.

[2]Received April 18, 2015, Accepted May 13, 2016.

isomorphism), then we write as $G \bullet H$ for this graph.

A subset D of V is called a *dominating set* of G if every vertex in $V - D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all minimal dominating sets of G. (See Ore [12]).

A subset D of V is called an *equitable dominating set* if for every $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such a dominating set is called the *equitable domination number* of G and is denoted by $\gamma^e(G)$. For more details about graph valued functions, domination number and their related parameters we refer [1-4, 6 - 10, 12]. The opposite of equitable dominating set is the *Smarandachely dominating set* with $|deg(u) - deg(v)| \leq 1$ for $\forall uv \in E(G)$.

The purpose of this paper is to introduce a new graph valued function in the field of domination theory in graphs.

§2. Entire Equitable Dominating Graph

Definition 2.1 *The entire equitable dominating graph $EE_qD(G)$ of a graph G with vertex set $V \cup S$, where S is the collection of all minimal equitable dominating sets of G and two vertices $u, v \in V \cup S$ adjacent if u, v are not disjoint minimal equitable dominating sets in S or $u, v \in D$, where D is the minimal equitable dominating set in S or $u \in V$ and v is a minimal equitable dominating set in S containing u.*

In Fig.1, a graph G and its entire equitable dominating graph $EE_qD(G)$ are shown. Here $D_1 = \{1, 3\}$, $D_2 = \{1, 4\}$, $D_3 = \{2, 3\}$ and $D_4 = \{2, 4\}$ are minimal equitable dominating sets of G.

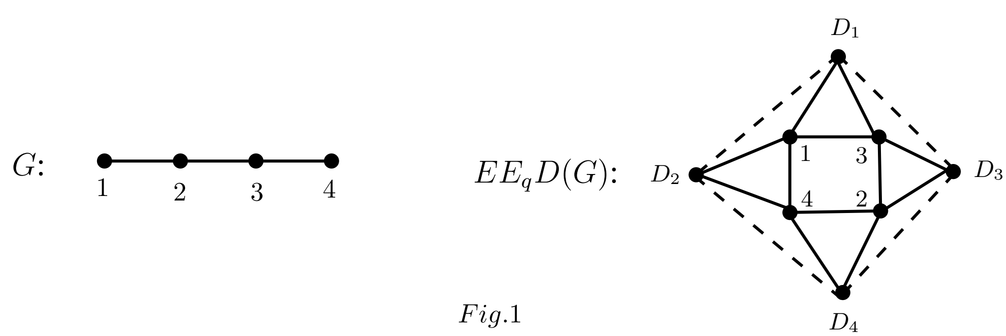

Fig.1

§3. Preliminary Results

The following will be useful in the proof of our results.

Theorem 3.1([5]) *For any nontrivial graph G, $\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1$.*

Theorem 3.2([5]) *A connected graph G is Eulerian if and only if every vertex of G has even degree.*

§4. Results

First we obtain a necessary and sufficient condition on a graph G such that the entire equitable dominating graph $EE_qD(G)$ is connected.

Theorem 4.1 *For any graph G with at least three vertices, the entire equitable dominating graph $EE_qD(G)$ is connected if and only if $\Delta(G) < p - 1$.*

Proof Let $\Delta(G) < p - 1$ and u, v be any two vertices in G. We consider the following cases:

Case 1. If u and v are adjacent vertices in G, then there exist two not disjoint minimal equitable dominating sets D_1 and D_2 containing u and v respectively. Therefore by the definition 2.1, u and v are adjacent in $EE_qD(G)$.

Case 2. Suppose there exist two vertices $u \in D_1$ and $v \in D_2$ such that u and v are not adjacent in G. Then there exists a minimal equitable dominating set D_3 containing both u and v and by definition 2.1, D_1 and D_2 are connected in $EE_qD(G)$.

Conversely, suppose $EE_qD(G)$ is connected. Suppose $\Delta(G) = p - 1$ and u is a vertex of degree $p - 1$. Then the degree of u in $EE_qD(G)$ is minimum. If every vertex of G has degree $p - 1$, then every vertex of G forms a minimal equitable dominating set. Therefore $EE_qD(G)$ has at least two components, a contradiction. Thus $\Delta(G) < p - 1$. □

Proposition 4.1 *$EE_qD(G) = pk_2$ if and only if $G = K_p; p \geq 2$.*

Proof Suppose $G = K_p; p \geq 2$. Then clearly each vertex of G will form a minimal equitable dominating set. Hence by definition 2.1, $EE_qD(G) = pK_2$.

Conversely, suppose $EE_qD(G) = pK_2$ and $G \neq K_p$. Then there exists at least one minimal equitable dominating set D containing two vertices of G. Then D will form C_3 in $EE_qD(G)$, a contradiction. Hence $G = K_p; p \geq 2$. □

Theorem 4.2 *For any graph G, $EE_qD(G)$ is either connected or it has at least one component which is K_2.*

Proof If $\Delta(G) < p - 1$, then by Theorem 4.1, $EE_qD(G)$ is connected. If G is complete graph $K_p; p \leq 2$ and by Proposition 4.1, then each component of $EE_qD(G)$ is K_2.

Next, we must prove that $\delta(G) < \Delta(G) = p - 1$. Let $v_1, v_2, \cdots, v_n$ be the set of vertices in G such that $deg(v_i) = p - 1$, then it is clear that $\{v_i\}$ forms a minimal equitable dominating set and which forms a component isomorphic to K_2. Hence $EE_qD(G)$ has at least one component which is K_2. □

In the next theorem, we characterize the graphs G for which $EE_qD(G)$ is complete.

Theorem 4.3 *$EE_qD(G) = K_{p+2}$ if and only if G is $K_{1,p}; p \geq 3$.*

Proof Suppose $G = K_{1,p}; p \geq 3$. Then there exists a minimal equitable dominating set D

contains all the vertices of G i.e $|D| = |\{u, v_1, v_2, v_3, \cdots, v_p\}| = p+1$. Hence $EE_qD(G) = K_{p+2}$.

Conversely, $EE_qD(G) = K_{p+2}$, then we prove that G is $K_{1,p}; p \geq 3$. Let us suppose that, $G \neq K_{1,p}; p \geq 3$. Then there exists a minimal equitable dominating set D of cardinality is maximum p i.e $|D| = |\{v_1, v_2, v_3, \cdots, v_p\}| = p$, a contradiction. Therefore G must be $K_{1,p}; p \geq 3$. □

Theorem 4.4 *Let G be a nontrivial connected graph of order p and size q. The entire equitable dominating graph is a graph with order $2p$ and size p if and only if $G = K_p; p \geq 2$.*

Proof Let G be a complete graph with $p \geq 2$, then by Proposition 4.1, $G = K_p;\ p \geq 2$.

Conversely, suppose $EE_qD(G)$ be a $(2p, p)$ graph. Then pK_2 is the only graph with order $2p$ and size q. □

In the next results, we obtain the bounds on the order and size of $EE_qD(G)$.

Theorem 4.5 *For any graph G, $2p \leq p' \leq \frac{p(p-1)}{2} + 1$, where p' denotes the number of vertices in $EE_qD(G)$. Further, the lower bound is attained if and only if G is either P_4 or $K_p; p \geq 2$ and upper bound is attained if and only if G is $K_3 \cup K_2$, $K_3 \bullet K_2$ or $C_4 \cup K_1$.*

Proof The lower bound follows from the fact that the twice the number of vertices in G and the upper bound follows that the maximum number of edges in G.

Suppose the lower bound is attained. Then every vertex of G forms a minimal equitable dominating set or every vertex of G is in exactly two minimal equitable dominating sets. This implies that the necessary condition.

Conversely, suppose G is P_4 or $K_p; p \geq 2$. Then by definition of entire equitable dominating graph, $V(EE_qD(G)) = 2p$. If the upper bound is attained. Then G must be one of the following graphs are $K_3 \cup K_2$, $K_3 \bullet K_2$ or $C_4 \cup K_1$.

If $G = K_3 \cup K_2$, then every vertex of G is in exactly two minimal equitable dominating sets hence

$$V(EE_qD(G)) \quad = \quad \frac{p(p-1)}{2} + 1 = \frac{pq}{2} + 1.$$

Suppose $G = K_3 \bullet K_2$. Then the pendant vertex of G is in all the minimal equitable dominating sets and forms $(p-1)$ minimal equitable dominating sets. Therefore the upper bound holds.

Now if G is $C_4 \cup K_1$. Then every equitable dominating sets contains an isolated vertex and they are not disjoint sets and by definition 2.1. Therefore upper bound holds.

Conversely, suppose G is one of the following graphs $K_3 \cup K_2$, $K_3 \bullet K_2$ or $C_4 \cup K_1$. Then it is obvious that $V(EE_qD(G)) = \frac{p(p-1)}{2} + 1$. □

Theorem 4.6 *For any graph G, $p \leq q' \leq \frac{p(p+1)}{2} + 1$, where q' denotes the number of edges in $EE_qD(G)$. Further, the lower bound is attained if and only $G = K_p \geq 2$ and the upper bound is attained if and only if G is $K_3 \cup K_1$.*

Proof The proof follows from Theorem 4.5. □

In the next result, we find the diameter of $EE_qD(G)$.

Theorem 4.7 *Let G be any graph with $\Delta(G) < p-1$, then $diam(EE_qD(G)) \leq 2$, where $diam(G)$ is the diameter of G.*

Proof Let G be any graph with $\Delta(G) < p-1$, then by Theorem 4.1, $EE_qD(G)$ is connected. Let u, v be any arbitrary vertices in $EE_qD(G)$. We consider the following cases.

Case 1. Suppose $u, v \in V$, u and v are nonadjacent in G. Then there exists a minimal equitable dominating set containing u and v and by definition 2.1, $d_{EE_qD(G)}(u,v) = 1$. If u and v are adjacent in G and there is no minimal equitable dominating set containing u and v, then there exists another vertex $w \in V$ which is not adjacent to both u and v. Let D_1 and D_2 be two minimal equitable dominating sets containing (u,w) and (w,v) respectively. This implies that $d_{EE_qD(G)}(u,v) = 2$.

Case 2. Suppose $u \in V$ and $v \in S$. Then $v = D$ is a minimal equitable dominating set of G. If $u \in S$, then u and v are adjacent in $EE_qD(G)$. Otherwise, there exists another vertex $w \in D$. This implies that

$$d_{EE_qD(G)}(u,v) \leq d_{EE_qD(G)}(u,w) + d_{EE_qD(G)}(w,v) = 2.$$

Case 3. Suppose $u, v \in S$. Then $u \in D_1$ and $v \in D_2$ are two minimal equitable dominating sets of G and by Definition 2.1, $d_{EE_qD(G)}(u,v) = 1$. □

We now characterize graphs G for which $SE_qD(G) = EE_qDG$. A *semientire equitable dominating graph* $SE_qD(G)$ of a graph G is the graph with vertex set $V \cup S$ and two vertices $u, v \in V \cup S$ adjacent if $u, v \in D$, where D is a minimal equitable dominating set or $u \in V$ and $v = D$ is a minimal equitable dominating set containing u ([1]).

Proposition 4.2([3]) *The semientire equitable dominating graph $SE_qD(G)$ is pK_2 if and only if $G = K_p$; $p \geq 2$.*

Remark 4.1([3]) For any graph G, $SE_qD(G)$ is a subgraph of $EE_qD(G)$.

Theorem 4.8 *For any graph G, $SE_qD(G) \subseteq EE_qD(G)$. Further, equality G, $SE_qD(G) = EE_qD(G)$ if and only if G has exactly one minimal equitable dominating set containing all vertices of G.*

Proof By Remark 4.1, $SE_qD(G) \subseteq EE_qD(G)$. Suppose $SE_qD(G) = EE_qD(G)$. Then by Theorem 4.3, D is the only minimal equitable dominating set contains all the vertices of G. Therefore G must be $K_{1,n}; n \geq 3$.

The converse is obvious. □

In the next results, we discuss about α_0 and β_0 of $EE_qD(G)$.

Theorem 4.9 *For any graph G with no isolated vertices,*

(1) $\alpha_0(EE_qD(G)) = |S| + 1$, *where S is the collection of all minimal equitable dominating*

sets of G;

(2) $\beta_0(EE_qD(G)) = \gamma(G)$.

Proof (i) Let G be graph of order p. Let $S = \{s_1, s_2, \cdots s_i\}$ be the set of all minimal equitable dominating sets. Then by definition 2.1 and Theorem **??**. Therefore the minimum number of vertices in $EE_qD(G)$ which covers all the edges. Hence $\alpha_0(EE_qD(G)) = |S| + 1$.

(ii) By definition of $EE_qD(G)$, for any vertex v_i ; $1 \leq i \leq p$ of $EE_qD(G)$ are not adjacent. Hence these vertices forms a maximum independent set of $EE_qD(G)$. Hence (ii) follows. □

In the next two results, we prove the vertex connectivity and edge- connectivity of $EE_qD(G)$.

Theorem 4.10 *For any graph G, $\kappa(EE_qD(G)) = min\{min(deg_{EE_qD(G)_{1\leq i\leq p}} v_i), min_{1\leq j\leq n}|S_j|\}$, where S_j's is the collection of all minimal equitable dominating sets of G.*

Proof Let G be any graph with order p and size q. We consider the following cases.

Case 1. Let $u \in v_i'(EE_qD(G))$ for some i, having the minimum degree among all v_i' in $EE_qD(G)$. If the degree of u is less than any other vertex in $EE_qD(G)$, then by deleting the vertices which are adjacent to u, results a disconnected graph.

Case 2. Let $v \in S_j$ for some j, having the minimum degree among all S_j's in $EE_qD(G)$. If degree of v is less than any other vertex in $EE_qD(G)$, then by deleting all the vertices which are adjacent to v. This results the graph is disconnected. Hence the result follows. □

Theorem 4.11 *For any graph G, $\lambda(EE_qD(G)) = min\{min(deg_{EE_qD(G)_{1\leq i\leq p}} v_i), min_{1\leq j\leq n}|S_j|\}$, where S_j's is the collection of all minimal equitable dominating sets of G.*

Proof Let G be any (p, q) graph. We consider two cases.

Case 1. Let $u \in v_i'(EE_qD(G))$, having minimum degree among all v_i' in $EE_qD(G)$. If the degree of u is less than any other vertex in $EE_qD(G)$, then by deleting those edges of $EE_qD(G)$ which are incident with u, results a disconnected graph.

Case 2. Let $v \in S_j$, having the minimum degree among all vertices of S_j. If degree of v is less than any other vertex in $EE_qD(G)$, then by deleting those edges which are adjacent to v, results in a disconnected. Hence the result follows. □

Next, we prove the necessary and sufficient condition for $EE_qD(G)$ to be Eulerian.

Theorem 4.12 *For any graph G, $EE_qD(G)$ is Eulerian if and only if one of the following conditions are satisfied:*

(1) *There exists a vertex $u \in V$ is in all minimal equitable dominating sets and cardinality of every minimal equitable dominating set D of G is even;*

(2) *If $v \in V$ is a vertex of odd degree, then it is in odd number of minimal equitable dominating sets, otherwise it is in even number of minimal equitable dominating sets of G.*

Proof Suppose $\Delta < p-1$ and by Theorem 4.1, $EE_qD(G)$ is connected. Suppose $EE_qD(G)$

is Eulerian. on the contrary if condition (i) is not satisfied, then there exists a minimal equitable dominating set contains odd number of vertices and does not contains a vertex of odd degree, a contradiction. Therefore by Theorem 3.2, $EE_qD(G)$ is Eulerian. Hence condition (1) holds.

Suppose (2) does not hold. Then there exists $v \in V$ of even degree which is in odd number of minimal equitable dominating sets, a contradiction. Hence (ii) hold.

Conversely, suppose the conditions (1) and (2) are satisfied. Then every vertex of $EE_qD(G)$ has even degree and hence $EE_qD(G)$ is Eulerian. □

§5. Domination in $EE_qD(G)$

We calculate the domination number of $EE_qD(G)$ of some standard class of graphs.

Theorem 5.1 *For any graph G with no isolated vertices.*

(1) *If $G = K_p; p \geq 2$, then $\gamma(EE_qD(K_p) = p$;*

(2) *If $G = K_{1,p}; p \geq 3$, then $\gamma(EE_qD(K_{1,p}) = 1$;*

(3) *If $G = C_p$, $p \geq 3$, then $\gamma(EE_qD(C_p) = 2$.*

Theorem 5.2 *For any graph G, $\gamma(EE_qD(G)) = 1$, if and only if G is $K_{1,p}; p \geq 3$.*

Proof If G is $K_{1,p}; p \geq 3$, then there exists a minimal equitable dominating set D contains all the vertices of G and by Theorem **??**, it is clear that, $EE_qD(G)$ is complete. Hence $\gamma(EE_qD(G)) = 1$.

Conversely, suppose $\gamma(EE_qD(G)) = 1$ and $G \neq K_{1,p}; p \geq 3$. Then there exists a minimal dominating set D in $EE_qD(G)$ of cardinality greater than or equal to 2, a contradiction. Therefore G must be $K_{1,p}; p \geq 3$. □

We conclude this paper by exploring one open problem on $EE_qD(G)$.

Problem 1. *Give necessary and sufficient condition for a given graph G is entire equitable dominating graph of some graph.*

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International J.Math. Combin. Vol.2(2016), 51-64

Radio Mean Number of Certain Graphs

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Abstract: A *radio mean labeling* of a connected graph G is a one to one map f from the vertex set $V(G)$ to the set of natural numbers N such that for each distinct vertices u and v of G, $d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam }(G)$. The radio mean number of f, $rmn(f)$, is the maximum number assigned to any vertex of G. The radio mean number of G, $rmn(G)$ is the minimum value of $rmn(f)$ taken over all radio mean labeling f of G. In this paper we find the radio mean number of Jelly fish, subdivision of jelly fish, book with n pentagonal pages and $\langle K_{1,n} : m \rangle$.

Key Words: Radio mean number, subdivision of a graph, complete bipartite graph.

AMS(2010): 05C78.

§1. Introduction

For standard terminology and notion we follow Harary [6] and Gallian [4]. Unless or otherwise mentioned, $G = (V(G), E(G))$ is a simple, finite, connected and undirected graph. A graph labeling is an assignment of integers to the vertices, or edges, or both, subject to certain conditios. Graph labeling used for several areas of science and few of them are communication network, coding theory, database management ete. In particular, radio labeling applied for channel assignment problem. The concept of radio labeling was introduced by Chatrand et al. [1] in 2001. Also in [2, 3], radio number of several graphs were found. Motivated by the above labeling, Ponraj et al. [7] introduced the notion of radio mean labeling of G. A *radio mean labeling* is a one to one mapping f from $V(G)$ to N satisfying the condition

$$d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam }(G) \tag{1.1}$$

for every $u, v \in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of Graph G. The radio mean number of G, $rmn(G)$ is the lowest span taken over all radio mean labelings of the graph G. The condition 1.1 is called radio mean condition. In [7, 8, 9], they have found the radio mean number of some graphs like three diameter graphs, lotus inside a circle, gear graph, Helms, Sunflower graphs, subdivision of complete bipartite, corona of complete graph with path, one point union of cycle C_6 and wheel related graphs. In

[1]Received October 26, 2015, Accepted May 15, 2016.

this article we find the radio mean number of Jelly fish, subdivision of jelly fish, book with n pentogonal pages and $\langle K_{1,n} : m\rangle$. We write $d(u, v)$ for the distance between the vertices u and v in G. The maximum distance between any pair of vertices is called the diameter of G and denoted by $diam(G)$. Let x be any real number. Then $\lceil x\rceil$ stands for smallest integer greater than or equal to x.

§2. Main Results

First we look into the Jelly fish graphs. Jelly fish graphs $J(m, n)$ obtained from a cycle C_4 : $uvxyu$ by joining x and y with an edge and appending m pendent edges to u and n pendent edges to v.

Theorem 2.1 *The radio mean number of a jelly fish graph $J(m, n)$ is $m + n + 4$.*

Proof Let $V(J(m, n)) = \{u, v, x, y\} \cup \{u_i, v_j : 1 \le i \le m; 1 \le j \le n\}$ and $E(J(m, n)) = \{uy, yv, vx, xu, xy\} \cup \{uu_i, vv_j : 1 \le i \le m; 1 \le j \le n\}$. It is clear that $diam(J(m, n)) = 4$. The vertex labeling of $J(1, 1)$, $J(1, 2)$ given in Figure 1 shows that their radio mean numbers are 6, 7 respectively.

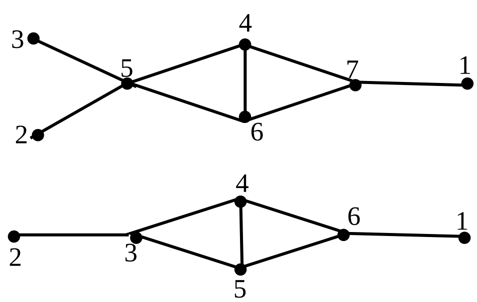

Figure 1

Assume $m \ge 2$ and $n \ge 3$. We define a vertex labeling f as follows. Assign the label 1 to u_1. Then put the label 2 to v_1, 3 to v_2 and so on. In this sequence v_n received the label $n + 1$. Then assign the label $n + 2$ to u_2, $n + 3$ to u_3 and so on. Clearly label of u_m is $m + n$. Then assign the labels $m + n + 3$, $m + n + 1$, $m + n + 2$, $m + n + 4$ respectively to the vertices u, v, x, y. Now we check the radio mean condition

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \ge 1 + \text{diam } (J(m, n))$$

for all $u, v \in V(J(m, n))$. It is easy to verify that the vertices u, v, x, y are mutually satisfies the radio mean condition.

Case 1. Check the pair (u, u_i).

$$d(u, u_i) + \left\lceil \frac{f(u) + f(u_i)}{2} \right\rceil \ge 1 + \left\lceil \frac{m + n + 3 + 1}{2} \right\rceil \ge 6.$$

Case 2. Consider the pair (u, u_i).

$$d(u, v_j) + \left\lceil \frac{f(u) + f(v_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{m + n + 3 + 2}{2} \right\rceil \geq 8.$$

Case 3. Check the pair (x, u_i).

$$d(x, u_i) + \left\lceil \frac{f(x) + f(u_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{m + n + 2 + 1}{2} \right\rceil \geq 6.$$

Case 4. Verify the pair (x, v_j).

$$d(x, v_j) + \left\lceil \frac{f(x) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{m + n + 2 + 2}{2} \right\rceil \geq 7.$$

Case 5. Consider the pair (y, u_i).

$$d(y, u_i) + \left\lceil \frac{f(y) + f(u_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{m + n + 4 + 1}{2} \right\rceil \geq 7.$$

Case 6. Check the pair (y, v_j).

$$d(y, v_j) + \left\lceil \frac{f(y) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{m + n + 4 + 2}{2} \right\rceil \geq 8.$$

Case 7. Check the pair (v, v_j).

$$d(v, v_j) + \left\lceil \frac{f(v) + f(v_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{m + n + 1 + 2}{2} \right\rceil \geq 5.$$

Case 8. Verify the pair (v, u_i).

$$d(v, u_i) + \left\lceil \frac{f(v) + f(u_i)}{2} \right\rceil \geq 3 + \left\lceil \frac{m + n + 1 + 1}{2} \right\rceil \geq 7.$$

Case 9. Consider the pair (u_i, v_j).

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{1 + 2}{2} \right\rceil \geq 6.$$

Hence $rmn(J(m, n)) = m + n + 4$. □

Now, we find the radio mean number of subdivision of jelly fish graph. If $x = uv$ is an edge of G and w is not a vertex of of G, then x is subdivided when it is replaced by the lines uw and wv. If every edges of G is subdivided, the resulting graph is the *subdivision graph* $S(G)$.

Theorem 2.2 *For a subdivision of graph* $J_{m,n}$*,*

$$rmn(S(J_{m,n})) = \begin{cases} 16 & \text{if} \quad m = n = 1 \\ 2m + 2n + 11 & \text{otherwise} \end{cases}$$

Proof Let $V(S(J_{m,n})) = \{z_i : 1 \leq i \leq 9\} \cup \{u_i, u_i' : 1 \leq i \leq m\} \cup \{v_j, v_j' : 1 \leq j \leq n\}$ and $E(S(J_{m,n})) = \{z_i z_{i+1} : 1 \leq i \leq 7\} \cup \{z_8 z_1, z_7 z_9, z_9 z_3\} \cup \{z_1 u_i, u_i u_i', z_5 v_j, v_j v_j' : 1 \leq i \leq m, 1 \leq j \leq n\}$. Clearly $diam(S(J_{m,n})) = 8$.

Case 1. $m = n = 1$.

In this case 1 should be a label of the vertex u_1 or u_1' or v_1 or v_1'. If not, 1 is a label of any one of the remaining vertices , say x, and suppose 2 is a label of any other vertex, say x'. Then

$$d(x, x') + \left\lceil \frac{f(x) + f(x')}{2} \right\rceil \leq 6 + \left\lceil \frac{1+2}{2} \right\rceil \leq 8,$$

a contradiction.

Subcase 1. u_1 receives the label 1.

Then 2 should be a label of v_1' otherwise we get a contradiction as previously. For satisfying the radio mean condition, 3 should be a label of a vertex which is at least at a distance 6 from the vertex v_1' and 7 from u_1, such a vertex doesn't exists. Therefore, in this case, $rmn(S(J_{1,1})) \geq 14$.

Subcase 2. u_1' receives the label 1.

Then 2 should be a label of either v_1 or v_1'. Otherwise as in subcase a, the radio mean condition is not satisfied. If v_1 or v_1' receives the label 2 then 3 can not be a label of any of the remaining vertices. Suppose 3 is a label of any other vertices , say x, then

$$d(u_1', x) + \left\lceil \frac{f(u_1') + f(x)}{2} \right\rceil \leq 8.$$

or

$$d(v_1', x) + \left\lceil \frac{f(v_1') + f(x)}{2} \right\rceil \leq 8.$$

or

$$d(v_1, x) + \left\lceil \frac{f(v_1) + f(x)}{2} \right\rceil \leq 8,$$

a contradiction. Thus, here also $rmn(S(J_{1,1})) \geq 14$. By symmetry, the same case arises when v_1 or v_1' receives the label 1. Therefore in all the cases $rmn(S(J_{1,1})) \geq 14$. Now we will try to label the vertices of $S(J_{1,1})$ with the property that the sum of the distance between the any pair of vertices and the mean value of labels of that pair of vertices exceeds the integer 9. We drop the label 1 from the set of integers $\{1, 2, \cdots, 13\}$ and add a new label 14 . Thus the labels are $\{2, 3, \cdots, 14\}$. Suppose l, m, n are any three vertices of $S(J_{1,1})$ with their respective labels are 2, 3, 4. Then $d(l, m) \geq 6$, $d(l, n) \geq 6$ and $d(m, n) \geq 5$. It is clear that, such type of vertices l, m, n doesn't exists. So $rmn(S(J_{1,1})) \geq 15$. Now consider the labels from the set $\{3, 4, \cdots, 15\}$. Since the vertices with labels 3 and 4 are at least at a distance 5, any one of the vertices with these label should be a pendent vertex and the other is either z_6 or z_9 or z_4. Now suppose either 3 or 4 is a label of z_6 or z_4 then 5 can not be a label of any of the rest vertices. So 3 or

4 should be a label of z_9. Suppose 3, 4 are the labels of u_1', z_9 then 5 should be the label of v_1'. Then 6 can not be a label of any of the remaining vertices. The same problem arises when 4, 3 are the labels of u_1', z_9. By symmetry, if we assign the label 3 or 4 to the vertex v_1' , then 6 can not be a label of any other vertices as discussed above. Hence $rmn(S(J_{1,1})) \geq 16$. Consider the labeling given in Figure 2.

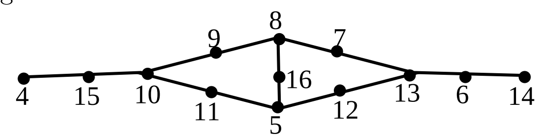

Figure 2

From Figure 2, it is clear that $rmn(S(J_{1,1})) \leq 16$. Hence $rmn(S(J_{1,1})) = 16$.

Case 2. $m \neq 1$, $n \neq 1$.

Subcase 1. $m + n \leq 4$.

As discussed in case 1, clearly it is not possible to label the vertices of $S(J_{m,n})$ from the sets $\{1, 2, \cdots, 2m+2n+9\}$ and $\{1, 2, \cdots, 2m+2n+10\}$. That is $rmn(S(J_{m,n})) \geq 2m+2n+11$. The following Figure 3 shows that $rmn(S(J_{m,n})) \leq 2m + 2n + 11$ where $m + n \leq 4$.

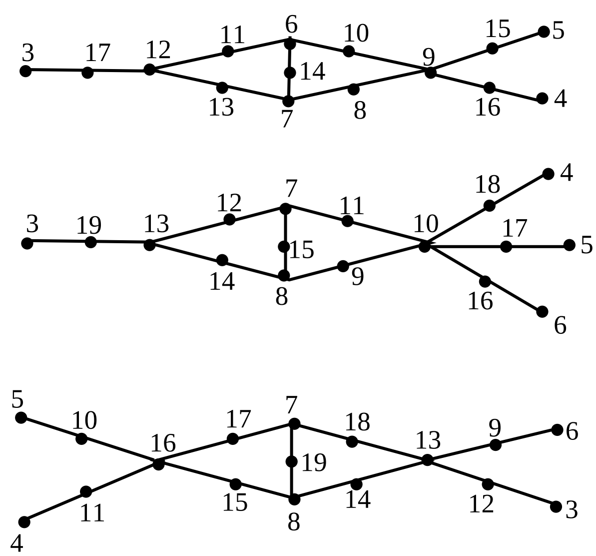

Figure 3

Hence $rmn(S(J_{m,n})) = 2m + 2n + 11$ for $m + n \leq 4$ and $m \neq 1$, $n \neq 1$.

Subcase 2. $m + n > 4$.

Define an injective map $f : V(S(J_{m,n})) \to \{1, 2, \cdots, 2m + 2n + 11\}$ by $f(v_1') = 3$, $f(v_1) =$

$2m+2n+2$,

$$\begin{aligned}
f(u'_i) &= i+3, & 1 \le i \le m\\
f(v'_i) &= m+2+i, & 2 \le i \le n\\
f(v_{n-i+1}) &= m+n+2+i, & 1 \le i \le n-1\\
f(u_{m-i+1}) &= m+2n+1+i, & 1 \le i \le m
\end{aligned}$$

$f(z_3) = 2m+2n+3,\ f(z_2) = 2m+2n+4,\ f(z_1) = 2m+2n+5,\ f(z_8) = 2m+2n+6,$ $f(z_7) = 2m+2n+7,\ f(z_6) = 2m+2n+8,\ f(z_5) = 2m+2n+9,\ f(z_4) = 2m+2n+10$ and $f(z_9) = 2m+2n+11$. Now we check the radio mean condition that

$$d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 9.$$

for every pair of vertices $u, v \in V(S(J_{m,n}))$.

Case 1. Consider the pair (z_i, z_j).

$$d(z_i,z_j) + \left\lceil \frac{f(z_i)+f(z_j)}{2} \right\rceil \ge 1 + \left\lceil \frac{2m+2n+3+2m+2n+4}{2} \right\rceil \ge 15.$$

Case 2. Check the pair (u_i, u'_i).

$$d(u_i,u'_i) + \left\lceil \frac{f(u_i)+f(u'_i)}{2} \right\rceil \ge 1 + \left\lceil \frac{2m+2n+5}{2} \right\rceil \ge 9.$$

Case 3. Check the pair (u'_i, u_j), $i \ne j$.

$$d(u'_i,u_j) + \left\lceil \frac{f(u'_i)+f(u_j)}{2} \right\rceil \ge 3 + \left\lceil \frac{4+m+2n+2}{2} \right\rceil \ge 9.$$

Case 4. Examine the pair (u_i, u_j).

$$d(u_i,u_j) + \left\lceil \frac{f(u_i)+f(u_j)}{2} \right\rceil \ge 2 + \left\lceil \frac{m+2n+2+m+2n+3}{2} \right\rceil \ge 11.$$

Case 5. Verify the pair (u'_i, u'_j).

$$d(u'_i,u'_j) + \left\lceil \frac{f(u'_i)+f(u'_j)}{2} \right\rceil \ge 4 + \left\lceil \frac{4+5}{2} \right\rceil \ge 9.$$

Case 6. Check the pair (u'_i, z_j).

$$d(u'_i,z_j) + \left\lceil \frac{f(u'_i)+f(z_j)}{2} \right\rceil \ge 2 + \left\lceil \frac{2m+2n+3+4}{2} \right\rceil \ge 11.$$

Case 7. Examine the pair (u_i, z_j).

$$d(u_i,z_j) + \left\lceil \frac{f(u_i)+f(z_j)}{2} \right\rceil \ge 1 + \left\lceil \frac{2m+2n+3++m+2n+2}{2} \right\rceil \ge 12.$$

Case 8. Verify the pair (u_i, v_j).

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 6 + \left\lceil \frac{m + 2n + 2 + m + n + 3}{2} \right\rceil \geq 14.$$

Case 9. Consider the pair (u_i, v'_j).

$$d(u_i, v'_j) + \left\lceil \frac{f(u_i) + f(v'_j)}{2} \right\rceil \geq 7 + \left\lceil \frac{m + 2n + 3 + 3}{2} \right\rceil \geq 13.$$

Case 10. Examine the pair (u'_i, v'_j).

$$d(v'_i, v'_j) + \left\lceil \frac{f(u'_i) + f(v'_j)}{2} \right\rceil \geq 8 + \left\lceil \frac{4 + 3}{2} \right\rceil \geq 12.$$

Case 11. Verify the pair (u'_i, v_j), $i \neq j$.

$$d(u'_i, v_j) + \left\lceil \frac{f(u'_i) + f(v_j)}{2} \right\rceil \geq 7 + \left\lceil \frac{4 + m + n + 3}{2} \right\rceil \geq 13.$$

Case 12. Check the pair (v_i, v_j), $i \neq j$.

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{m + n + 3 + m + n + 4}{2} \right\rceil \geq 11.$$

Case 13. Verify the pair (v'_i, v'_j), $i \neq j$. In this case, obviously $m \geq 2$.

$$d(v'_i, v'_j) + \left\lceil \frac{f(v'_i) + f(v'_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{3 + m + 4}{2} \right\rceil \geq 9.$$

Case 14. Consider the pair (v_i, v'_i).

$$d(v_i, v'_i) + \left\lceil \frac{f(v_i) + f(v'_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2m + 2n + 5}{2} \right\rceil \geq 9.$$

Case 15. Check the pair (v_i, v'_j).

$$d(v_i, v'_j) + \left\lceil \frac{f(v_i) + f(v'_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{3 + m + n + 3}{2} \right\rceil \geq 9.$$

Case 16. Verify the pair (v_i, z_j), $i \neq j$.

$$d(v_i, z_j) + \left\lceil \frac{f(v_i) + f(z_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{m + n + 3 + 2m + 2n + 3}{2} \right\rceil \geq 12.$$

Case 17. Check the pair (v'_i, z_j), $i \neq j$.

$$d(v'_i, z_j) + \left\lceil \frac{f(v'_i) + f(z_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{3 + m + 2n + 3}{2} \right\rceil \geq 10.$$

Hence $rmn(S(J_{m,n})) \leq 2m + 2n + 11$ where $m + n > 4$. As in argument in case 1, $rmn(S(J_{m,n})) \geq 2m + 2n + 11$ for this case also. Hence $rmn(S(J_{m,n})) = 2m + 2n + 11$ when $m + n > 4$. □

Next investigation is about book with n pentogonal pages. n copies of the cylce C_5 with one edge common is called book with n pentogonal pages.

Theorem 2.3 *The radio mean number of a book with n pentogonal pages, BP_n, is $3n + 2$.*

Proof Let $V(BP_n) = \{u, v\} \cup \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(BP_n) \cup \{u, v\} \cup \{uu_i, u_iw_i, w_iv_i, v_iv : 1 \leq i \leq n\}$. Note that

$$diam(BP_n) = \begin{cases} 2 & \text{if } n = 1 \\ 4 & \text{otherwise} \end{cases}$$

For $n =$ 1,2, the labeling given in Figure 4 satisfies the radio mean condition.

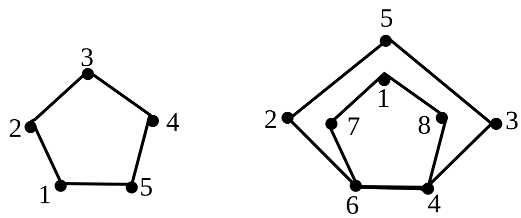

Figure 4

For $n \geq 3$, define an injective map $f : V(BP_n) \to \{1, 2, \cdots, 3n + 2\}$ by

$$\begin{aligned} f(w_i) &= i, & 1 \leq i \leq n \\ f(v_{n-i+1}) &= n + i, & 1 \leq i \leq n \\ f(u_{n-i+1}) &= 2n + i & 1 \leq i \leq n \end{aligned}$$
$$f(u) = 3n + 1, \text{ and } \quad f(v) = 3n + 2$$

Now we check the condition

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 5$$

for every pair of vertices $u, v \in V(BP_n)$.

Case 1. Check the pair (u_i, w_i).

$$d(u_i, w_i) + \left\lceil \frac{f(u_i) + f(w_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{3n + 1}{2} \right\rceil \geq 6.$$

Case 2. Verify the pair (v_i, w_i).

$$d(v_i, w_i) + \left\lceil \frac{f(v_i) + f(w_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n + 1}{2} \right\rceil \geq 5.$$

Case 3. Examine the pair (u_i, v_i).

$$d(u_i, v_i) + \left\lceil \frac{f(u_i) + f(v_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{n+1+2n+1}{2} \right\rceil \geq 8.$$

Case 4. Consider the pair (w_i, u_j).

$$d(w_i, u_j) + \left\lceil \frac{f(w_i) + f(u_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{1+2n+1}{2} \right\rceil \geq 7.$$

Case 5. Consider the pair (v_i, w_j).

$$d(v_i, w_j) + \left\lceil \frac{f(v_i) + f(w_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{n+1+1}{2} \right\rceil \geq 6.$$

Case 6. Verify the pair (u_i, v_j).

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{2n+1+n+2}{2} \right\rceil \geq 9.$$

Case 7. Check the pair (w_i, w_j).

$$d(w_i, w_j) + \left\lceil \frac{f(w_i) + f(w_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{1+2}{2} \right\rceil \geq 6.$$

Case 8. Examine the pair (u_i, u_j).

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{2n+1+2n+2}{2} \right\rceil \geq 10.$$

Case 9. Consider the pair (v_i, v_j).

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{n+1+n+2}{2} \right\rceil \geq 7.$$

Since $\left\lceil \frac{f(u)+f(x)}{2} \right\rceil = \left\lceil \frac{3n+1+f(x)}{2} \right\rceil \geq 6$, the pair (u, x) for every $x \in V(BP_n)$ satisfy the radio mean condition. Similarly the pair (v, y) for every $y \in V(BP_n)$ also satisfy the condition. Hence $rmn(BP_n) \leq 3n + 2$. Since f is injective, $rmn(BP_n) = 3n + 2$. □

The following result is used for the next theorem.

Theorem 2.4([7]) *Let G be a (p, q)-connected graph with diameter $= 2$. Then $rmn(G) = p$.*

Let $\langle K_{1,n} : m\rangle$ denotes the graph obtained by taking m disjoint copies of $K_{1,n}$ and joining a new vertex to the centers of the m copies of $K_{1,n}$. Let $V(\langle K_{1,n} : m\rangle) = \{v\} \cup \{v_i : 1 \leq i \leq m\} \cup \{u^i_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(\langle K_{1,n} : m\rangle) = \{vv_i : 1 \leq i \leq m\} \cup \{v_i u^i_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Theorem 2.5 *For integers $m, n \geq 1$,*

$$rmn(\langle K_{1,n} : m\rangle) = \begin{cases} 6 & \text{if} \quad m = 2, n = 1 \\ mn + m + 1 & \text{otherwise} \end{cases}$$

Proof First we observe that

$$diam(\langle K_{1,n} : m\rangle) = \begin{cases} 2 & \text{if } \ m = 1 \\ 4 & \text{otherwise} \end{cases}$$

Case 1. $m = 1$.

In this case $\langle K_{1,n} : 1\rangle \cong K_{1,n+1}$, which is a 2-diameter graph and hence by Theorem 2.4, $rmn(\langle K_{1,n} : 1\rangle) = n + 2$.

Case 2. $m = 2$.

Subcase 1. $n = 1$.

Since 1 and 2 are labels of the vertices which are at least at a distance 3, either 1 or 2 is a label of a pendent vertex. Assume that the label of u_1^1 is 1. Then 2 is a label of either v_2 or u_1^2. Then 3 can not be a label of the remaining vertices. Similarly we can show that if 2 is a label of u_1^1, 1 is a label of either v_2 or u_1^2 and then 3 can not be a label of the remaining vertices. Hence $rmn(\langle K_{1,1} : 2\rangle) \geq 6$. Obviously, Figure 5 shows that $rmn(\langle K_{1,1} : 2\rangle) \leq 6$.

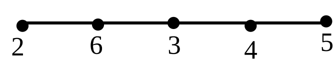

Figure 5

Hence $rmn(\langle K_{1,1} : 2\rangle) = 6$.

Subcase 2. $n \geq 2$.

Define an injective function $f : V(\langle K_{1,n} : 2\rangle) \to \{1, 2, \cdots, 2n + 3\}$ by $f(u_1^1) = 1$, $f(v) = 2n + 1$, $f(v_1) = 2n + 3$, $f(v_2) = 2n + 2$,

$$\begin{aligned} f(u_i^2) &= i + 1, \quad 1 \leq i \leq n \\ f(u_i^1) &= n + i, \quad 1 \leq 2 \leq n. \end{aligned}$$

We now check whether the labeling f is a required labeling. It is easy to check that the pairs (v_1, v_2), (v_1, v) and (v_2, v) satisfy the radio mean condition.

Subcase 1. Check the pair (u_i^2, u_j^2), $i \neq j$.

$$d(u_i^2, u_j^2) + \left\lceil \frac{f(u_i^2) + f(u_j^2)}{2} \right\rceil \geq 2 + \left\lceil \frac{2 + 3}{2} \right\rceil \geq 5.$$

Subcase 2. Verify the pair (u_i^2, u_j^1).

$$d(u_i^2, u_j^1) + \left\lceil \frac{f(u_i^2) + f(u_j^1)}{2} \right\rceil \geq 4 + \left\lceil \frac{2+1}{2} \right\rceil \geq 6.$$

Subcase 3. Consider the pair (u_i^1, u_j^1), $i \neq j$.

$$d(u_i^1, u_j^1) + \left\lceil \frac{f(u_i^1) + f(u_j^1)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+n+2}{2} \right\rceil \geq 5.$$

Subcase 4. Examine the pair (u_i^1, v_1).

$$d(u_i^1, v_1) + \left\lceil \frac{f(u_i^1) + f(v_1)}{2} \right\rceil \geq 1 + \left\lceil \frac{1+2n+3}{2} \right\rceil \geq 5.$$

Subcase 5. Check the pair (u_i^1, v).

$$d(u_i^1, v) + \left\lceil \frac{f(u_i^1) + f(v)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+2n+1}{2} \right\rceil \geq 5.$$

Subcase 6. Consider the pair (u_i^1, v_2).

$$d(u_i^1, v_2) + \left\lceil \frac{f(u_i^1) + f(v_2)}{2} \right\rceil \geq 3 + \left\lceil \frac{1+2n+2}{2} \right\rceil \geq 7.$$

Subcase 7. Verify the pair (u_i^2, v_2).

$$d(u_i^2, v_2) + \left\lceil \frac{f(u_i^2) + f(v_2)}{2} \right\rceil \geq 1 + \left\lceil \frac{2+2n+2}{2} \right\rceil \geq 5.$$

Subcase 8. Check the pair (u_i^2, v).

$$d(u_i^2, v) + \left\lceil \frac{f(u_i^2) + f(v)}{2} \right\rceil \geq 2 + \left\lceil \frac{2+2n+1}{2} \right\rceil \geq 6.$$

Subcase 9. Examine the pair (u_i^2, v_1).

$$d(u_i^2, v_1) + \left\lceil \frac{f(u_i^2) + f(v_1)}{2} \right\rceil \geq 3 + \left\lceil \frac{2+2n+3}{2} \right\rceil \geq 8.$$

Therefore, $rmn(\langle K_{1,n} : 2\rangle) \leq 2n+3$. But $rmn(\langle K_{1,n} : 2\rangle) \geq 2n+3$ and hence

$$rmn(\langle K_{1,n} : 2\rangle) = 2n+3.$$

Case 3. $m \geq 3$.

For $n = 1$, $m = 3$, Figure 6 shows that $rmn(\langle K_{1,1} : 3\rangle) = 7$. Now we consider the cases $n \geq 2$, $m = 3$ and $n \geq 1$, $m \geq 4$. Define a function $f : V(\langle K_{1,n} : m\rangle) \to \{1, 2, \cdots, mn+m+1\}$

by $f(v) = mn + 1$,

$$\begin{aligned} f(u_j^i) &= (j-1)m + i, \quad 1 \le i \le m, 1 \le j \le n \\ f(v_i) &= mn + 1 + i, \quad\quad 1 \le i \le m. \end{aligned}$$

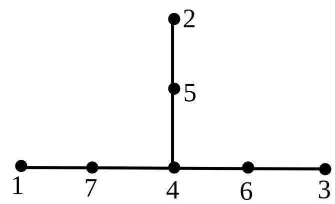

Figure 6

We show that f is a valid radio mean labeling.

Subcase 1. Check the pair (u_i^j, u_k^j).

$$d(u_i^j, u_k^j) + \left\lceil \frac{f(u_i^j) + f(u_k^j)}{2} \right\rceil \ge 2 + \left\lceil \frac{1 + m + 1}{2} \right\rceil \ge 5.$$

Subcase 2. Consider the pair (u_i^j, u_k^r), $j \neq r$.

$$d(u_i^j, u_k^r) + \left\lceil \frac{f(u_i^j) + f(u_k^r)}{2} \right\rceil \ge 4 + \left\lceil \frac{1 + 2}{2} \right\rceil \ge 6.$$

Subcase 3. Verify the pair (v_i, v_j).

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \ge 2 + \left\lceil \frac{mn + 2 + mn + 3}{2} \right\rceil \ge 8.$$

Subcase 4. Examine the pair (v, u_i^j).

$$d(v, u_i^j) + \left\lceil \frac{f(v) + f(u_i^j)}{2} \right\rceil \ge 2 + \left\lceil \frac{mn + 1 + 1}{2} \right\rceil \ge 5.$$

Subcase 5. Verify the pair (v, v_i).

$$d(v, v_i) + \left\lceil \frac{f(v) + f(v_i)}{2} \right\rceil \ge 1 + \left\lceil \frac{mn + 1 + mn + 2}{2} \right\rceil \ge 6.$$

Subcase 6. Check the pair (v_i, u_j^i).

For $n \ge 2$ and $m = 3$,

$$d(v_i, u_j^i) + \left\lceil \frac{f(v_i) + f(u_j^i)}{2} \right\rceil \ge 1 + \left\lceil \frac{mn + 2 + 1}{2} \right\rceil \ge 6.$$

If $n \geq 1$ and $m \geq 4$ then,

$$d(v_i, u_j^i) + \left\lceil \frac{f(v_i) + f(u_j^i)}{2} \right\rceil \geq 1 + \left\lceil \frac{mn + 2 + 1}{2} \right\rceil \geq 5.$$

Subcase 7. Consider the pair (v_i, u_j^k), $i \neq k$.

$$d(v_i, u_j^k) + \left\lceil \frac{f(v_i) + f(u_j^k)}{2} \right\rceil \geq 3 + \left\lceil \frac{mn + 2 + 2}{2} \right\rceil \geq 7.$$

Hence $rmn(\langle K_{1,n} : m \rangle) = mn + m + 1$. □

Example 2.6 A radio mean labeling of $\langle K_{1,5} : 4 \rangle$ is given in Figure 7.

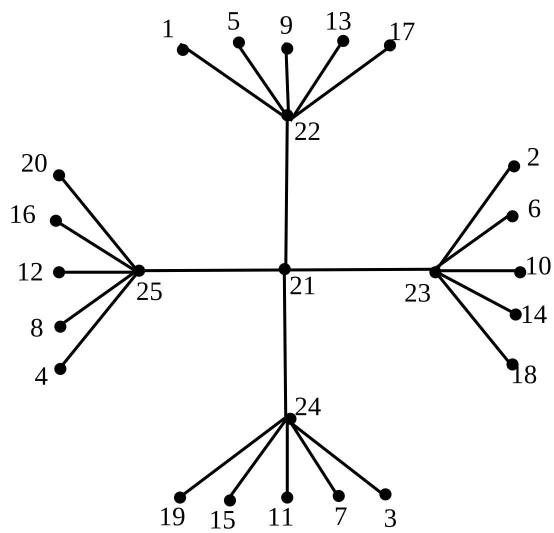

Figure 7

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International J.Math. Combin. Vol.2(2016), 65-82

Some Results on
Vertex Version and Edge Versions of Modified Schultz Index

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Abstract: Let G_1 and G_2 be two simple connected graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$, respectively. For given vertices $a_1 \in V(G_1)$ and $a_2 \in V(G_2)$, a splice of G_1 and G_2 by vertices a_1 and a_2 is defined by identifying the vertices a_1 and a_2 in the union of G_1 and G_2 and a link of G_1 and G_2 by vertices a_1 and a_2 is obtained by joining a_1 and a_2 by an edge in the union of these graphs. The modified Schultz index of a simple connected graph G is defined as the sum of the terms $d(u|G)d(v|G)d(u,v|G)$ over all unordered pairs $\{u,v\}$ of vertices in G, where $d(u|G)$ and $d(u,v|G)$ denote the degree of the vertex u of G and the distance between the vertices u and v of G, respectively. In this paper, explicit formulas for computing the vertex and edge-modified Schultz indices of splice and link of graphs are presented.

Key Words: Distance, vertex-degree, modified Schultz index, edge-modified Schultz index, splice, link.

AMS(2010): 05C07, 05C12, 05C76.

§1. Introduction

In this paper, we consider connected finite graphs without any loops or multiple edges. A *topological index* $Top(G)$ of a graph G is a real number with the property that for every graph H isomorphic to G, $Top(H) = Top(G)$. There are numerous topological indices that have been found to be useful in chemical documentation, isomer discrimination, quantitative structure-property relationships (QSPR), quantitative structure-activity relationships (QSAR), and pharmaceutical drug design [7, 10]. The *Wiener index* is the first reported distance-based topological index which was introduced in 1947 by Wiener [19, 20] who used it for modeling the shape of organic molecules and for calculating several of their physico-chemical properties. The Wiener index of a graph G is defined as the sum of distances between all pairs of vertices of G,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v\,|G),$$

where $d(u,v\,|G)$ denotes the distance between the vertices u and v of G which is defined as the

[1]Received May 24, 2015, Accepted May 6, 2016.

length of any shortest path in G connecting them.

The *molecular topological index* or *Schultz index* [17] was introduced by Harry Schultz in 1989. The Schultz index of a graph G is defined as

$$S(G)=\sum_{\{u,v\}\subseteq V(G)}\big[d(u\,|G)+d(v\,|G)\big]d(u,v\,|G),$$

where $d(u\,|G)$ denotes the degree of the vertex u of G.

The vertex version of the *modified Schultz index* [9]of a graph G was introduced by Ivan Gutman in 1994 as

$$S^*(G)=\sum_{\{u,v\}\subseteq V(G)}d(u\,|G)d(v\,|G)d(u,v\,|G).$$

The modified Schultz index is also known as *Gutman index*.

The edge versions of the modified Schultz index [15] were introduced by Khormali *et al.* in 2010. Two possible distances between the edges $e=uv$ and $f=zt$ of a graph G can be considered. The first distance is denoted by $d_0(e,f\,|G)$ and defined as

$$d_0(e,f\,|G)=\begin{cases} d_1(e,f\,|G)+1 & e\neq f,\\ 0 & e=f,\end{cases}$$

where $d_1(e,f\,|G)=\min\{d(u,z\,|G),d(u,t\,|G),d(v,z\,|G),d(v,t\,|G)\}$. It is easy to see that, $d_0(e,f\,|G)=d(e,f\,|L(G))$, where $L(G)$ is the line graph of G.

The second distance is denoted by $d_4(e,f\,|G)$ and defined as

$$d_4(e,f\,|G)=\begin{cases} d_2(e,f\,|G) & e\neq f,\\ 0 & e=f,\end{cases}$$

where $d_2(e,f\,|G)=\max\{d(u,z\,|G),d(u,t\,|G),d(v,z\,|G),d(v,t\,|G)\}$.

Related to the distances d_0 and d_4, two edge versions of the modified Schultz index can be defined. The first and second *edge-modified Schultz indices* of G are denoted by $(W_*)_{e_0}(G)$ and $(W_*)_{e_4}(G)$, respectively and defined as

$$(W_*)_{e_i}(G)=\sum_{\{e,f\}\subseteq E(G)}d(e\,|G)d(f\,|G)d_i(e,f\,|G),\qquad i\in\{0,4\},$$

where $d(e\,|G)$ denotes the degree of the edge e in G which is the degree of the vertex e in the line graph $L(G)$. For more information on the edge-modified Schultz indices, see [14].

In this paper, we compute the vertex version and edge versions of the modified-Schultz index for splice and link of graphs. Readers interested in more information on computing topological indices of splice and link of graphs can be referred to [1 C 6, 8, 12, 13, 16, 18].

§2. Definitions and Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $V(e)$ the set of two end-vertices of the edge e of G. For $u \in V(G)$ and $e = ab \in E(G)$, we define

$$D_1(u, e\,|G) = \min\{d(u, a\,|G), d(u, b\,|G)\}, \qquad D_2(u, e\,|G) = \max\{d(u, a\,|G), d(u, b\,|G)\}.$$

Note that, $D_1(u, e\,|G)$ is a nonnegative integer and $D_1(u, e\,|G) = 0$ if and only if $u \in V(e)$. Also, $D_2(u, e\,|G)$ is a positive integer and $D_2(u, e\,|G) = 1$ if and only if $u \in V(e)$ or u and the end vertices of e form a 3-cycle in G.

For $u \in V(G)$, let $N(u\,|G)$ denote the set of all first neighbors of u in G. We denote by $\delta(u\,|G)$, the sum of degrees of all neighbors of u in G, i.e.,

$$\delta(u\,|G) = \sum_{v \in N(u|G)} d(v\,|G).$$

We denote by $M_1(G)$, the *first Zagreb index* [11] of G which is defined as

$$M_1(G) = \sum_{u \in V(G)} d(u\,|G)^2.$$

The first Zagreb index can also be expressed as a sum over edges of G,

$$M_1(G) = \sum_{uv \in E(G)} \big[d(u\,|G) + d(v\,|G)\big].$$

Let e be an edge of G with $V(e) = \{a, b\}$. It is easy to see that, $d(e\,|G) = d(a\,|G) + d(b\,|G) - 2$. Therefore,

$$\sum_{e \in E(G)} d(e\,|G) = M_1(G) - 2\,|E(G)|\,.$$

Also for $u \in V(G)$, we have

$$\sum_{e \in E(G); u \in V(e)} d(e\,|G) = d(u\,|G)\big(d(u\,|G) - 2\big) + \delta(u\,|G),$$

$$\sum_{\{e,f\} \subseteq E(G); u \in V(e) \cap V(f)} \big[d(e\,|G) + d(f\,|G)\big] = \big(d(u\,|G) - 1\big)\big[d(u\,|G)(d(u\,|G) - 2\big) + \delta(u\,|G)\big].$$

§3. Results and Discussion

In this section, we compute the vertex version and edge versions of the modified Schultz index for splice and link of graphs.

3.1 Splice

Let G_1 and G_2 be two graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, respectively. For given vertices $a_1 \in V(G_1)$ and $a_2 \in V(G_2)$, a *splice* [8] of G_1 and G_2 by vertices a_1 and a_2 is denoted by $(G_1.G_2)(a_1, a_2)$ and defined by identifying the vertices

a_1 and a_2 in the union of G_1 and G_2. We denote by n_i, m_i and α_i, the order and size of the graph G_i and the degree of a_i in G_i, respectively, where $i \in \{1, 2\}$. It is easy to see that, $\left|V\big((G_1.G_2)(a_1, a_2)\big)\right| = n_1 + n_2 - 1$ and $\left|E\big((G_1.G_2)(a_1, a_2)\big)\right| = m_1 + m_2$.

In the following lemmas, the degree of an arbitrary vertex of $(G_1.G_2)(a_1, a_2)$ and the distance between two arbitrary vertices of this graph are computed. The results follow easily from the definition of the splice of graphs, so their proofs are omitted.

Lemma 3.1 *Let* $u \in V\big((G_1.G_2)(a_1, a_2)\big)$. *Then*

$$d\big(u\,|(G_1.G_2)(a_1, a_2)\big) = \begin{cases} d(u\,|G_1) & u \in V(G_1) \setminus \{a_1\}, \\ d(u\,|G_2) & u \in V(G_2) \setminus \{a_2\}, \\ \alpha_1 + \alpha_2 & u = a_1 \text{ or } u = a_2. \end{cases}$$

Lemma 3.2 *Let* $u, v \in V\big((G_1.G_2)(a_1, a_2)\big)$. *Then*

$$d\big(u, v\,|(G_1.G_2)(a_1, a_2)\big) = \begin{cases} d(u, v\,|G_1) & u, v \in V(G_1), \\ d(u, v\,|G_2) & u, v \in V(G_2), \\ d(u, a_1\,|G_1) + d(a_2, v\,|G_2) & u \in V(G_1), v \in V(G_2). \end{cases}$$

In the following theorem, the modified Schultz index of $(G_1.G_2)(a_1, a_2)$ is computed.

Theorem 3.3 *The modified Schultz index of* $G = (G_1.G_2)(a_1, a_2)$ *is given by*

$$\begin{aligned} S^*(G) =& S^*(G_1) + S^*(G_2) + 2m_2 \sum_{u \in V(G_1) \setminus \{a_1\}} d(u\,|G_1) d(u, a_1\,|G_1) \\ &+ 2m_1 \sum_{u \in V(G_2) \setminus \{a_2\}} d(u\,|G_2) d(u, a_2\,|G_2). \end{aligned}$$

Proof We partition the sum in the formula of $S^*(G)$ into three sums as follows.

The first sum S_1 consists of contributions to $S^*(G)$ of pairs of vertices from G_1. Using Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned} S_1 =& \sum_{\{u,v\} \subseteq V(G_1) \setminus \{a_1\}} d(u\,|G_1) d(v\,|G_1) d(u, v\,|G_1) \\ &+ \big(\alpha_1 + \alpha_2\big) \sum_{u \in V(G_1) \setminus \{a_1\}} d(u\,|G_1) d(u, a_1\,|G_1) \\ =& S^*(G_1) + \alpha_2 \sum_{u \in V(G_1) \setminus \{a_1\}} d(u\,|G_1) d(u, a_1\,|G_1). \end{aligned}$$

The second sum S_2 consists of contributions to $S^*(G)$ of pairs of vertices from G_2. Similar to the previous case, we obtain

$$S_2 = S^*(G_2) + \alpha_1 \sum_{u\in V(G_2)\setminus\{a_2\}} d(u\,|G_2\,)d(u,a_2\,|G_2\,).$$

The third sum S_3 is taken over all pairs $\{u,v\}$ of vertices in G such that $u \in V(G_1)\setminus\{a_1\}$ and $v \in V(G_2)\setminus\{a_2\}$. Using Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned} S_3 = & \sum_{u\in V(G_1)\setminus\{a_1\}} \sum_{v\in V(G_2)\setminus\{a_2\}} d(u\,|G_1\,)d(v\,|G_2\,)\big[d(u,a_1\,|G_1\,)+d(a_2,v\,|G_2\,)\big] \\ =& \big(2m_2-\alpha_2\big) \sum_{u\in V(G_1)\setminus\{a_1\}} d(u\,|G_1\,)d(u,a_1\,|G_1\,) \\ & + \big(2m_1-\alpha_1\big) \sum_{u\in V(G_2)\setminus\{a_2\}} d(u\,|G_2\,)d(u,a_2\,|G_2\,). \end{aligned}$$

The formula of $S^*(G)$ is obtained by adding S_1, S_2 and S_3 and simplifying the resulting expression. □

In the following lemmas, the degree of an arbitrary edge of $(G_1.G_2)(a_1,a_2)$ and the distances d_0 and d_4 between two arbitrary edges of this graph are computed. The results are direct consequences of Lemmas 3.1 and 3.2, respectively, so their proofs are omitted.

Lemma 3.4 *Let* $e \in E\big((G_1.G_2)(a_1,a_2)\big)$*. Then*

$$d\big(e\,|(G_1.G_2\,)(a_1,a_2)\big) = \begin{cases} d(e\,|G_1\,) & e\in E(G_1), a_1\notin V(e),\\ d(e\,|G_1\,)+\alpha_2 & e\in E(G_1), a_1\in V(e),\\ d(e\,|G_2\,) & e\in E(G_2), a_2\notin V(e),\\ d(e\,|G_2\,)+\alpha_1 & e\in E(G_2), a_2\in V(e). \end{cases}$$

Lemma 3.5 *Let* $G = (G_1.G_2)(a_1,a_2)$ *and* $e,f \in E\big(G\big)$*. Then*

$$(i)\quad d_0(e,f|G) = \begin{cases} d_0(e,f\,|G_1\,) & e,f\in E(G_1),\\ d_0(e,f\,|G_2\,) & e,f\in E(G_2),\\ D_1(a_1,e\,|G_1\,)+D_1(a_2,f\,|G_2\,)+1 & e\in E(G_1), f\in E(G_2), \end{cases}$$

$$(ii)\quad d_4(e,f|G) = \begin{cases} d_4(e,f\,|G_1\,) & e,f\in E(G_1),\\ d_4(e,f\,|G_2\,) & e,f\in E(G_2),\\ D_2(a_1,e\,|G_1\,)+D_2(a_2,f\,|G_2\,) & e\in E(G_1), f\in E(G_2). \end{cases}$$

In the following theorem, the first and second edge-modified Schultz indices of $(G_1\,.G_2)(a_1,a_2)$ are computed.

Theorem 3.6 *The first and second edge-modified Schultz indices of* $G = (G_1.G_2)(a_1,a_2)$ *are* *given by*

(i) $$\begin{aligned}(W_*)_{e_0}(G) &= (W_*)_{e_0}(G_1) + (W_*)_{e_0}(G_2)\\ &+ \alpha_2 \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(f\,|G_1\,)d_0(e,f\,|G_1\,)\\ &+ \alpha_1 \sum_{e,f\in E(G_2);a_2\in V(e)\setminus V(f)} d(f\,|G_2\,)d_0(e,f\,|G_2\,)\\ &+ \big(M_1(G_2)-2m_2+\alpha_1\alpha_2\big)\sum_{e\in E(G_1)} d(e\,|G_1\,)D_1(a_1,e\,|G_1\,)\\ &+ \big(M_1(G_1)-2m_1+\alpha_1\alpha_2\big)\sum_{f\in E(G_2)} d(f\,|G_2\,)D_1(a_2,f\,|G_2\,)\\ &+ \big(M_1(G_1)-2m_1\big)\big(M_1(G_2)-2m_2\big) + \alpha_2(\alpha_1-1)\delta(a_1\,|G_1\,) + \alpha_1(\alpha_2-1)\delta(a_2\,|G_2\,)\\ &+ \alpha_1\alpha_2\Big[M_1(G_1)+M_1(G_2)+\big(\alpha_1+\alpha_2\big)^2 - \frac{7}{2}\big(\alpha_1+\alpha_2\big) - 2(m_1+m_2-2)\Big].\end{aligned}$$

(ii) $$\begin{aligned}(W_*)_{e_4}(G) &= (W_*)_{e_4}(G_1) + (W_*)_{e_4}(G_2)\\ &+ \alpha_2 \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(f\,|G_1\,)d_4(e,f\,|G_1\,)\\ &+ \alpha_1 \sum_{e,f\in E(G_2);a_2\in V(e)\setminus V(f)} d(f\,|G_2\,)d_4(e,f\,|G_2\,)\\ &+ \big(M_1(G_2)-2m_2+\alpha_1\alpha_2\big)\sum_{e\in E(G_1)} d(e\,|G_1\,)D_2(a_1,e\,|G_1\,)\\ &+ \big(M_1(G_1)-2m_1+\alpha_1\alpha_2\big)\sum_{f\in E(G_2)} d(f\,|G_2\,)D_2(a_2,f\,|G_2\,)\\ &+ \alpha_2 \sum_{\{u,v\}\subseteq N(a_1|G_1\,)} \big[d(u\,|G_1\,)+d(v\,|G_1\,)\big]d(u,v\,|G_1\,)\\ &+ \alpha_1 \sum_{\{u,v\}\subseteq N(a_2|G_2\,)} \big[d(u\,|G_2\,)+d(v\,|G_2\,)\big]d(u,v\,|G_2\,)\\ &+ \alpha_2\big(\alpha_2+2\alpha_1-4\big)\sum_{\{u,v\}\subseteq N(a_1|G_1\,)} d(u,v\,|G_1\,)\\ &+ \alpha_1\big(\alpha_1+2\alpha_2-4\big)\sum_{\{u,v\}\subseteq N(a_2|G_2\,)} d(u,v\,|G_2\,)\\ &+ \alpha_1\alpha_2\big[M_1(G_1)+M_1(G_2)-2(m_1+m_2-\alpha_1\alpha_2)\big].\end{aligned}$$

Proof We prove the first part of the theorem. The second part can be proved by a similar method. At first, we partition the sum in the formula of $(W_*)_{e_0}(G)$ into three sums as follows.

The first sum S_1 consists of contributions to $(W_*)_{e_0}(G)$ of pairs of edges from G_1. In order to compute S_1, we partition it into three sums S_{11}, S_{12} and S_{13} as follows.

The sum S_{11} is equal to

$$S_{11} = \sum_{\{e,f\}\subseteq E(G_1);a_1\notin V(e)\cup V(f)} d(e\,|G\,)d(f\,|G\,)d_0(e,f\,|G\,).$$

Using Lemmas 3.4 and 3.5, we obtain

$$S_{11} = \sum_{\{e,f\}\subseteq E(G_1);a_1\notin V(e)\cup V(f)} d(e\,|G_1\,)d(f\,|G_1\,)d_0(e,f\,|G_1\,).$$

The sum S_{12} is equal to

$$S_{12} = \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} d(e\,|G\,)d(f\,|G\,)d_0(e,f\,|G\,).$$

Note that, for every pairs $\{e,f\}$ of edges in G_1 such that $a_1 \in V(e)\cap V(f)$, $d_0(e,f\,|G_1\,) = 1$. Now, using Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned}S_{12} &= \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} \big(d(e\,|G_1\,)+\alpha_2\big)\big(d(f\,|G_1\,)+\alpha_2\big)\\ &= \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} d(e\,|G_1\,)d(f\,|G_1\,)\\ &\quad + \alpha_2(\alpha_1-1)\big[\alpha_1(\alpha_1-2)+\delta(a_1\,|G_1\,)\big] + {\alpha_2}^2\binom{\alpha_1}{2}.\end{aligned}$$

The sum S_{13} is equal to

$$S_{13} = \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(e\,|G\,)d(f\,|G\,)d_0(e,f\,|G\,).$$

Using Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned}S_{13} &= \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} \big(d(e\,|G_1\,)+\alpha_2\big)d(f\,|G_1\,)d_0(e,f\,|G_1\,)\\ &= \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(e\,|G_1\,)d(f\,|G_1\,)d_0(e,f\,|G_1\,)\\ &\quad + \alpha_2 \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(f\,|G_1\,)d_0(e,f\,|G_1\,).\end{aligned}$$

By adding S_{11}, S_{12} and S_{13}, we obtain

$$\begin{aligned}S_1 =& (W_*)_{e_0}(G_1) + {\alpha_2}^2\binom{\alpha_1}{2} + \alpha_2(\alpha_1-1)\big[\alpha_1(\alpha_1-2)+\delta(a_1\,|G_1\,)\big]\\ &+ \alpha_2 \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(f\,|G_1\,)d_0(e,f\,|G_1\,).\end{aligned}$$

The second sum S_2 consists of contributions to $(W_*)_{e_0}(G)$ of pairs of edges from G_2.

Using the same argument as in the computation of S_1, we obtain

$$S_2 = (W_*)_{e_0}(G_2) + {\alpha_1}^2 \binom{\alpha_2}{2} + \alpha_1(\alpha_2 - 1)\big[\alpha_2(\alpha_2 - 2) + \delta(a_2 \,|G_2\,)\big] + \alpha_1 \sum_{e,f \in E(G_2); a_2 \in V(e) \setminus V(f)} d(f \,|G_2\,) d_0(e, f \,|G_2\,).$$

The third sum S_3 is taken over all pairs $\{e, f\}$ of edges in G such that $e \in E(G_1)$ and $f \in E(G_2)$. In order to compute S_3, we partition it into four sums S_{31}, S_{32}, S_{33} and S_{34} as follows.

The sum S_{31} is equal to

$$S_{31} = \sum_{e \in E(G_1); a_1 \notin V(e)} \; \sum_{f \in E(G_2); a_2 \notin V(f)} d(e \,|G\,) d(f \,|G\,) d_0(e, f \,|G\,).$$

Using Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned} S_{31} = & \sum_{e \in E(G_1); a_1 \notin V(e)} \; \sum_{f \in E(G_2); a_2 \notin V(f)} d(e \,|G_1\,) d(f \,|G_2\,)\big[D_1(a_1, e \,|G_1\,) + D_1(a_2, f \,|G_2\,) + 1\big] \\ = & \big[M_1(G_2) - 2m_2 - \alpha_2(\alpha_2 - 2) - \delta(a_2 \,|G_2\,)\big] \sum_{e \in E(G_1)} d(e \,|G_1\,) D_1(a_1, e \,|G_1\,) \\ & + \big[M_1(G_1) - 2m_1 - \alpha_1(\alpha_1 - 2) - \delta(a_1 \,|G_1\,)\big] \sum_{f \in E(G_2)} d(f \,|G_2\,) D_1(a_2, f \,|G_2\,) \\ & + \big[M_1(G_1) - 2m_1 - \alpha_1(\alpha_1 - 2) - \delta(a_1 \,|G_1\,)\big] \\ & \times \big[M_1(G_2) - 2m_2 - \alpha_2(\alpha_2 - 2) - \delta(a_2 \,|G_2\,)\big]. \end{aligned}$$

The sum S_{32} is equal to

$$S_{32} = \sum_{e \in E(G_1); a_1 \notin V(e)} \; \sum_{f \in E(G_2); a_2 \in V(f)} d(e \,|G\,) d(f \,|G\,) d_0(e, f \,|G\,).$$

Using Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned} S_{32} = & \sum_{e \in E(G_1); a_1 \notin V(e)} \; \sum_{f \in E(G_2); a_2 \in V(f)} d(e \,|G_1\,)\big(d(f \,|G_2\,) + \alpha_1\big)\big[D_1(a_1, e \,|G_1\,) + 1\big] \\ = & \big[\alpha_2(\alpha_2 - 2) + \delta(a_2 \,|G_2\,) + \alpha_1\alpha_2\big]\Big[\sum_{e \in E(G_1)} d(e \,|G_1\,) D_1(a_1, e \,|G_1\,) \\ & + M_1(G_1) - 2m_1 - \alpha_1(\alpha_1 - 2) - \delta(a_1 \,|G_1\,)\Big]. \end{aligned}$$

The sum S_{33} is equal to

$$S_{33} = \sum_{e \in E(G_1); a_1 \in V(e)} \; \sum_{f \in E(G_2); a_2 \notin V(f)} d(e \,|G\,) d(f \,|G\,) d_0(e, f \,|G\,).$$

Using the same argument as in the computation of S_{32}, we obtain

$$S_{33} = \big[\alpha_1(\alpha_1 - 2) + \delta(a_1 \,|G_1\,) + \alpha_1\alpha_2\big]\Big[\sum_{f\in E(G_2)} d(f\,|G_2\,)D_1(a_2, f\,|G_2\,) + M_1(G_2) - 2m_2 - \alpha_2(\alpha_2 - 2) - \delta(a_2\,|G_2\,)\Big].$$

The sum S_{34} is equal to

$$S_{34} = \sum_{e\in E(G_1);a_1\in V(e)} \sum_{f\in E(G_2);a_2\in V(f)} d(e\,|G\,)d(f\,|G\,)d_0(e, f\,|G\,).$$

Using Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned} S_{34} &= \sum_{e\in E(G_1);a_1\in V(e)} \sum_{f\in E(G_2);a_2\in V(f)} \big(d(e\,|G_1\,) + \alpha_2\big)\big(d(f\,|G_2\,) + \alpha_1\big) \\ &= \big[\alpha_1(\alpha_1 - 2) + \delta(a_1\,|G_1\,) + \alpha_1\alpha_2\big]\big[\alpha_2(\alpha_2 - 2) + \delta(a_2\,|G_2\,) + \alpha_1\alpha_2\big]. \end{aligned}$$

By adding S_{31}, S_{32}, S_{33} and S_{34}, we obtain

$$\begin{aligned} S_3 =& \big(M_1(G_2) - 2m_2 + \alpha_1\alpha_2\big) \sum_{e\in E(G_1)} d(e\,|G_1\,)D_1(a_1, e\,|G_1\,) \\ &+ \big(M_1(G_1) - 2m_1 + \alpha_1\alpha_2\big) \sum_{f\in E(G_2)} d(f\,|G_2\,)D_1(a_2, f\,|G_2\,) \\ &+ \big(M_1(G_1) - 2m_1\big)\big(M_1(G_2) - 2m_2\big) \\ &+ \alpha_1\alpha_2\big[M_1(G_1) - 2m_1 + M_1(G_2) - 2m_2 + \alpha_1\alpha_2\big]. \end{aligned}$$

The formula of $(W_*)_{e_0}(G)$ is obtained by adding S_1, S_2 and S_3 and simplifying the resulting expression. □

3.2 Link

Let G_1 and G_2 be two graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, respectively. For vertices $a_1 \in V(G_1)$ and $a_2 \in V(G_2)$, a *link* [8] of G_1 and G_2 by vertices a_1 and a_2 is denoted by $(G_1 \sim G_2)(a_1, a_2)$ and obtained by joining a_1 and a_2 by an edge in the union of these graphs. We denote by n_i, m_i and α_i, the order and size of the graph G_i and the degree of a_i in G_i, respectively, where $i \in \{1, 2\}$. It is easy to see that, $\big|V\big((G_1 \sim G_2)(a_1, a_2)\big)\big| = n_1 + n_2$ and $\big|E\big((G_1 \sim G_2)(a_1, a_2)\big)\big| = m_1 + m_2 + 1$.

In the following lemmas, the degree of an arbitrary vertex of $(G_1 \sim G_2)(a_1, a_2)$ and the distance between two arbitrary vertices of this graph are computed. The results follow easily from the definition of the link of graphs, so their proofs are omitted.

Lemma 3.7 *Let* $u \in V\big((G_1 \sim G_2)(a_1, a_2)\big)$. *Then*

$$d\big(u\,|(G_1 \sim G_2)(a_1,a_2)\big) = \begin{cases} d(u\,|G_1) & u \in V(G_1) \setminus \{a_1\}, \\ d(u\,|G_2) & u \in V(G_2) \setminus \{a_2\}, \\ \alpha_1 + 1 & u = a_1, \\ \alpha_2 + 1 & u = a_2. \end{cases}$$

Lemma 3.8 *Let* $u, v \in V\big((G_1 \sim G_2)(a_1, a_2)\big)$. *Then*

$$d\big(u, v\,|(G_1 \sim G_2)(a_1,a_2)\big) = \begin{cases} d(u, v\,|G_1) & u, v \in V(G_1), \\ d(u, v\,|G_2) & u, v \in V(G_2), \\ d(u, a_1\,|G_1) + d(a_2, v\,|G_2) + 1 & u \in V(G_1), v \in V(G_2). \end{cases}$$

In the following theorem, the modified Schultz index of $(G_1 \sim G_2)(a_1, a_2)$ is computed.

Theorem 3.9 *The modified Schultz index of* $G = (G_1 \sim G_2)(a_1, a_2)$ *is given by*

$$\begin{aligned} S^*(G) =& S^*(G_1) + S^*(G_2) + 2(m_2+1) \sum_{u \in V(G_1)\setminus\{a_1\}} d(u\,|G_1)d(u, a_1\,|G_1) \\ &+ 2(m_1+1) \sum_{u \in V(G_2)\setminus\{a_2\}} d(u\,|G_2)d(u, a_2\,|G_2) + 2(m_1 + m_2 + 2m_1m_2) + 1. \end{aligned}$$

Proof We partition the sum in the formula of $S^*(G)$ into three sums as follows.

The first sum S_1 consists of contributions to $S^*(G)$ of pairs of vertices from G_1. By Lemmas 3.7 and 3.8, we obtain

$$\begin{aligned} S_1 =& \sum_{\{u,v\} \subseteq V(G_1)\setminus\{a_1\}} d(u\,|G_1)d(v\,|G_1)d(u, v\,|G_1) \\ &+ \big(\alpha_1 + 1\big) \sum_{u \in V(G_1)\setminus\{a_1\}} d(u\,|G_1)d(u, a_1\,|G_1) \\ =& S^*(G_1) + \sum_{u \in V(G_1)\setminus\{a_1\}} d(u\,|G_1)d(u, a_1\,|G_1). \end{aligned}$$

The second sum S_2 consists of contributions to $S^*(G)$ of pairs of vertices from G_2. Similar to the previous case, we obtain

$$S_2 = S^*(G_2) + \sum_{u \in V(G_2)\setminus\{a_2\}} d(u\,|G_2)d(u, a_2\,|G_2).$$

The third sum S_3 is taken over all pairs $\{u, v\}$ of vertices in G such that $u \in V(G_1)$ and $v \in V(G_2)$. In order to compute S_3, we partition it into four sums S_{31}, S_{32}, S_{33} and S_{34} as follows.

The sum S_{31} is taken over all pairs $\{u,v\}$ of vertices in G such that $u \in V(G_1) \setminus \{a_1\}$ and $v = a_2$. Using Lemmas 3.7 and 3.8, we obtain

$$\begin{aligned} S_{31} =& \big(\alpha_2+1\big) \sum_{u\in V(G_1)\setminus\{a_1\}} d(u\,|G_1\,)\big[d(u,a_1\,|G_1\,)+1\big] \\ =& \big(\alpha_2+1\big)\big[\sum_{u\in V(G_1)\setminus\{a_1\}} d(u\,|G_1\,)d(u,a_1\,|G_1\,)+2m_1-\alpha_1\big]. \end{aligned}$$

The sum S_{32} is taken over all pairs $\{u,v\}$ of vertices in G such that $u \in V(G_2) \setminus \{a_2\}$ and $v = a_1$. Similar to the previous case, we obtain

$$S_{32} = \big(\alpha_1+1\big)\big[\sum_{u\in V(G_2)\setminus\{a_2\}} d(u\,|G_2\,)d(u,a_2\,|G_2\,)+2m_2-\alpha_2\big].$$

The sum S_{33} is taken over all pairs $\{u,v\}$ of vertices in G such that $u \in V(G_1) \setminus \{a_1\}$ and $v \in V(G_2) \setminus \{a_2\}$. By Lemmas 3.7 and 3.8, we obtain

$$\begin{aligned} S_{33} =& \sum_{u\in V(G_1)\setminus\{a_1\}} \sum_{v\in V(G_2)\setminus\{a_2\}} d(u\,|G_1\,)d(v\,|G_2\,)\big[d(u,a_1\,|G_1\,)+d(a_2,v\,|G_2\,)+1\big] \\ =& \big(2m_2-\alpha_2\big) \sum_{u\in V(G_1)\setminus\{a_1\}} d(u\,|G_1\,)d(u,a_1\,|G_1\,) \\ &+ \big(2m_1-\alpha_1\big) \sum_{v\in V(G_2)\setminus\{a_2\}} d(v\,|G_2\,)d(a_2,v\,|G_2\,) + \big(2m_1-\alpha_1\big)\big(2m_2-\alpha_2\big). \end{aligned}$$

The sum S_{34} is equal to

$$S_{34} = d(a_1\,|G\,)d(a_2\,|G\,)d(a_1,a_2\,|G\,) = \big(\alpha_1+1\big)\big(\alpha_2+1\big).$$

By adding S_{31}, S_{32}, S_{33} and S_{34}, we obtain

$$\begin{aligned} S_3 =& \big(2m_2+1\big) \sum_{u\in V(G_1)\setminus\{a_1\}} d(u\,|G_1\,)d(u,a_1\,|G_1\,) \\ &+ \big(2m_1+1\big) \sum_{v\in V(G_2)\setminus\{a_2\}} d(v\,|G_2\,)d(a_2,v\,|G_2\,) \\ &+ 2(m_1+m_2+2m_1m_2)+1. \end{aligned}$$

Now, the formula of $S^*(G)$ is obtained by adding S_1, S_2 and S_3 and simplifying the resulting expression. □

In the following lemmas, the degree of an arbitrary edge of $(G_1 \sim G_2)(a_1,a_2)$ and the distances d_0 and d_4 between two arbitrary edges of this graph are computed. The results are direct consequences of Lemmas 3.7 and 3.8, respectively, so their proofs are omitted.

Lemma 3.10 *Let* $e \in E\big((G_1 \sim G_2)(a_1,a_2)\big)$. *Then*

$$d\big(e\,|(G_1 \sim G_2)(a_1,a_2)\big) = \begin{cases} d(e\,|G_1) & e \in E(G_1), a_1 \notin V(e), \\ d(e\,|G_1) + 1 & e \in E(G_1), a_1 \in V(e), \\ d(e\,|G_2) & e \in E(G_2), a_2 \notin V(e), \\ d(e\,|G_2) + 1 & e \in E(G_2), a_2 \in V(e), \\ \alpha_1 + \alpha_2 & e = a_1a_2. \end{cases}$$

Lemma 3.11 *Let* $G = (G_1 \sim G_2)(a_1,a_2)$ *and* $e, f \in E(G)$. *Then*

$$(i)\ \ d_0\big(e,f|G\big) = \begin{cases} d_0(e,f\,|G_1) & e,f \in E(G_1), \\ d_0(e,f\,|G_2) & e,f \in E(G_2), \\ D_1(a_1,e\,|G_1) + 1 & e \in E(G_1), f = a_1a_2, \\ D_1(a_2,e\,|G_2) + 1 & e \in E(G_2), f = a_1a_2, \\ D_1(a_1,e\,|G_1) + D_1(a_2,f\,|G_2) + 2 & e \in E(G_1), f \in E(G_2), \end{cases}$$

$$(ii)\ \ d_4\big(e,f|G\big) = \begin{cases} d_4(e,f\,|G_1) & e,f \in E(G_1), \\ d_4(e,f\,|G_2) & e,f \in E(G_2), \\ D_2(a_1,e\,|G_1) + 1 & e \in E(G_1), f = a_1a_2, \\ D_2(a_2,e\,|G_2) + 1 & e \in E(G_2), f = a_1a_2, \\ D_2(a_1,e\,|G_1) + D_2(a_2,f\,|G_2) + 1 & e \in E(G_1), f \in E(G_2). \end{cases}$$

In the following theorem, the first and second edge-modified Schultz indices of $(G_1 \sim G_2)(a_1,a_2)$ are computed.

Theorem 3.12 *The first and second edge-modified Schultz indices of* $G = (G_1 \sim G_2)(a_1,a_2)$ *are given by*

$$\begin{aligned} (i)\ \ (W_*)_{e_0}(G) &= (W_*)_{e_0}(G_1) + (W_*)_{e_0}(G_2) + \sum_{e,f \in E(G_1);\, a_1 \in V(e)\setminus V(f)} d(f\,|G_1)d_0(e,f\,|G_1) \\ &+ \sum_{e,f \in E(G_2);\, a_2 \in V(e)\setminus V(f)} d(f\,|G_2)d_0(e,f\,|G_2) \\ &+ \big(M_1(G_2) - 2m_2 + 2\alpha_2 + \alpha_1\big) \sum_{e \in E(G_1)} d(e\,|G_1)D_1(a_1,e\,|G_1) \\ &+ \big(M_1(G_1) - 2m_1 + 2\alpha_1 + \alpha_2\big) \sum_{f \in E(G_2)} d(f\,|G_2)D_1(a_2,f\,|G_2) \\ &+ 2\big(M_1(G_1) - 2m_1\big)\big(M_1(G_2) - 2m_2\big) + (\alpha_1 - 1)\delta(a_1\,|G_1) + (\alpha_2 - 1)\delta(a_2\,|G_2) \\ &+ (3\alpha_2 + \alpha_1)\big(M_1(G_1) - 2m_1\big) + (3\alpha_1 + \alpha_2)\big(M_1(G_2) - 2m_2\big) \\ &+ {\alpha_1}^3 + {\alpha_2}^3 + 4\alpha_1\alpha_2 - 3\binom{\alpha_1}{2} - 3\binom{\alpha_2}{2}. \end{aligned}$$

$$
\begin{aligned}
(ii)\quad (W_*)_{e_4}(G) &= (W_*)_{e_4}(G_1) + (W_*)_{e_4}(G_2) + \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(f\,|G_1\,)d_4(e,f\,|G_1\,)\\
&+ \sum_{e,f\in E(G_2);a_2\in V(e)\setminus V(f)} d(f\,|G_2\,)d_4(e,f\,|G_2\,)\\
&+ \big(M_1(G_2) - 2m_2 + 2\alpha_2 + \alpha_1\big) \sum_{e\in E(G_1)} d(e\,|G_1\,)D_2(a_1,e\,|G_1\,)\\
&+ \big(M_1(G_1) - 2m_1 + 2\alpha_1 + \alpha_2\big) \sum_{f\in E(G_2)} d(f\,|G_2\,)D_2(a_2,f\,|G_2\,)\\
&+ \sum_{\{u,v\}\subseteq N(a_1|G_1\,)} \big[d(u\,|G_1\,) + d(v\,|G_1\,)\big]d(u,v\,|G_1\,)\\
&+ \sum_{\{u,v\}\subseteq N(a_2|G_2\,)} \big[d(u\,|G_2\,) + d(v\,|G_2\,)\big]d(u,v\,|G_2\,)\\
&+ (2\alpha_1 - 3) \sum_{\{u,v\}\subseteq N(a_1|G_1\,)} d(u,v\,|G_1\,) + (2\alpha_2 - 3) \sum_{\{u,v\}\subseteq N(a_2|G_2\,)} d(u,v\,|G_2\,)\\
&+ \big(M_1(G_1) - 2m_1\big)\big(M_1(G_2) - 2m_2\big) + (3\alpha_2 + \alpha_1)\big(M_1(G_1) - 2m_1\big)\\
&+ (3\alpha_1 + \alpha_2)\big(M_1(G_2) - 2m_2\big) + (\alpha_1 + \alpha_2)^2 + 3\alpha_1\alpha_2.
\end{aligned}
$$

Proof We prove the second part of the theorem. The first part can be proved by a similar method. At first, we partition the sum in the formula of $(W_*)_{e_4}(G)$ into three sums as follows.

The first sum S_1 consists of contributions to $(W_*)_{e_4}(G)$ of pairs of edges from G_1. In order to compute S_1, we partition it into three sums S_{11}, S_{12} and S_{13} as follows.

The sum S_{11} is equal to

$$S_{11} = \sum_{\{e,f\}\subseteq E(G_1);a_1\notin V(e)\cup V(f)} d(e\,|G\,)d(f\,|G\,)d_4(e,f\,|G\,).$$

Using Lemmas 3.10 and 3.11, we obtain

$$S_{11} = \sum_{\{e,f\}\subseteq E(G_1);a_1\notin V(e)\cup V(f)} d(e\,|G_1\,)d(f\,|G_1\,)d_4(e,f\,|G_1\,).$$

The sum S_{12} is equal to

$$S_{12} = \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} d(e\,|G\,)d(f\,|G\,)d_4(e,f\,|G\,).$$

Using Lemmas 3.10 and 3.11, we obtain

$$\begin{aligned}S_{12} =& \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} \big(d(e\,|G_1\,)+1\big)\big(d(f\,|G_1\,)+1\big)d_4(e,f\,|G_1\,)\\
=& \sum_{\{e,f\}\subseteq E(G_1);a_1\in V(e)\cap V(f)} d(e\,|G_1\,)d(f\,|G_1\,)d_4(e,f\,|G_1\,)\\
&+ \sum_{\{u,v\}\subseteq N(a_1|G_1\,)} \big[d(u\,|G_1\,)+d(v\,|G_1\,)\big]d(u,v\,|G_1\,)\\
&+(2\alpha_1-3)\sum_{\{u,v\}\subseteq N(a_1|G_1\,)} d(u,v\,|G_1\,).\end{aligned}$$

The sum S_{13} is equal to

$$S_{13} = \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(e\,|G)d(f\,|G)d_4(e,f\,|G).$$

Using Lemmas 3.10 and 3.11, we obtain

$$\begin{aligned}S_{13} =& \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} \big(d(e\,|G_1\,)+1\big)d(f\,|G_1\,)d_4(e,f\,|G_1\,)\\
=& \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(e\,|G_1\,)d(f\,|G_1\,)d_4(e,f\,|G_1\,)\\
&+ \sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(f\,|G_1\,)d_4(e,f\,|G_1\,).\end{aligned}$$

By adding S_{11}, S_{12} and S_{13}, we obtain

$$\begin{aligned}S_1 =&(W_*)_{e_4}(G_1)+\sum_{\{u,v\}\subseteq N(a_1|G_1\,)} \big[d(u\,|G_1\,)+d(v\,|G_1\,)\big]d(u,v\,|G_1\,)\\
&+(2\alpha_1-3)\sum_{\{u,v\}\subseteq N(a_1|G_1\,)} d(u,v\,|G_1\,)+\sum_{e,f\in E(G_1);a_1\in V(e)\setminus V(f)} d(f\,|G_1\,)d_4(e,f\,|G_1\,).\end{aligned}$$

The second sum S_2 consists of contributions to $(W_*)_{e_4}(G)$ of pairs of edges from G_2. Using the same argument as in the computation of S_1, we obtain

$$\begin{aligned}S_2 =&(W_*)_{e_4}(G_2)+\sum_{\{u,v\}\subseteq N(a_2|G_2\,)} \big[d(u\,|G_2\,)+d(v\,|G_2\,)\big]d(u,v\,|G_2\,)\\
&+(2\alpha_2-3)\sum_{\{u,v\}\subseteq N(a_2|G_2\,)} d(u,v\,|G_2\,)+\sum_{e,f\in E(G_2);a_2\in V(e)\setminus V(f)} d(f\,|G_2\,)d_4(e,f\,|G_2\,).\end{aligned}$$

The third sum S_3 is taken over all pairs $\{e,f\}$ of edges in G such that $e\in E(G_1)$ and $f=a_1a_2$. In order to compute S_3, we partition it into two sums S_{31} and S_{32} as follows.

The sum S_{31} is equal to

$$S_{31} = \sum_{e\in E(G_1);a_1\notin V(e),f=a_1a_2} d(e\,|G)d(f\,|G)d_4(e,f\,|G).$$

Using Lemmas 3.10 and 3.11, we obtain

$$\begin{aligned} S_{31} =& \sum_{e\in E(G_1);a_1\notin V(e)} d(e\,|G_1\,)(\alpha_1+\alpha_2)\big[D_2(a_1,e\,|G_1\,)+1\big] \\ =&(\alpha_1+\alpha_2)\big[\sum_{e\in E(G_1)} d(e\,|G_1\,)D_2(a_1,e\,|G_1\,)+M_1(G_1)-2m_1-2\alpha_1(\alpha_1-2)-2\delta(a_1\,|G_1\,)\big]. \end{aligned}$$

The sum S_{32} is equal to

$$S_{32} = \sum_{e\in E(G_1);a_1\in V(e),f=a_1a_2} d(e\,|G\,)d(f\,|G\,)d_4(e,f\,|G\,).$$

Using Lemmas 3.10 and 3.11, we obtain

$$S_{32} =2\sum_{e\in E(G_1);a_1\in V(e)} \big(d(e\,|G_1\,)+1\big)(\alpha_1+\alpha_2)=2(\alpha_1+\alpha_2)\big[\alpha_1(\alpha_1-2)+\delta(a_1\,|G_1\,)+\alpha_1\big].$$

By adding S_{31} and S_{32}, we obtain

$$S_3=(\alpha_1+\alpha_2)\big[\sum_{e\in E(G_1)} d(e\,|G_1\,)D_2(a_1,e\,|G_1\,)+M_1(G_1)-2m_1+2\alpha_1\big].$$

The fourth sum S_4 is taken over all pairs $\{e,f\}$ of edges in G such that $e\in E(G_2)$ and $f=a_1a_2$. Using the same argument as in the computation of S_3, we obtain

$$S_4=(\alpha_1+\alpha_2)\big[\sum_{e\in E(G_2)} d(e\,|G_2\,)D_2(a_2,e\,|G_2\,)+M_1(G_2)-2m_2+2\alpha_2\big].$$

The fifth sum S_5 is taken over all pairs $\{e,f\}$ of edges in G such that $e\in E(G_1)$ and $f\in E(G_2)$. In order to compute S_5, we partition it into four sums S_{51}, S_{52}, S_{53} and S_{54} as follows.

The sum S_{51} is equal to

$$S_{51} = \sum_{e\in E(G_1);a_1\notin V(e)}\ \sum_{f\in E(G_2);a_2\notin V(f)} d(e\,|G\,)d(f\,|G\,)d_4(e,f\,|G\,).$$

Using Lemmas 3.10 and 3.11, we obtain

$$\begin{aligned} S_{51} =& \sum_{e\in E(G_1);a_1\notin V(e)}\ \sum_{f\in E(G_2);a_2\notin V(f)} d(e\,|G_1\,)d(f\,|G_2\,)\big[D_2(a_1,e\,|G_1\,)+D_2(a_2,f\,|G_2\,)+1\big] \\ =&\big[M_1(G_2)-2m_2-\alpha_2(\alpha_2-2)-\delta(a_2\,|G_2\,)\big] \\ &\big[\sum_{e\in E(G_1)} d(e\,|G_1\,)D_2(a_1,e\,|G_1\,)-\alpha_1(\alpha_1-2)-\delta(a_1\,|G_1\,)\big] \\ &+\big[M_1(G_1)-2m_1-\alpha_1(\alpha_1-2)-\delta(a_1\,|G_1\,)\big] \end{aligned}$$

$$\times \Big[\sum_{f\in E(G_2)} d(f\,|G_2\,)D_2(a_2,f\,|G_2\,) - \alpha_2(\alpha_2-2) - \delta(a_2\,|G_2\,)\Big]$$
$$+ \big[M_1(G_1) - 2m_1 - \alpha_1(\alpha_1-2) - \delta(a_1\,|G_1\,)\big] \times \big[M_1(G_2) - 2m_2 - \alpha_2(\alpha_2-2) - \delta(a_2\,|G_2\,)\big].$$

The sum S_{52} is equal to

$$S_{52} = \sum_{e\in E(G_1);a_1\notin V(e)} \sum_{f\in E(G_2);a_2\in V(f)} d(e\,|G)d(f\,|G)d_4(e,f\,|G).$$

Using Lemmas 3.10 and 3.11, we obtain

$$\begin{aligned} S_{52} =& \sum_{e\in E(G_1);a_1\notin V(e)} \sum_{f\in E(G_2);a_2\in V(f)} d(e\,|G_1\,)\big(d(f\,|G_2\,)+1\big)\big[D_2(a_1,e\,|G_1\,)+2\big] \\ =&\big[\alpha_2(\alpha_2-1)+\delta(a_2\,|G_2\,)\big]\Big[\sum_{e\in E(G_1)} d(e\,|G_1\,)D_2(a_1,e\,|G_1\,) - \alpha_1(\alpha_1-2) - \delta(a_1\,|G_1\,) \\ &+ 2\big(M_1(G_1) - 2m_1 - \alpha_1(\alpha_1-2) - \delta(a_1\,|G_1\,)\big)\Big]. \end{aligned}$$

The sum S_{53} is equal to

$$S_{53} = \sum_{e\in E(G_1);a_1\in V(e)} \sum_{f\in E(G_2);a_2\notin V(f)} d(e\,|G)d(f\,|G)d_4(e,f\,|G).$$

Using the same argument as in the computation of S_{52}, we obtain

$$\begin{aligned} S_{53} =&\big[\alpha_1(\alpha_1-1)+\delta(a_1\,|G_1\,)\big]\Big[\sum_{f\in E(G_2)} d(f\,|G_2\,)D_1(a_2,f\,|G_2\,) - \alpha_2(\alpha_2-2) - \delta(a_2\,|G_2\,) \\ &+ 2\big(M_1(G_2) - 2m_2 - \alpha_2(\alpha_2-2) - \delta(a_2\,|G_2\,)\big)\Big]. \end{aligned}$$

The sum S_{54} is equal to

$$S_{54} = \sum_{e\in E(G_1);a_1\in V(e)} \sum_{f\in E(G_2);a_2\in V(f)} d(e\,|G)d(f\,|G)d_4(e,f\,|G).$$

Using Lemmas 3.10 and 3.11, we obtain

$$\begin{aligned} S_{54} =&3 \sum_{e\in E(G_1);a_1\in V(e)} \sum_{f\in E(G_2);a_2\in V(f)} \big(d(e\,|G_1\,)+1)(d(f\,|G_2\,)+1\big) \\ =&3\big[\alpha_1(\alpha_1-1)+\delta(a_1\,|G_1\,)\big]\big[\alpha_2(\alpha_2-1)+\delta(a_2\,|G_2\,)\big]. \end{aligned}$$

By adding S_{51}, S_{52}, S_{53} and S_{54}, we obtain

$$\begin{aligned} S_5 =&\big(M_1(G_2) - 2m_2 + \alpha_2\big) \sum_{e\in E(G_1)} d(e\,|G_1\,)D_2(a_1,e\,|G_1\,) \\ &+ \big(M_1(G_1) - 2m_1 + \alpha_1\big) \sum_{f\in E(G_2)} d(f\,|G_2\,)D_2(a_2,f\,|G_2\,) \end{aligned}$$

$$+ \big(M_1(G_1) - 2m_1\big)\big(M_1(G_2) - 2m_2\big) + 2\alpha_2\big(M_1(G_1) - 2m_1\big)$$
$$+ 2\alpha_1\big(M_1(G_2) - 2m_2\big) + 3\alpha_1\alpha_2.$$

The formula of $(W_*)_{e_4}(G)$ is obtained by adding S_1, S_2, S_3, S_4 and S_5 and simplifying the resulting expression. □

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International J.Math. Combin. Vol.2(2016), 83-91

Folding of Cayley Graphs

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Abstract: The aim of this paper is to discuss the folding of Cayley graphs of finite group.We prove that, for any finite group $G, |G\ | = n$ and H is a subgroup of G. Then Cayley graph $\Gamma = \text{Cay}(G, S)$ of G with respect to $S = H \setminus \{1_G\}$ can be folded into a complete graph K_r where $r = |H|$. Hence every Cayley graph $\Gamma = \text{Cay}(G, S)$ of valency $n-1$ can not be folded. Also every Cayley graph $\Gamma = \text{Cay}(G, S)$ of valency one can be folded and $\Gamma = \text{Cay}(G, S)$, where S is generating set, every elements in it is self inverse and $|S| = \frac{1}{2}|G|$, can be folded to an edge. Theorems governing these types of foldings are achieved.

Key Words: Cayley graph, folding, graph folding.

AMS(2010): 54C05, 54A05.

§1. Introduction

It was Robertson, S.A. [7] who in 1977 introduced the idea of folding on manifolds. Following this first paper there has been huge progress in the folding theory. All are focusing on topology and manifolds .Many other studies on the folding of different types of manifolds introduced by many others [5], [6], [8]. Also a graph folding has been introduced by E. El-Kholy [4]. But EL-Ghoul in [3], turns this idea to algebras branch by giving a definition of the folding of abstract rings and studying its properties. Zeen El-Deen in [9] introduced the folding of groups and studying its properties. Some applications on the folding of a manifold into itself was introduced by P. Di.Francesco [2].

Graph Theory began with Leonhard Euler in his study of the Bridges of Kӧnigsburg problem. Since Euler solved this very first problem in Graph Theory, the field has exploded, becoming one of the most important areas of applied mathematics we currently study. Generally speaking, Graph Theory is a branch of Combinatorics but it is closely connected to Applied Mathematics, Topology and Computer Science.

There are frequent occasions for which graphs with a lot of symmetry are required. One such family of graphs is constructed using groups. The study of graphs of groups is innovative because through this description one can immediately look at the graph and deduce many properties of this group. Cayley graphs are an example where graphs theory can be applied to groups. These graphs are useful for studying the structure of groups and the relationships

[1]Received September 14, 2015, Accepted May 18, 2016.

between elements with respect to subsets of these groups (for example, generating sets, inverse closed sets,$\cdots$, etc.).

Cayley (1878) used graphs to draw picture of groups as we will see in the following definitions.

§2. Definitions and Notations.

We will start putting down some definitions which are needed in this paper. We begin with a short review of some basic definitions and properties of graphs. *A graph* Γ consists of a set of elements called vertices $V(\Gamma)$, and a set of unordered pairs of these elements, called edges $E(\Gamma)$. We will write (x, y) for directed edge, and xy or $\{x, y\}$ for an undirected edge. we will only deal with *simple graphs*; that is, graph with no loops and no multiple edges and we will define all graphs to have a nonempty vertex set. A graph with no edges, but at least one vertex, is called *empty graph*.

It is important to note that a graph may have many different geometric representation, but we just use these as a visualization tools and focus on $V(\Gamma)$ and $E(\Gamma)$ for our analysis.

A graph is said to be *connected* if every pair of vertices has a path connecting them. Otherwise the graph is disconnected. *The valency* of a vertex is the number of edges with the vertex as an end point.If all the vertices of a graph have the same valency then it called *a regular* graph. A graph is *complete* if every vertex is connected to every other vertex, and we denote the complete graph on n vertices by K_n. A graph is said to be *bipartite* if its vertex set can be partitioned into two sets, V_1 and V_2 , such that there are no edges of the form $\{x, y\}$ where $x, y \in V_1$ *or* $x, y \in V_2$. The *complete bipartite* graph $K_{m,n}$ is a bipartite graph with vertex set $V_1 \bigcup V_2$, such that V_1 and V_2 have size m and n respectively, and edge set $\{\{x, y\},\ x \in V_1\ , y \in V_2\}$. A clique of a graph is its maximal complete subgraph.

Definition 2.1 *Let S be a subset of a finite group G . The Cayley digraph $\Gamma = Cay(G, S)$ of G with respect to S is the directed graph given as follows. The vertices of $\Gamma = Cay(G, S)$ are the elements of the group G . There is an arc between two vertices g and h if and only if $g^{-1}h \in S$. In other words, for every vertex $g \in G$ and element $s \in S$, there is an arc from g to gs .*

Notice that if the identity 1 of G is in S , then there is a loop at every vertex, while if $1 \notin S$, the digraph has no loops. For convenience we will assume that the latter case holds; it makes no difference to the results. Also notice that since S is a set, it contains no multiple entries and hence there are no multiple arcs.

Definition 2.2 *A Cayley digraph can be consider to be a* Cayley graph *if whenever S is closed inverse, that is ; if $s \in S$, we also have $s^{-1} \in S$, since in this case every arc is a part of a digon, and we can replace a digons with undirected edges*

Definition 2.3 *A non empty subset S of a group G is called a* Cayley subset *if $S = S^{-1}$ and $1_G \notin S$.*

It should be noted that, the Cayley graph depends very much on the given Cayley subset as well as on the group. Also Cayley graph $\Gamma = Cay(G,S)$ has valency $|S|$ and that $\Gamma = Cay(G,S)$ is connected if and only if S is generating set for G i.e., $\langle S\rangle = G$.

The *complement* $\overline{S}$ of Cayley subset S with respect to $G^* = G \setminus \{1_G\}$ is also a Cayley subset .Because if $x \in \overline{S}$ then $x \notin S$ and since S is a Cayley subset then $x^{-1} \notin S$ .Hence $x^{-1} \in \overline{S}$,i.e., $\overline{S}$ is a Cayley subset. It is clear that The $\overline{\Gamma} = Cay(G,\overline{S})$ and $\Gamma = Cay(G,S)$ have the same vertex set as G , where vertex g and h are adjacent in $\overline{\Gamma} = Cay(G,\overline{S})$ if and only if they are not adjacent in $\Gamma = Cay(G,S)$.

Definition 2.4([4]) *A graph map* $f : \Gamma_1 \longrightarrow \Gamma_2$ *between two graphs* Γ_1 *and* Γ_2 *is a* graph folding *if and only if* f *maps vertices to vertices and edges to edges, i.e., if,*

(1) *for each* $v \in V(\Gamma_1), f(v)$ *is a vertex in* $V(\Gamma_2)$*;*

(2) *for each* $e \in E(\Gamma_1), f(e)$ *is an edge in* $E(\Gamma_2)$.

Note that if the vertices of an edge $e = (u, v) \in E(\Gamma_1)$ are mapped to the same vertex, then the edge e will collapse to this vertex and hence we cannot get a graph folding.

In the case of a graph folding f the set of singularities, $\sum f$, consists of vertices only. The graph folding is non trivial iff $\sum f \neq \phi$. In this case the no. $V(f(\Gamma_1)) \leq$ no. $V(\Gamma_1)$, also no. $E(f(\Gamma_1)) \leq$ no. $E(\Gamma_1)$.

§3. Folding of Cayley Graphs

In this section we will discuss the folding of Cayley graph $\Gamma = Cay(G,S)$ to a subgraph Γ^* of it. We notice that not every Cayley graph $\Gamma = Cay(G,S)$ can be folded into a subgraph of it, for example, Let $G = S_3$ be the Symmetric group of order 6, $G = \{(),(1\ 2),(1\ 3),(2\ 3),(1\ 2\ 3),(1\ 3\ 2)\}$ and let $S = \{(1\ 2),(1\ 2\ 3),(1\ 3\ 2)\}$ since S is generating set and $|S| = 3$, so $\Gamma = Cay(S_3, S)$ is connected graph of valency 3 . This graph cannot be folded into the induced subgraph Γ^* which shown in Figure 1. Because the vertex (2 3) is adjacent with the vertices (1 2 3) , (1 2) and (1 3) then it can not mapped by any folding to these vertices. Also the vertex (2 3) can not mapped to the vertex ( ) because the edge $\{(1\ 3),(2\ 3)\}$ has no image in Γ^*. Finally the vertex (2 3) can not mapped to the vertex (1 3 2) because the edge $\{(1\ 2),(2\ 3)\}$ has no image in Γ^*.

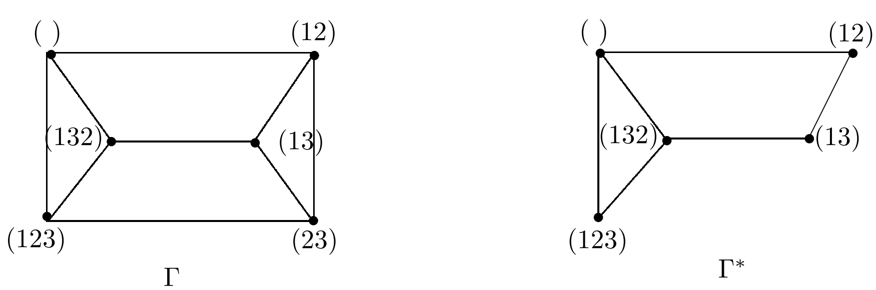

Figure 1 $\Gamma = Cay(G,S)$ can not be folded into Γ^*

It is known that, up to isomorphism , for any finite group G , there is 1 Cayley graph of G of valency zero. which is a trivial graph and this graph has a trivial foldings since there are no edges, then any graph map on the vertices is a graph folding.

Theorem 3.1 *Let G be a finite group and H is a subgroup of G. Then Cayley graph $\Gamma = Cay(G,S)$ of G with respect to $S = H \setminus \{1_G\}$ can be folded and the end of these foldings is a complete subgraph (clique) K_r where $r = |H|$ i.e.,*

$$\text{the map } \phi : \Gamma = Cay(G,S) \longrightarrow K_r \text{ is graph folding.}$$

Proof Let G be a finite group and H is a subgroup of G. Define the set S to be the subgroup H with the identity removed $S = H \setminus \{1_G\}$, so S is closed inverse. Then we can define Cayley graph $\Gamma^* = Cay(H,S)$ as follows, from the definition, there exist an edge between $\{1_G\}$ and every element $x \in S$. Also for all $x, y \in S$ there exist an edge between them, because, since S is inverse closed, then

$$x^{-1}, y^{-1} \in S\ ,\ xy^{-1} \in H \text{ and } yx^{-1} \in H \implies xy^{-1} \neq 1 \text{ or } yx^{-1} \neq 1$$

$$\implies xy^{-1} \in S \text{ or } yx^{-1} \in S$$

Then Cayley graph of H with respect to S is a complete graph $K_{|H|}$.

Also we can define Cayley graph $\Gamma = Cay(G,S)$ such that $\{g,h\}$ be an edge of $\Gamma = Cay(G,S)$ if $gh^{-1} \in S$ and hence gh^{-1} an element of H . From the properties of cosets $gh^{-1} \in H$ implies that $Hg = Hh$. This means that, two vertices are adjacent if and only if they are in the same cosets .Thus $\Gamma = Cay(G,S)$ is a graph depicting or describing the cosets of H in G .

Since the number of the cosets of H in G is the index $m = |G : H|$, thus there are m components in $\Gamma = Cay(G,S)$ each of which is a clique of size $|H|$ i.e., $K_{|H|}$. Then $\Gamma = Cay(G,S) = \{H, Hx_1, Hx_2, ..., Hx_{m-1}\}$ where $x_i \notin H,\ i = 1, 2, \cdots, m-1$

We can define a graph maps $\phi_i : \Gamma = Cay(G,S) \longrightarrow \Gamma^* = Cay(H,S)$ by

$$\phi_i : V(Hx_i) \longrightarrow V(H) \quad , \quad \text{where,} \quad \phi_i(ax_i) = a \text{ for all } a \in H \ .$$

These are graph foldings since any edge in Hx_i be in the form $e = \{ax_i,\ bx_i\}$ where $a,\ b \in H$ will mapped under ϕ into the edge $e' = \{a,b\}$ in H . Then ϕ_i preserves the edges between vertices. The end of these foldings is the folding $\phi : \Gamma = Cay(G,S) \longrightarrow K_r$ where K_r is a complete subgraph (clique) and $r = |H|$. □

Example 3.1 Let $G = D_8 = \{\ \alpha,\ \beta \mid \alpha^4 = \beta^2 = (\alpha\beta)^2 = 1\ ;\ \alpha\beta\alpha = \beta\ \}$ be the dihedral group of order 8, $G = \{1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \beta,\ \alpha\beta,\ \alpha^2\beta,\ \alpha^3\beta\ \}$.

(1) Let $H = \langle\ \alpha\ \rangle = \{1,\ \alpha,\ \alpha^2,\ \alpha^3\ \}$ be a subgroup of G . Since H is closed inverse but not generating set of G and $|G : H| = 2$ .So there exist two cosets of H in G i.e., H , $H\beta = \{\beta,\ \alpha\beta,\ \alpha^2\beta,\ \alpha^3\beta\ \}$. Let $S = H \setminus \{1\}$, then $\Gamma = Cay(D_8, S)$ has two disjoint component $\{\ H\ ,\ H\beta\ \}$ each of which is a clique of size $r = |H| = 4$, see Figure 2.

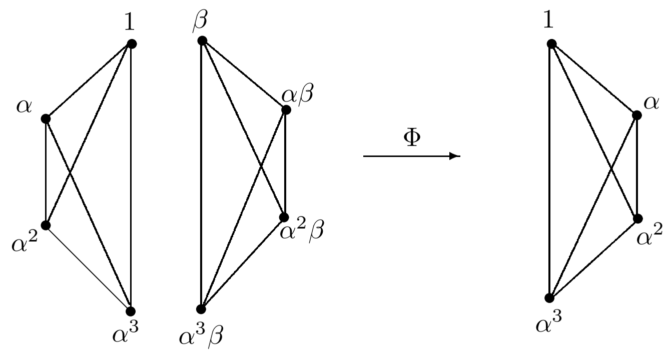

Figure 2 Folding of Cayley graph $\Gamma = Cay(D_8, \{\alpha,\ \alpha^2,\ \alpha^3\ \}\ )$

Then, the map $\phi : \Gamma = Cay(G,S) \longrightarrow\ \Gamma^* = Cay(H,S)\ \cong\ K_4$ defined by

$$\phi : V(H\beta) \longrightarrow V(H)\ ,\ \text{where}\ \phi\ (\alpha^i\ \beta) =\ \alpha^i\quad , i = 0,1,2,3\ \ \text{where}\ \ \alpha^i \in H$$

This is a graph folding since it preserves the edges between vertices and the limit of the foldings is a clique of order 4.

(2) Let $H = \langle\ \beta\ \rangle = \{1, \beta\}$ be a subgroup of G , H is closed inverse but not generating set of G . Since $|G : H| = 4$, there exist four cosets of H in G i.e., $\{\ H\ , \alpha H, \alpha^2 H, \alpha^3 H\ \}$ .Let $S = H \setminus \{1\}$, then $\Gamma = Cay(D_8, S)$ has four disjoint component $\{\ H\ , \alpha H, \alpha^2 H, \alpha^3 H\ \}$ each of which is a clique of size $r = 2$, see Figure 3.

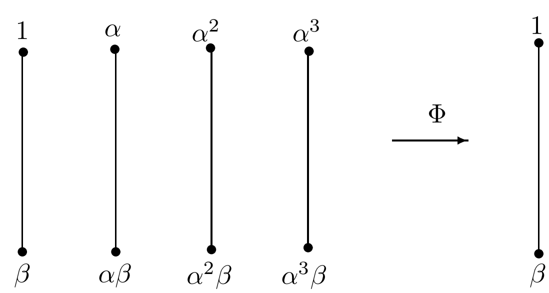

Figure 3 Folding of Cayley graph $\Gamma = Cay(D_8, \{\beta\ \}\ )$

Then the maps $\phi_i : \Gamma = Cay(G,S) \longrightarrow\ \Gamma^* = Cay(H,S)\ \cong\ K_2$ defined by

$$\phi_i : V(\alpha^i H) \longrightarrow V(H)\ \ \text{where}\ \phi_i\ (\alpha^i a) =\ a\ \ ,\ i = 0,1,2,3\quad \text{where}\ \ a \in H$$

are graph foldings and the end of these foldings is a clique of order 2.

Theorem 3.2 *For any finite group* G *of order n,* $|G\ | = n$ *. Every Cayley graph* $\Gamma = Cay(G,S)$ *of valency* $n-2$ *can be folded to a clique of order* $\frac{n}{2}$.

Proof Let G be a finite group of order n , $G = \{1, a_1, a_2, \cdots, a_{n-1}\ \}$, let S be a Cayley subset of G , since the identity element $1\ \notin S$ and valency of Cayley graph $\Gamma = Cay(G,S)$

is the valency of S , if Γ has valency $n-2$ so $|S| = n-2$. From the definition of Cayley graph, the identity element is adjacent to all the elements in S and since $\Gamma = Cay(G,S)$ has no loop, then there exists exactly one element $y \in G$, $y \notin S$ not adjacent with the identity element. this means that the two elements which is not in S is not adjacent and they adjacent to all elements in S . For any element $a_i \in S$, since $|a_i| = n-2$, $\{1, a_i\} \in E(\Gamma)$ and $\{y, a_i\} \in E(\Gamma)$ then there exist one element $b \in S$ such that $\{b, a_i\} \notin E(\Gamma)$ and a_i must adjacent to all other elements in S. Then, the vertices of $\Gamma = Cay(G,S)$ can be partitioned into $\frac{n}{2}$ sets $\{A_1, A_2, \cdots, A_{\frac{n}{2}}\}$ each set has two vertices which are not adjacent, for example $A_1 = \{1, y\}$, $y \notin S$ and $A_i = \{a_k, a_r\}$, $a_k, a_r \in S$ such that a_k and a_r not adjacent and each elements in A_i are adjacent to all elements in A_j , $i \neq j$, $i, j = 1, 2, \cdots, \frac{n}{2}$. Then we have a complete $\frac{n}{2}$ partite graph $K_{2,2,\cdots,2}$, so we can define $\frac{n}{2}$ foldings on Γ as follows $\phi_k : \Gamma \longrightarrow \Gamma^*$ defined by

$$\phi_k(x) = y \ , \ if \ x, y \in A_i \ , \quad i, k = 1, 2, \cdots, \frac{n}{2}$$

are graph foldings and the end of these foldings is a clique of order $\frac{n}{2}$. □

Example 3.2 Let $G = D_{2.3} = \{ \alpha, \beta \mid \alpha^3 = \beta^2 = (\alpha\beta)^2 = 1 ; \alpha\beta\alpha = \beta \}$ be the dihedral group of order 6 , $G = \{1, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta\}$, and $S = \{\alpha, \alpha^2, \beta, \alpha\beta\}$ be a Cayley subset of G . The Cayley graph $\Gamma = Cay(D_6, S)$ is shown in Figure 4.

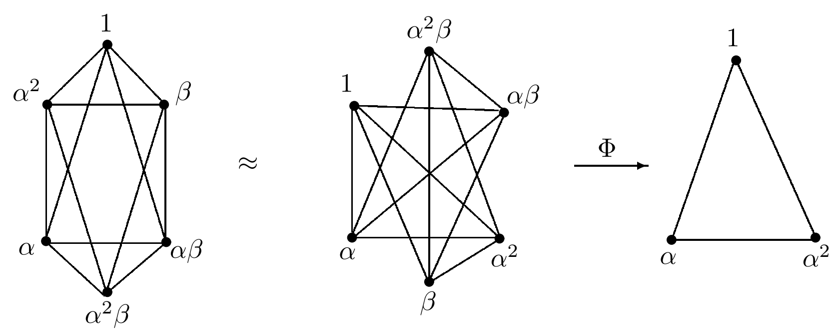

Figure 4 Folding of Cayley graph $\Gamma = Cay(D_6, \{ \alpha, \alpha^2, \beta, \alpha\beta \})$

The vertices of the graph $\Gamma = Cay(D_6, S)$ can divides into $\frac{n}{2} = 3$ sets, each set has two elements which are not adjacent in $\Gamma = Cay(D_6, S)$ .So we have $A_1 = \{1, \alpha^2\beta\}$, $A_2 = \{\alpha, \beta\}$, $A_3 = \{\alpha^2, \alpha\beta\}$. Then we can define three foldings $\phi_1(\alpha^2\beta) = 1$, $\phi_2(\beta) = \alpha^2$ and $\phi_3(\alpha\beta) = \alpha^2$. The composition of these foldings is the map $\phi : V(\Gamma = Cay(D_6, S)) \rightarrow V(\Gamma = Cay(D_6, S))$ defined by

$$\phi(x) = \begin{cases} 1 & \text{if} \quad x \in A_1 \\ \alpha & \text{if} \quad x \in A_2 \\ \alpha^2 & \text{if} \quad x \in A_3 \end{cases}$$

Since the image of any edge of $E(\Gamma)$ will be the edge, then ϕ is a graph folding.

Proposition 3.1 *For any finite group G , $|G| = n$. Every Cayley graph $\Gamma = Cay(G,S)$ of valency $n-1$ can not be folded.*

Proof Let G be a finite group $|G| = n$, let S be a Cayley subset of G, since valency of Cayley graph $\Gamma = Cay(G,S)$ is the valency of S, if Γ has valency $n-1$ so $|S| = n-1$. Then $S = G \setminus \{1_G\}$ is a generating and closed inverse which implies that Cayley graph $\Gamma = Cay(G,S)$ is connected. Let $H = S \cup \{1\} = G$, so H is a subgroup of G and the index $|G : H| = 1$. Then we have one component which is a clique of order $|S| = n-1$ i.e., $\Gamma = Cay(G,S) \cong K_{n-1}$, so every vertex of Γ is adjacent to all other vertices. Then we can not define any folding on Γ since mapping any vertex to another will collapse the edge between them. □

Proposition 3.2 *For any finite group G. Every Cayley graph $\Gamma = Cay(G,S)$ of valency one can be folded.*

Proof Let G be a finite group $|G| = n$. Up to isomorphism, there is one Cayley graph $\Gamma = \mathrm{Cay}(G,S)$ of valency one. This graph $\Gamma = \mathrm{Cay}(G,S)$ is disconnected graph consists of $\frac{n}{2}$ disconnected components each component is an edge between two vertices. Since if $S = \{a\}$, $a \in G$ we have two cases

(i) If a is self inverse, $a = a^{-1}$, then $H = \{1, a\}$ is subgroup of G. Then from Theorem 3.1 $\Gamma = \mathrm{Cay}(G,S)$ consists of disjoint components $\{H, Hx_i\}$, $x_i \notin H$, $i = 1, 2, \ldots, \frac{n}{2}-1$, where H is a clique of order two, i.e., there is an edge between the two vertices on $Hx_i = \{x_i, ax_i\}$. so each Hx_i is a clique of order two. This graph $\Gamma = \mathrm{Cay}(G,S)$ can be folded into H.

(ii) If a is not self inverse $a \neq a^{-1}$, then H is not subgroup of G. Let $S = H \setminus \{1_G\} = \{a\}$, then $\Gamma = \mathrm{Cay}(G,S)$ consists of an edge between elements of $H = \{1, a\}$. For any other vertex $x \in G$, then $x^{-1} \in G$ so there exists only one vertex $y \in G$ such that $xy^{-1} = a \in H$, which implies an edge between x and y. Then the graph $\Gamma = Cay(G,S)$ is disconnected graph consists of disconnected components each component is an edge between two vertices, if $|G| = n$ then Γ consists of $\frac{n}{2}$ disconnecting edges, which can be folded into one component. . □

§4. Folding Cayley Graph of Non-Abelian Group

In this section we will discuss the folding of Cayley graph of finite non-abelian groups.

Theorem 4.1 *For any finite non-abelian group G. Every Cayley graph of G with respect to a Cayley subset S, $\Gamma = Cay(G,S)$, where S is generating set, every elements in it is self inverse and $|S| = \frac{1}{2}|G|$, can be folded to an edge.*

Proof Let G be a finite non-abelian group, $|G| = n$ and $S \subseteq G$ such that S is generating set, every elements in it is self inverse and $|S| = \frac{1}{2}|G|$. Since the valency of Cayley graph $\Gamma = Cay(G,S)$ is equal to the valency of S then $|\Gamma| = |S| = \frac{1}{2}|G|$. If x and $y \in S$ then $x^{-1} = x \in S$ and $y = y^{-1} \in S$ but there is no edge between x and y in $\Gamma = Cay(G,S)$, since if there exist an edge between them this must implies that $x^{-1}y \in S$ and $x^{-1}y = xy$, but $xy \notin G$ because $(xy)^{-1} = y^{-1}x^{-1} = yx \neq xy$. So to have a graph $\Gamma = Cay(G,S)$ of valency $\frac{1}{2}|n|$, every element in S must connected to every element in $G-S$. Then we have a complete bipartite graph $\Gamma = Cay(G,S) = K_{\frac{n}{2},\frac{n}{2}}$.

Let $G = \{a_1, a_2, \cdots, a_{\frac{n}{2}}, b_1, b_2, \cdots, b_{\frac{n}{2}}\} = V(\Gamma)$ and let $S = \{a_1, a_2, \cdots, a_{\frac{n}{2}}\}$, then each vertex of S is joined to each vertex of $(G-S)$ by exactly one edge, thus

$E(\Gamma) = \{ (a_1, b_1), (a_1, b_2), \cdots, (a_1, b_{\frac{n}{2}}), (a_2, b_1), (a_2, b_2), \cdots, (a_2, b_{\frac{n}{2}}), \cdots, (a_{\frac{n}{2}}, b_1), (a_{\frac{n}{2}}, b_2), \cdots, (a_{\frac{n}{2}}, b_{\frac{n}{2}}) \}$

Now, let $\phi : V(\Gamma) \longrightarrow V(\Gamma)$ be a graph map defined by

$$\phi(x) = \begin{cases} a_1 & \text{if } x \in S \\ b_1 & \text{if } x \in G-S \end{cases}$$

Thus the image of any edge of $E(\Gamma)$ will be the edge (a_1, b_1), then ϕ is a graph folding.□

Example 4.1 Let $G = S_3$ be the Symmetric group of order 6, $G = \{(), (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$ and let $S = \{(1\,2), (1\,3), (2\,3)\}$ since S is generating set, every elements in it is self inverse and $|S| = \frac{1}{2}|G| = 3$, so $\Gamma = Cay(S_3, S)$ has valency 3 and the vertices of Γ can be partitioned into two sets S and $G-S$ such that there are no edges between any two vertices on the same set, see Figure 5. Then $\Gamma \cong K_{3,3}$ which can be folded by the function

$$\phi(x) = \begin{cases} (1\ 2) & \text{if } x \in S \\ (\,) & \text{if } x \in G-S \end{cases}$$

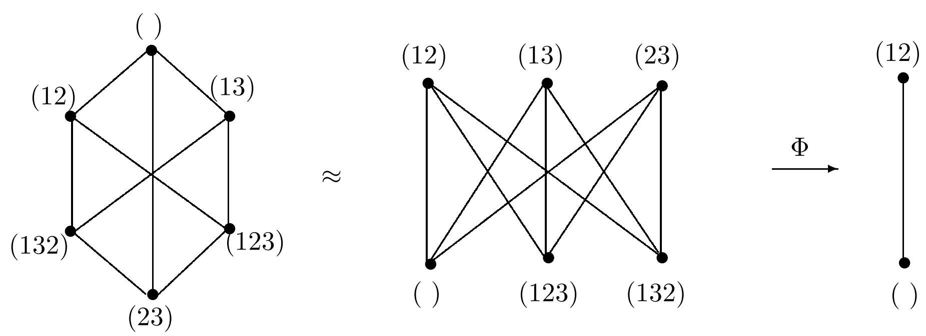

Figure 5 Folding of Cayley graph $\Gamma = Cay(S_3, \{(1\ 2), (1\ 3), (2\ 3)\})$

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International J.Math. Combin. Vol.2(2016), 92-100

The Merrifield-Simmons Indices of Triangle-Trees with k Pendant-Triangles

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Abstract: Triangle-trees are a kind of graphs derived from Koch networks. The Merrifield-Simmons index of a graph is the total number of the independent sets of the graph. We prove that $P^{\Delta}_{k,n-k}$ is the triangle-tree with maximal Merrifield-Simmons index among all the triangle-trees with n triangles and k pendant triangles.

Key Words: Triangle-tree; Merrifield-Simmons index; pendant-triangle.

AMS(2010): 05C25.

§1. Introduction

The Koch networks (see [10], [13]) are derived from the Koch fractals (see [4], [9]) and are constructed iteratively. Let $K_{m,g}$ (m is a natural number) denote the Koch network after g iterations. Then, the family of Koch networks can be generated in the following way: initially $(g = 0)$, $K_{m,0}$ consists of a triangle with three nodes labeled respectively by x, y, z, which have the highest degree among all nodes in the networks. For $g \geq 1$, $K_{m,g}$ is obtained from $K_{m,g-1}$ by performing the following operation. For each of the three nodes in every existing triangle in $K_{m,g-1}$, we add m groups of nodes. Each node group contains two nodes, both of which and their 'mother' node are connected to one another forming a new triangle. In other words, to get $K_{m,g}$ from $K_{m,g-1}$, we can replace each triangle in $K_{m,g-1}$ by a connected cluster on the right-hand side of the arrow in Fig.1.

Note that a Koch network does not have any cycle except for the triangles, we can call such a graph a triangle-tree.

Definition 1.1 *Let T^{Δ}_n (n is a natural number) denote a triangle-tree with n triangles. The family of triangle-trees can be generated in the following way: initially $n = 1$, T^{Δ}_1 consists of a triangle with three vertices labeled respectively by x, y, z. For $n \geq 2$, T^{Δ}_n is obtained from T^{Δ}_{n-1} by adding a pair of new vertices u, v, both of them are joined to a vertex of T^{Δ}_{n-1} and the edge uv is also added to form a new triangle. In other words, to get T^{Δ}_n from T^{Δ}_{n-1}, we add a new triangle to T^{Δ}_{n-1} by identifying a vertex of the new triangle with a vertex of T^{Δ}_{n-1}.*

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[1]Supported by National Natural Science Foundation of China, No. 11401576.

[1]Received June 4, 2015, Accepted May 20, 2016.

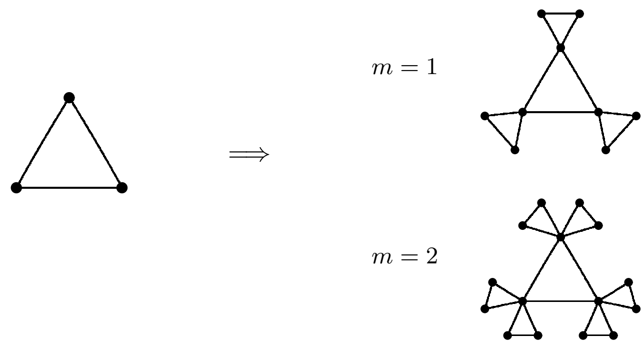

Fig.1

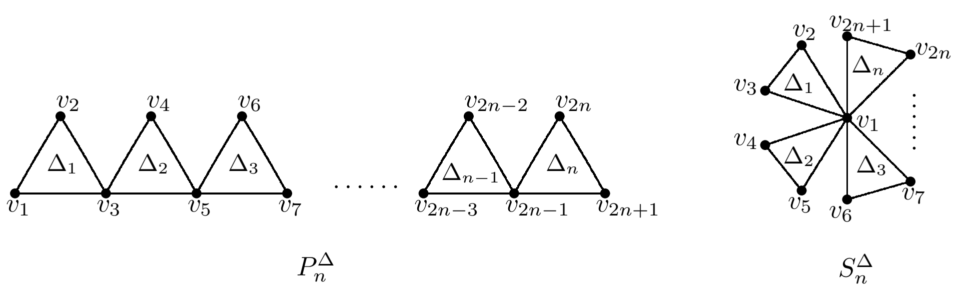

Fig.2

Obviously Koch networks are all triangle-trees. Suppose T^Δ is a triangle-tree, Δ is a triangle of T, if there are two vertices with degree two in Δ, we call the triangle Δ a pendant triangle of T^Δ. The triangle-path P_n^Δ (see Fig.2) is the only triangle-tree with only two pendant triangles and the triangle-star S_n^Δ (see Fig.2) is the only triangle-tree with n pendant triangles. For any two triangles Δ_1 and Δ_2 of T^Δ, if Δ_1 and Δ_2 have a common vertex, we say Δ_1 and Δ_2 are adjacent, and the distance between Δ_1 and Δ_2 is 1, denoted by $d(\Delta_1, \Delta_2) = 1$. If Δ_1 and Δ_2 do not have a common vertex, there is only one triangle-path between them. If the triangle-path between Δ_1 and Δ_2 contains d triangles, we say the distance between Δ_1 and Δ_2 is $d-1$, denoted by $d(\Delta_1, \Delta_2) = d - 1$. The diameter of a triangle-tree is denoted by d^Δ, defined as

$$d^\Delta(T_n^\Delta) = \max\{d(\Delta, \Delta') \mid \Delta, \Delta' \text{are two triangles of } T_n^\Delta\}.$$

Throughout this paper $G = (V, E)$ is a finite simple undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The neighborhood of a vertex $v \in V$ is the set $N_G(v) = \{w : w \in V, vw \in E\}$, $d_G(v) = |N_G(v)|$, and $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V$, we use $G - S$ for the subgraph induced by $V(G) \setminus S$, $G[S]$ for the subgraph of G induced by S and $N_S(v) = \{w : w \in S, vw \in E(G)\}$. For $F \subseteq E(G)$, we use $G - F$ for the subgraph of G obtained by deleting F.

Let G be a graph on n vertices. Two vertices of G are said to be independent if they are not adjacent in G. A k-independent set of G is a set of k-mutually independent vertices. Denote by $f_k(G)$ the number of the k-independent sets of G. For convenience, we regard the empty vertex set as an independent set. Then $f_0(G) = 1$ for any graph G. Let $\alpha(G)$ denote the cardinality of a maximal independent set of G.

The *Merrifield-Sommons index* was introduced by Prodinger and Tichy in 1982, which is defined by

$$i(G) = \sum_{s=0}^{\alpha(G)} f_s(G),$$

although it is called Fibonacci number of a graph in [8]. It is one of the most popular topological indices in chemistry, which was extensively studied in monograph [7]. Now there have been many papers studying the Merrifield-Simmons index. In [8], Prodinger and Tichy showed that, for trees with order n, the star has the maximal Merrifield-Simmons index and the path has the minimal Merrifield-Simmons index. In [6], Li et al characterized the tree with the maximal Merrifield-Simmons index among the trees with given diameter. In [11], Yu and Lv characterized the trees with maximal Merrifield-Simmons indices, among the trees with given pendant vertices. For more results on Merrifield-Simmons index, see [1-3], [5] and [12].

Due to the similarity of triangle-trees and ordinary trees, it is very interesting to study the Merrifield-Simmons indices of triangle-trees. It is easy verify that, among all the triangle-trees with n triangles, S_n^{Δ} is the triangle-tree with maximal Merrifield-Simmons index and P_n^{Δ} is the triangle-tree with minimal Merrifield-Simmons index. As noting this result is similar to the result of ordinary trees, we consider all the triangle-trees with n triangles and k pendant triangles. It is very interesting to find that $P_{k,n-k}^{\Delta}$ (as shown in Fig. 3) is the triangle-tree with maximal Merrifield-Simmos index among all such triangle-trees, and this result is also similar to the result of ordinary trees.

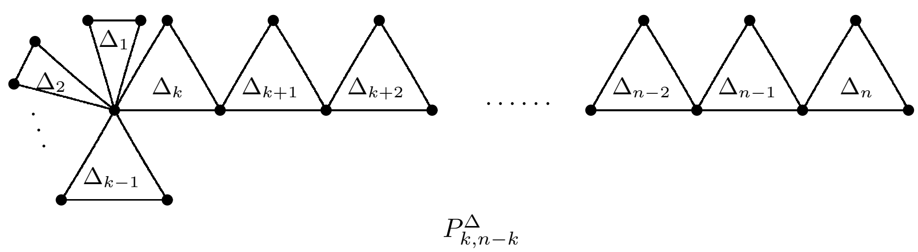

Fig.3

§2. Lemmas and Results

We first introduce the following lemma, which is obvious and well-known.

Lemma 2.1 *For a graph G, we have*

(1) $i(G) = i(G - v) + i(G - N[v])$ *for any* $v \in V(G)$;
(2) $i(G) = i(G - e) - i(G - N[e])$ *for any* $e \in E(G)$;
(3) *If* $G = G_1 \cup G_2$, *then* $i(G) = i(G_1)i(G_2)$.

Using the above lemma, we can derive some recursion formulas on the Merrifield-Simmons index of the triangle-path P_n^{Δ}. Denote $a_n = i(P_n^{\Delta})$. It is easy to see that $a_1 = 4, a_2 = 10, a_3 = 24$. Let $Q_n = P_n^{\Delta} - v_1$, where v_1 is one of the vertices with degree two of the pendant-triangle of P_n^{Δ}(as shown in Fig 2) and $b_n = i(Q_n)$. It is easy to see that $b_1 = 3, b_2 = 7, b_3 = 17$. Let $R_n = Q_n - v_{2n+1}$, where v_{2n+1} is one of the vertices with degree two of another pendant-triangle of P_n^{Δ} (as shown in Fig.2). It is easy to see that $c_1 = 2, c_2 = 5, c_3 = 12$.

By Lemma 2.1, we know

$$a_n = b_n + b_{n-1},$$
$$b_n = a_{n-1} + b_{n-1} = c_n + c_{n-1},$$
$$c_n = b_{n-1} + c_{n-1}.$$

So we have

$$b_{n+1} = 2b_n + b_{n-1},$$
$$a_{n+1} = 2a_n + a_{n-1},$$
$$c_{n+1} = 2c_n + c_{n-1}.$$

Let $P_k^{\Delta} = \Delta_1\Delta_2\cdots\Delta_k$ be a path of a triangle-tree T^{Δ}, where $\Delta_i = v_{2i-1}v_{2i}v_{2i+1}$. If $d_{T^{\Delta}}(v_1) \geq 6$, $d_{T^{\Delta}}(v_{2k+1}) \geq 6$, $d_{T^{\Delta}}(v_{2i}) = 2$ $(1 \leq i \leq k)$ and $d_{T^{\Delta}}(v_{2i+1}) = 4$ $(1 \leq i \leq k-1)$, we call P_k^{Δ} an internal triangle-path of T^{Δ}. If the triangle $\Delta_1 = v_1v_2v_3$ is a pendant triangle of T^{Δ}, $d_{T^{\Delta}}(v_{2k+1}) \geq 6$, $d_{T^{\Delta}}(v_{2i}) = 2$ $(1 \leq i \leq k)$ and $d_{T^{\Delta}}(v_{2i+1}) = 4$ $(1 \leq i \leq k-1)$, we call P^{Δ} a pendant triangle-path of T^{Δ}. Let $s(T^{\Delta})$ be the number of vertices in T^{Δ} with degree not less than 6 and $p(T^{\Delta})$ be the number of pendant triangle-paths in T^{Δ} with length not less than 1.

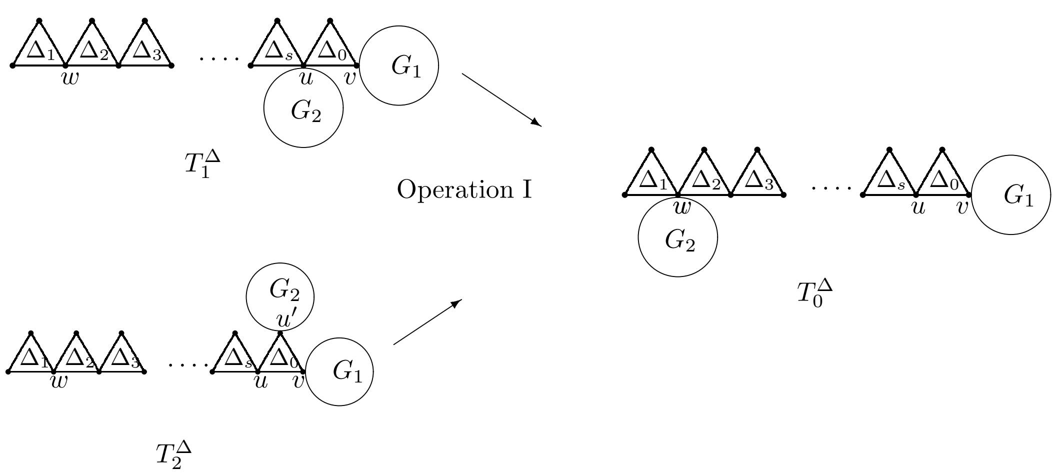

Fig.4

Denote $\mathcal{T}_{n,k}^{\Delta}$ $(3 \leq k \leq n-1)$ be the set of all triangle-trees with n triangles and k pendant triangles. In the following, we shall define two kinds of operations of $T^\Delta \in \mathcal{T}^\Delta_{n,k}$ and show that these two kinds of operations make the Merrifield-Simmons indices of the triangle-tree increase strictly.

If $T^\Delta \in \mathcal{T}^\Delta_{n,k}, T^\Delta \ncong P^\Delta_{k,n-k}$ and $p(T^\Delta) \neq 0$, then T^Δ can be seen as the triangle-trees T_1^Δ or T_2^Δ as shown in Fig.4, where $\Delta_1\Delta_2\cdots\Delta_s$ $(s \geq 2)$ is a pendant path of T^Δ with s triangles, G_1 and G_2 are two subtriangle-trees of T^Δ and $|V(G_1)| \geq 3, |V(G_2)| \geq 3$. If T_0^Δ is obtained from T_1^Δ or T_2^Δ by Operation I (as shown in Fig.4), it is easy to see that $T_0^\Delta \in \mathcal{T}^\Delta_{n,k}$.

Now we show that operation I makes the Merrifield-Simmons indices of the triangle-trees increase strictly.

Lemma 2.2 *If T_0^Δ is obtained from T_1^Δ or T_2^Δ by operation I, then $i(T_0^\Delta) > i(T_1^\Delta)$ and $i(T_0^\Delta) > i(T_2^\Delta)$.*

Proof Let $N_{G_1}[v] = V_1$, $N_{G_2}[u] = V_2$ in T_1^Δ, $N_{G_2}[u'] = V_2'$ in T_2^Δ and $N_{G_2}[w] = V_3$ in T_0^Δ.

If $s \geq 3$, by Lemma 2.1, we have

$$\begin{aligned} i(T_1^\Delta) &= i(T_1^\Delta - v) + i(T_1^\Delta - N_{T_1^\Delta}[v]) \\ &= i(G_1 - v)(2i(G_2 - u)b_s + i(G_2 - V_2)b_{s-1}) + i(G_1 - V_1)i(G_2 - u)b_s, \end{aligned}$$

$$\begin{aligned} i(T_2^\Delta) &= i(T_2^\Delta - v) + i(T_2^\Delta - N_{T_2^\Delta}[v]) \\ &= i(G_1 - v)(i(G_2 - u')a_s + i(G_2 - V_2')b_s) + i(G_1 - V_1)i(G_2 - u')b_s, \end{aligned}$$

$$\begin{aligned} i(T_0^\Delta) &= i(T_0^\Delta - v) + i(T_0^\Delta - N_{T_0^\Delta}[v]) \\ &= i(G_1 - v)(3i(G_2 - w)c_s + i(G_2 - V_3)c_{s-1}) \\ &\quad + i(G_1 - V_1)(3i(G_2 - w)c_{s-1} + i(G_2 - V_3)c_{s-2}). \end{aligned}$$

Obviously, $i(G_2 - w) = i(G_2 - u') = i(G_2 - u)$ and $i(G_2 - V_3) = i(G_2 - V_2') = i(G_2 - V_2)$, so we have

$$\begin{aligned} & i(T_0^\Delta) - i(T_1^\Delta) \\ &= i(G_1 - v)i(G_2 - u)(3c_s - 2b_s) + i(G_1 - v)i(G_2 - V_2)(c_{s-1} - b_{s-1}) \\ &\quad + i(G_1 - V_1)i(G_2 - u)(3c_{s-1} - b_s) + i(G_1 - V_1)i(G_2 - V_2)c_{s-2} \\ &= i(G_1 - v)i(G_2 - u)c_{s-2} - i(G_1 - v)i(G_2 - V_2)c_{s-2} \\ &\quad - i(G_1 - V_1)i(G_2 - u)c_{s-2} + i(G_1 - V_1)i(G_2 - V_2)c_{s-2} \\ &= c_{s-2}(i(G_1 - v) - i(G_1 - V_1))(i(G_2 - u) - i(G_2 - V_2)). \end{aligned}$$

Since $s \geq 3, c_{s-2} > 0$, $i(G_1 - v) - i(G_1 - V_1) > 0$ and $i(G_2 - u) - i(G_2 - V_2) > 0$, we know $i(T_0^\Delta) - i(T_1^\Delta) > 0$ when $s \geq 3$.

Similarly,

$$\begin{aligned}
&i(T_0^{\Delta}) - i(T_2^{\Delta}) \\
&= i(G_1 - v)i(G_2 - u')(3c_s - a_s) + i(G_1 - v)i(G_2 - V_2')(c_{s-1} - b_s) \\
&\quad + i(G_1 - V_1)i(G_2 - u')(3c_{s-1} - b_s) + i(G_1 - V_1)i(G_2 - V_2')c_{s-2} \\
&= i(G_1 - v)i(G_2 - u')(c_{s-2} + 2c_{s-1}) + i(G_1 - v)i(G_2 - V_2')(-c_{s-2} - 2c_{s-1}) \\
&\quad + i(G_1 - V_1)i(G_2 - u')(-c_{s-2}) + i(G_1 - V_1)i(G_2 - V_2')c_{s-2} \\
&= c_{s-2}(i(G_1 - v) - i(G_1 - V_1))(i(G_2 - u') - i(G_2 - V_2')) \\
&\quad + 2c_{s-1}i(G_1 - v)(i(G_2 - u') - i(G_2 - V_2')) > 0.
\end{aligned}$$

Therefore, $i(T_0^{\Delta}) - i(T_2^{\Delta}) > 0$ when $s \geq 3$. If $s = 2$, similarly, we have

$$\begin{aligned}
&i(T_0^{\Delta}) - i(T_1^{\Delta}) \\
&= i(G_1 - v)i(G_2 - u)(3c_2 - 2b_2) + i(G_1 - v)i(G_2 - V_2)(c_1 - b_1) \\
&\quad + i(G_1 - V_1)i(G_2 - u)(3c_1 - b_2) + i(G_1 - V_1)i(G_2 - V_2) \\
&= (i(G_1 - v) - i(G_1 - V_1))(i(G_2 - u) - i(G_2 - V_2)).
\end{aligned}$$

$$\begin{aligned}
&i(T_0^{\Delta}) - i(T_2^{\Delta}) \\
&= i(G_1 - v)i(G_2 - u')(3c_2 - a_2) + i(G_1 - v)i(G_2 - V_2')(c_1 - b_2) \\
&\quad + i(G_1 - V_1)i(G_2 - u')(3c_1 - b_2) + i(G_1 - V_1)i(G_2 - V_2') \\
&= (i(G_1 - v) - i(G_1 - V_1))(i(G_2 - u') - i(G_2 - V_2')) \\
&\quad + 4i(G_1 - v)(i(G_2 - u') - i(G_2 - V_2')) > 0.
\end{aligned}$$

Therefore, $i(T_0^{\Delta}) - i(T_1^{\Delta}) > 0$ and $i(T_0^{\Delta}) - i(T_2^{\Delta}) > 0$ when $s = 2$. □

From Lemma 2.2, we can immediately get the following result.

Lemma 2.3 *Let $T^{\Delta} \in \mathcal{T}_{n,k}^{\Delta}(3 \leq k \leq n-1)$, $T^{\Delta} \ncong P_{k,n-k}^{\Delta}$ and $p(T^{\Delta}) \geq 1$.*

(1) If $s(T^{\Delta}) = 1$, we can finally get a triangle-tree T'^{Δ} by operation I with $i(T'^{\Delta}) > i(T^{\Delta})$, and $p(T'^{\Delta}) = 1$; it is easy to see that $T'^{\Delta} \cong P_{k,n-k}^{\Delta}$;

(2) If $s(T^{\Delta}) > 1$, we can finally get a triangle-tree T'^{Δ} by operation I with $i(T'^{\Delta}) > i(T^{\Delta})$ and $p(T'^{\Delta}) = 0$.

If $T^{\Delta} \in \mathcal{T}_{n,k}^{\Delta}(3 \leq k \leq n-1), T^{\Delta} \ncong P_{n,k}^{\Delta}$ and $p(T^{\Delta}) = 0$, then we can find two pendant triangles Δ_1 and Δ_1' of T^{Δ} such that $d(\Delta_1, \Delta_1') = d^{\Delta}(T^{\Delta})$. Suppose $\Delta_1 = uu_1u_1'$ and $\Delta_1' = vv_1v_1'$, where u_1, u_1', v_1, v_1' are the vertices with degree 2 and $d(u) \geq 6$, $d(v) \geq 6$. Then the triangle-tree can be seen as the triangle-tree T^{Δ} shown in Fig 5, where $\Delta_1, \Delta_2, \cdots, \Delta_s$ are pendant triangles with common vertex u, $\Delta_1', \Delta_2', \cdots, \Delta_t'$ are pendant triangles with common vertex v, G_1 is the subgraph of T^{Δ} induced by $V(T^{\Delta}) \setminus \Big(\bigcup\limits_{i=1}^{s} V(\Delta_i) \cup \bigcup\limits_{i=1}^{t} V(\Delta_i')\Big)$.

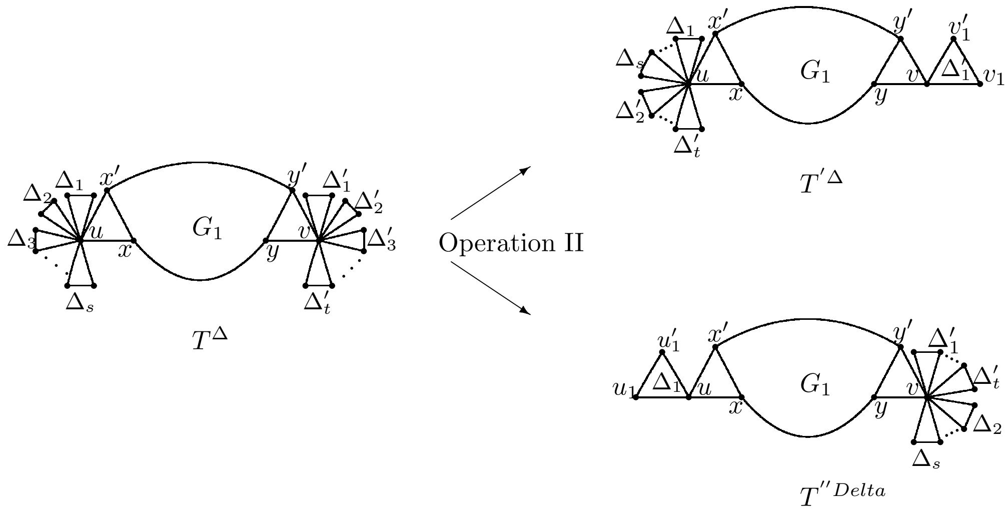

Fig.5

Note that if $d(\Delta_1,\Delta_2)=3$, then $x=y$; if $d(\Delta_1,\Delta_2)\geq 4$, then $|V(G_1)|\geq 5$. $T^{'\Delta}$ is a triangle-tree got from T^{Δ} by moving the pendant triangles $\Delta_2',\Delta_3',\cdots,\Delta_t'$ from v to u, and $T^{''\Delta}$ is a triangle-tree got from T^{Δ} by moving the pendant triangles $\Delta_2,\Delta_3,\cdots,\Delta_s$ from u to v. We say both of $T^{'\Delta}$ and $T^{''\Delta}$ are obtained from T^{Δ} by Operation II. It is easy to see that $T^{'\Delta},T^{''\Delta}\in\Gamma^{\Delta}_{n,k}$ $p(T^{'\Delta})=p(T^{''\Delta})=1$ and $s(T^{'\Delta})=s(T^{''\Delta})=s(T^{\Delta})-1$.

Lemma 2.4 *If $T^{'\Delta}$ and $T^{''\Delta}$ are obtained from T^{Δ} by Operation II, then either $i(T^{'\Delta})>i(T^{\Delta})$ or $i(T^{''\Delta})>i(T^{\Delta})$.*

Proof If $d(\Delta_1,\Delta_1')\geq 3$, then $N_{G_1}(u)=\{x,x'\}$ and $N_{G_1}[v]=\{y,y'\}$. Note that if $d(\Delta_1,\Delta_1')=3$, then $x=y$. By Lemma 2.2, we have

$$\begin{aligned} i(T^{\Delta}) &= i(T^{\Delta}-u)+i(T^{\Delta}-N_{T^{\Delta}}[u]) \\ &= 3^s(3^t i(G_1)+i(G_1-\{y,y'\}))+3^t i(G_1-\{x,x'\}\})+i(G_1-\{x,x',y,y'\}), \end{aligned}$$

$$\begin{aligned} i(T^{'\Delta}) &= i(T^{'\Delta}-u)+i(T^{'\Delta}-N_{T^{'\Delta}}[u]) \\ &= 3^{s+t-1}(3i(G_1)+i(G_1-\{y,y'\}))+3i(G_1-\{x,x'\})+i(G_1-\{x,x',y,y'\}), \end{aligned}$$

$$\begin{aligned} i(T^{''\Delta}) &= i(T^{''\Delta}-u)+i(T^{''\Delta}-N_{T^{''\Delta}}[u]) \\ &= 3(3^{s+t-1}i(G_1)+i(G_1-\{y,y'\}))+3^{s+t-1}i(G_1-\{x,x'\})+i(G_1-\{x,x',y,y'\}). \end{aligned}$$

It is easy to see that

$$i(T^{'\Delta})-i(T^{\Delta})=3(3^{t-1}-1)(3^{s-1}i(G_1-\{y,y'\})-i(G_1-\{x,x'\})),$$

$$i(T^{''\Delta}) - i(T^{\Delta}) = 3(3^{s-1}-1)(3^{t-1}i(G_1 - \{x,x'\}) - i(G_1 - \{y,y'\})).$$

Note that $s, t \geq 2$. If $i(T^{'\Delta}) - i(T^{\Delta}) \leq 0$, we have $3^{s-1}i(G_1 - \{y,y'\}) \leq i(G_1 - \{x,x'\})$. Then we have

$$i(T^{''\Delta}) - i(T^{\Delta}) \geq 3(3^{s-1}-1)(3^{s-1}3^{t-1}-1)i(G_1 - \{y,y'\}) > 0.$$

If $d(\Delta_1, \Delta_1') = 2$, we have $T^{'\Delta} \cong T^{''\Delta}$. Suppose $N_{G_2}(u) = \{v, w\}$, $N_{G_2}(v) = \{u, w\}$, then

$$i(T^{'\Delta}) - i(T^{\Delta}) = 3(3^{t-1}-1)(3^{s-1}-1)i(G_1 - w) > 0.$$

Therefore, if $T^{'\Delta}$ and $T^{''\Delta}$ are obtained from T^{Δ} by operation II, then either $i(T^{'\Delta}) > i(T^{\Delta})$ or $i(T^{''\Delta}) > i(T^{\Delta})$. □

Theorem 2.5 *Let $T^{\Delta} \in \mathcal{T}^{\Delta}_{n,k}$. Then $i(T^{\Delta}) \leq 3^{k-1}b_{n-k+1} + b_{n-k}$, the equality holds if and only if $T^{\Delta} \cong P^{\Delta}_{k,n-k}$.*

Proof By Lemma 2.1, it is easy to see that

$$i(P^{\Delta}_{k,n-k}) = 3^{k-1}b_{n-k+1} + b_{n-k}.$$

Since $\mathcal{T}^{\Delta}_{n,2} = \{P^{\Delta}_n\}$ and $P^{\Delta}_n \cong P^{\Delta}_{n,0}, \mathcal{T}^{\Delta}_{n,n} = \{S^{\Delta}_n\}$ and $S^{\Delta}_n \cong P^{\Delta}_{2,n-2}$, we may assume $3 \leq k \leq n-1$. It is sufficient to show that $i(T^{\Delta}) < i(P^{\Delta}_{k,n-k})$ for any $T^{\Delta} \in \mathcal{T}^{\Delta}_{n,k}$ and $T^{\Delta} \ncong P^{\Delta}_{k,n-k}$.

For $T^{\Delta} \in \mathcal{T}^{\Delta}_{n,k}(3 \leq k \leq n-1)$ and $T^{\Delta} \ncong P^{\Delta}_{k,n-k}$, we know $1 \leq s(T^{\Delta}) \leq n-k$, we shall show $i(T^{\Delta}) \leq i(P^{\Delta}_{k,n-k})$ by induction on $s(T^{\Delta})$. When $s(T^{\Delta}) = 1$, since $T^{\Delta} \ncong P^{\Delta}_{k,n-k}$, we have $p(T^{\Delta}) \geq 2$. By (1) of Lemma 2.3, we have $i(T^{\Delta}) < i(P^{\Delta}_{k,n-k})$. Suppose the result holds for any triangle-tree $T^{'\Delta}$ with $s(T^{'\Delta}) = s-1$. Let $s(T^{\Delta}) = s \geq 2$. If $p(T^{\Delta}) \neq 0$, by (2) of Lemma 2.3, we can get a triangle-tree $T^{\Delta}_1 \in \mathcal{T}^{\Delta}_{n,k}$ such that $p(T^{\Delta}_1) = 0$, $s(T^{\Delta}_1) = s$ and $i(T^{\Delta}_1) > i(T^{\Delta})$. Then by Lemma 2.4, we can get a triangle-tree $T^{\Delta}_2 \in \mathcal{T}^{\Delta}_{n,k}$ from T^{Δ}_1 such that $p(T^{\Delta}_2) = 1, s(T^{\Delta}_2) = s-1$ and $i(T^{\Delta}_2) > i(T^{\Delta}_1)$. By the induction hypothesis, we have

$$i(T^{\Delta}) < i(T^{\Delta}_1) < i(T^{\Delta}_2) < i(P^{\Delta}_{k,n-k}).$$

Therefore, if $T^{\Delta} \in \mathcal{T}^{\Delta}_{n,k}$, then $i(T^{\Delta}) \leq 3^{k-1}b_{n-k+1} + b_{n-k} = i(P^{\Delta}_{k,n-k})$ and the equality holds if and only if $T^{\Delta} \cong P^{\Delta}_{k,n-k}$. □

Lemma 2.6 *For $3 \leq k \leq n$, $i(P^{\Delta}_{n-k+2,k-2}) > i(P^{\Delta}_{n-k+3,k-3})$.*

Proof By Lemma 2.1, it is easy to see that

$$i(P^{\Delta}_{n-k+2,k-2}) = 3^{k-1}b_{n-k+1} + b_{n-k},$$

$$i(P^{\Delta}_{n-k+3,k-3}) = 3^{k-2}b_{n-k+2} + b_{n-k+1}.$$

Since $b_{n+1} = 2b_n + b_{n-1}$, we have

$$\begin{aligned} i(P^{\Delta}_{k,n-k}) - i(P^{\Delta}_{n-k+3,k-3}) &= 3^{k-1}b_{n-k+1} + b_{n-k} - 3^{k-2}b_{n-k+2} + b_{n-k+1} \\ &= (3^{k-2}-1)(b_{n-k+1} - b_{n-k}) > 0. \end{aligned}$$

Hence $i(P^{\Delta}_{n-k+2,k-2}) > i(P^{\Delta}_{n-k+3,k-3})$ for $3 \le k \le n$. □

From Theorem 2.5 and Lemma 2.6, we can immediately get the following result.

Corollary 2.7 *Let T^{Δ} be a triangle-tree with $2n+1$ vertices and n triangles. Then*

(1) $i(T^{\Delta}) \le 3^n + 1$ *and the equality holds if and only if* $T^{\Delta} \cong S^{\Delta}_n$*;*
(2) *If* $T^{\Delta} \ncong S^{\Delta}_n$*, then* $i(T^{\Delta}) \le 7\times 3^{n-2}+3$ *and the equality holds if and only if* $T^{\Delta} \cong P^{\Delta}_{3,n-3}$.

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International J.Math. Combin. Vol.2(2016), 101-108

On Linear Codes Over a Non-Chain Ring

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Abstract: In this paper, we study skew cyclic and quasi cyclic codes over the ring $S = F_2 + uF_2 + vF_2$ where $u^2 = u, v^2 = v, uv = vu = 0$. We investigate the structural properties of them. Using a Gray map on S we obtain the MacWilliams identities for codes over S. The relationships between Symmetrized, Lee and Hamming weight enumerator are determined.

Key Words: Line code, chain ring, non-chain ring, Gray map, MacWilliams identity.

AMS(2010): 94B05.

§1. Introduction

Since the revelation in 1994 [10], there are a lot of works on codes over finite rings. The structure of certain type of codes over many finite rings are determined such as cyclic, quasi-cyclic. Recently, it is introduced the class of skew codes which are generalized the notion cyclic, quasi-cyclic in [5,6,12,14].

In [1], T. Abualrub, P. Seneviratre studied skew cyclic codes over $F_2 + vF_2$, $v^2 = v$. In [2], T. Abualrub, A. Ghrayeb, N. Aydın, I. Siap introduced skew quasi-cyclic codes. They obtained several new codes with Hamming distance exceeding the distance of the previously best known linear codes with comparable parameters.

In [4], they investigated the structures of skew cyclic and skew quasi-cyclic of arbitrary length over Galois rings. They shown that the skew cyclic codes are equivalent to either cyclic and quasi-cyclic codes over Galois rings. Moreover, they gave a necessary and sufficient condition for skew cyclic codes over Galois rings to be free.

Jian Gao, L.Shen, F. W. Fu studied a class of generalized quasi–cyclic codes called skew generalized quasi-cyclic codes. They gave the Chinese Remainder Theorem over the skew polynomial ring which lead to a canonical decomposition of skew generalized quasi-cyclic codes. Moreover, they focused on 1-generator skew generalized quasi-cyclic code in [7]. J.Gao also presented skew cyclic codes over $F_p + vF_p$ in [8].

The MacWilliams identity supplies the relationship between the weight enumerator of a linear code and that of its dual code [11]. The distribution of weights for a linear code is important for its performance analysis such as linear programming bound, error correcting

[1]Received August 20, 2015, Accepted May 22, 2016.

capabilities, etc. There are a lot of work about the MacWilliams identities in [3,9,15].

This paper is organized as follows. In section 2, we give some basic knowledges about the finite ring S. In section 3, we define a new Gray map from S to F_2^3, Lee weights of elements of S and Lee distance in the linear codes over S. In section 4, we define a new non trivial automorphism and we introduce skew codes over S. In section 5, we obtain the MacWilliams identities and give an example.

§2. **Preliminaries**

Let S be the ring $F_2 + uF_2 + vF_2$ where $u^2 = u$, $v^2 = v$, $uv = vu = 0$ and $F_2 = \{0, 1\}$, a finite commutative ring with 8 elements. S is semi local ring with three maximal ideals and a principal ideal ring. It is not finite chain ring.

The ideals are follows;

$$\begin{aligned} I_0 &= \{0\},\ I_1 = S \\ I_u &= \{0, u\},\ I_v = \{0, v\},\ I_{1+u+v} = \{0, 1 + u + v\} \\ I_{u+v} &= \{0, u, v, u + v\},\ I_{1+u} = \{0, v, 1 + u, 1 + u + v\} \\ I_{1+v} &= \{0, u, 1 + v, 1 + u + v\} \end{aligned}$$

A linear code C over S length n is a $S-$submodule of S^n. An element of C is called a codeword.

For any $x = (x_0, x_1, \cdots, x_{n-1})$, $y = (y_0, y_1, \cdots, y_{n-1})$ the inner product is defined as

$$x.y = \sum_{i=0}^{n-1} x_i y_i$$

If $x.y = 0$ then x and y are said to be orthogonal. Let C be linear code of length n over S, the dual code of C

$$C^{\perp} = \{x : \forall y \in C, x.y = 0\},$$

which is also a linear code over S of length n. A code C is self orthogonal if $C \subseteq C^{\perp}$ and self dual if $C = C^{\perp}$.

A cyclic code C over S is a linear code with the property that if $c = (c_0, c_1, \cdots, c_{n-1}) \in C$ then $\sigma(C) = (c_{n-1}, c_0, \cdots, c_{n-2}) \in C$. A subset C of S^n is a linear cyclic code of length n iff it is polynomial representation is an ideal of $S[x] / \langle x^n - 1\rangle$.

Let C be code over F_2 of length n and $\acute{c} = (\acute{c}_0, \acute{c}_1, \cdots, \acute{c}_{n-1})$ be a codeword of C. The Hamming weight of $\acute{c}$ is defined as $w_H(\acute{c}) = \sum_{i=0}^{n-1} w_H(\acute{c}_i)$ where $w_H(\acute{c}_i) = 1$ if $\acute{c}_i = 1$ and $w_H(\acute{c}_i) = 0$ if $\acute{c}_i = 0$. Hamming distance of C is defined as $d_H(C) = \min d_H(c, \acute{c})$, where for any $\acute{c} \in C$, $c \neq \acute{c}$ and $d_H(c, \acute{c})$ is Hamming distance between two codewords with $d_H(c, \acute{c}) = w_H(c - \acute{c})$.

Let $a \in F_2^{3n}$ with $a = (a_0, a_1, \cdots, a_{3n-1}) = \left(a^{(0)} \left| a^{(1)} \right| a^{(2)}\right)$, $a^{(i)} \in F_2^n$ for $i = 0, 1, 2$. Let φ be a map from F_2^{3n} to F_2^{3n} given by $\varphi(a) = \left(\sigma\left(a^{(0)}\right) \left| \sigma\left(a^{(1)}\right) \right| \sigma\left(a^{(2)}\right)\right)$ where σ is a cyclic shift from F_2^n to F_2^n given by $\sigma\left(a^{(i)}\right) = ((a^{(i,n-1)}), (a^{(i,0)}), (a^{(i,1)}) \ , \cdots, (a^{(i,n-2)}))$ for every $a^{(i)} = (a^{(i,0)}, \cdots, a^{(i,n-1)})$ where $a^{(i,j)} \in F_2$, $0 \leq j \leq n-1$. A code of length $3n$ over F_2 is said to be quasi cyclic code of index 3 if $\varphi(C) = C$.

§3. Gray Map

Let $x = a + ub + uc$ be an element of S where $a, b, c \in F_2$. We define Gray map Ψ from S to F_2^3 by

$$\begin{aligned} \Psi \quad &: \quad S \rightarrow F_2^3 \\ \Psi(a + ub + vc) \quad &= \quad (a, a+b, a+c) \end{aligned}$$

The Lee weight of elements of S are defined $w_L(a + ub + vc) = w_H(a, a+b, a+c)$ where w_H denotes the ordinary Hamming weight for binary codes. Hence, there is one element whose weight is 0, there are $u, v, 1+u+v$ elements whose weights are 1, there are $1+u, 1+v, u+v$ elements whose weights are 2, there is one element whose weight are 3.

Let C be a linear code over S of length n. For any codeword $c = (c_0, \cdots, c_{n-1})$ the Lee weight of c is defined as $w_L(c) = \sum_{i=0}^{n-1} w_L(c_i)$ and the Lee distance of C is defined as $d_L(C) = \min d_L(c, \acute{c})$, where for any $\acute{c} \in C$, $c \neq \acute{c}$ and $d_L(c, \acute{c})$ is Lee distance between two codewords with $d_L(c, \acute{c}) = w_L(c - \acute{c})$. Gray map Ψ can be extended to map from S^n to F_2^{3n}.

Theorem 3.1 *The Gray map Ψ is a weight preserving map from $(S^n$, Lee weight$)$ to $(F_2^{3n}$, Ham$-$ ming weight$)$. Moreover it is an isometry from S^n to F_2^{3n}.*

Theorem 3.2 *If C is an $[n, k, d_L]$ linear codes over S then $\Psi(C)$ is a $[3n, k, d_H]$ linear codes over F_2, where $d_H = d_L$.*

§4. Skew Codes over S

We are interested in studying skew codes using the ring $S = F_2 + uF_2 + vF_2$ where $u^2 = u, v^2 = v, uv = vu = 0.$. We define non-trivial ring automorphism θ on the ring S by $\theta(a + ub + vc) = a + vb + uc$ for all $a + ub + vc \in S$.

The ring $S[x, \theta] = \{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} : a_i \in S, n \in N\}$ is called a skew polynomial ring. This ring is a non-commutative ring. The addition in the ring $S[x, \theta]$ is the usual polynomial addition and multiplication is defined using the rule, $(ax^i)(bx^j) = a\theta^i(b)x^{i+j}$. Note that $\theta^2(a) = a$ for all $a \in S$. This implies that θ is a ring automorphism of order 2.

Definition 4.1 *A subset C of S^n is called a skew cyclic code of length n if C satisfies the following conditions,*

(i) C is a submodule of S^n;

(ii) If $c = (c_0, c_1, \cdots, c_{n-1}) \in C$, then $\sigma_\theta(c) = \big(\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2})\big) \in C$.

Let $(f(x)+(x^n-1))$ be an element in the set $S_n = S[x,\theta]/(x^n-1)$ and let $r(x) \in S[x,\theta]$. Define multiplication from left as follows:

$$r(x)(f(x)+(x^n-1)) = r(x)f(x)+(x^n-1)$$

for any $r(x) \in S[x,\theta]$.

Theorem 4.2 *S_n is a left $S[x,\theta]$-module where multiplication defined as in above.*

Theorem 4.3 *A code C in S_n is a skew cyclic code if and only if C is a left $S[x,\theta]$-submodule of the left $S[x,\theta]$-module S_n.*

Theorem 4.4 *Let C be a skew cyclic code in S_n and let $f(x)$ be a polynomial in C of minimal degree. If $f(x)$ is monic polynomial, then $C = (f(x))$ where $f(x)$ is a right divisor of (x^n-1).*

Theorem 4.5 *Let n be odd and C be a skew cyclic code of length n. Then C is equivalent to cyclic code of length n over S.*

Proof Since n is odd, $gcd(2,n)=1$. Hence there exist integers b,c such that $2b+nc=1$. So $2b = 1-nc = 1+zn$ where $z>0$. Let $a(x) = a_0+a_1x+\cdots+a_{n-1}x^{n-1}$ be a codeword in C. Note that $x^{2b}a(x) = \theta^{2b}(a_0)x^{1+zn}+\theta^{2b}(a_1)x^{2+zn}+\cdots+\theta^{2b}(a_{n-1})x^{n+zn} = a_{n-1}+a_0x+\cdots+a_{n-2}x^{n-2} \in C$. Thus C is a cyclic code of length n. □

Corollary 4.6 *Let n be odd. Then the number of distinct skew cyclic codes of length n over S is equal to the number of ideals in $S[x]/(x^n-1)$ because of Theorem 7. If $x^n-1 = \prod_{i=0}^{r} p_i^{s_i}(x)$ where $p_i(x)$ are irreducible polynomials over F_2. Then the number of distinct skew cyclic codes of length n over S is $\prod_{i=0}^{r}(s_i+1)^2$.*

Example 4.7 Let $n=15$ and $g(x) = x^4+x^3+x^2+x+1$. Then $g(x)$ generates a skew cyclic codes of length 15. This code is equivalent to a cyclic code of length 15. Since $x^{15}-1 = (x+1)(x^2+x+1)(x^4+x+1)(x^4+x^3+1)(x^4+x^3+x^2+x+1)$, it follows that there are 2^8 skew cyclic code of length 15.

Definition 4.8 *A subset C of S^n is called a skew quasi-cyclic code of length n if C satisfies the following conditions:*

(i) C is a submodule of S^n;

(ii) If $e = (e_{0,0},\cdots,e_{0,l-1},e_{1,0},\cdots,e_{1,l-1},\cdots,e_{s-1,0},\cdots,e_{s-1,l-1}) \in C$, then $\tau_{\theta,s,l}(e) = (\theta(e_{s-1,0}),\cdots,\theta(e_{s-1,l-1}),\theta(e_{0,0}),\cdots,\theta(e_{0,l-1}),\theta(e_{s-2,0}),\cdots,\theta(e_{s-2,l-1})) \in C$.

We note that x^s-1 is a two sided ideal in $S[x,\theta]$ if $m|s$ where $m=2$ is the order of θ. So $S[x,\theta]/(x^s-1)$ is well defined.

The ring $M_s^l = (S[x,\theta]/(x^s-1))^l$ is a left $M_s = S[x,\theta]/(x^s-1)$ module by the following multiplication on the left $f(x)(g_1(x),\cdots,g_l(x)) = (f(x)g_1(x),\cdots,f(x)g_l(x))$. If the map γ is defined by

$$\gamma : S^n \longrightarrow M_s^l$$

$(e_{0,0}, \cdots, e_{0,l-1}, e_{1,0}, \cdots, e_{1,l-1}, \cdots, e_{s-1,0}, \cdots, e_{s-1,l-1}) \mapsto (c_0(x), \cdots, c_{l-1}(x))$ such that $e_j(x) = \sum_{i=0}^{s-1} e_{i,j}x^i \in M_s^l$ where $j = 0, 1, \cdots, l-1$ then the map γ gives a one to one correspondence S^n and the ring M_s^l.

Theorem 4.9 *A subset C of S^n is a skew quasi-cyclic code of length $n = sl$ and index l if and only if $\gamma(C)$ is a left S_s-submodule of M_s^l.*

§5. MacWilliams Identities

Let the elements of S be represented as $S = \{f_1, f_2, \cdots, f_8\} = \{0, 1, u, v, 1+u, 1+v, u+v, 1+u+v\}$ where the order of elements is fixed.

Definition 5.1 *Define $\chi : S \longrightarrow \mathbb{C}^*$ by $\chi(a+ub+vc) = (-1)^{a+b+c}$. χ is a non-trivial character of each non-zero ideal I of S. Hence we have $\sum_{a\in I} \chi(a) = 0$.*

Lemma 5.2 *Let C be a linear code over S of length n. Then for any $m \in S^n$,*

$$\sum_{c\in C} \chi(c.m) = \begin{cases} 0, \text{ if } m \notin C^\perp \\ |C|, \text{ if } m \in C^\perp \end{cases}$$

Theorem 5.3([11]) *Let C be a linear code over S of length n and $\widehat{f}(c) = \sum_{m\in S^n} \chi(c.m)f(m)$. Then $\sum_{m\in C^\perp} f(m) = \frac{1}{|C|}\sum_{c\in C} \widehat{f}(c)$.*

Let A is a 8×8 matrix.A matrix defined by $A(i,j) = \chi(f_i f_j)$. The matrix A is given as follows

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix}$$

Definition 5.4 *Let C be a linear code of length n over S, then $Lee_C(x,y) = \sum_{c\in C} x^{3n-w_L(c)}y^{w_L(c)}$ can be called as the Lee weight enumerator of C and $Ham_C(x,y) = \sum_{c\in C} x^{n-w_H(c)}y^{w_H(c)}$ can be called as the Hamming weight enumerator of C. Besides,*

$$Swe_C(x,y,z,w) = \sum_{c\in C} x^{n_0(c)}y^{n_1(c)}z^{n_2(c)}w^{n_3(c)}$$

is the symmetric weight enumerator where $n_i(c)$ denote the number of elements of c with Lee *weight* $0, 1, 2, 3$*, respectively.*

Definition 5.5 *The complete weight enumerator of a linear code* C *over* R *is defined as* $cwe_C(x_1, x_2, \cdots, x_8) = \sum_{c \in C} x_1^{n_{f_1}(c)} \cdots x_8^{n_{f_8}(c)}$ *where* $n_{f_i}(c)$ *is the number of appearances of* f_i *in the vector* c.

The complete weight enumerator gives us a lot of information about the code, such as the size of the code, the minimum weight of the code and the weight enumerator of the code for any weight function.

We can define the symmetrized weight enumerator as follows.

Definition 5.6 *Let* C *be a linear code of length* n *over* S*. Then define the symmetrized weight enumerator of* C *as*

$$Swe_C(x, y, z, w) = cwe_C(x, w, y, y, z, z, z, y)$$

Here x represents the elements that have weight 0 (the 0 element), y represents the elements with weight 1 (the elements $u, v, 1+u+v$), z represents the elements with weight 2 (the elements $1+u, 1+v, u+v$), w represents the elements with weight 3 (the element 1).

Theorem 5.7 *Let* C *be a linear code of length* n *over* S *and let* $C^{\perp}$ *be its dual. Then* $cwe_{C^{\perp}}(x_1, x_2, \cdots, x_8) = \frac{1}{|C|} cwe_C(A.(x_1\ x_2\ \cdots\ x_8)^{\top})$ *where* $()^{\top}$ *denotes the transpose.*

Theorem 5.8 *Let* C *be a linear code of length* n *over* S *and let* $C^{\perp}$ *be its dual. Then* $Swe_{C^{\perp}}(x, y, z, w) = \frac{1}{|C|} Swe_C(x+w+3y+3z, x-w-3y+3z, x-w+y-z, x+w-y-z)$.

Proof The proof follows simply from calculating the matrix product

$$A.(x\ w\ y\ y\ z\ z\ z\ y)^{\top}$$

where $()^{\top}$ denotes the transpose. □

Theorem 5.9 *Let* C *be a linear code of length* n *over* S*. Then,*

(*i*) $Lee_C(x, y) = Swe_C(x^3, x^2y, y^2x, y^3)$;

(*ii*) $Lee_{C^{\perp}}(x, y) = \frac{1}{|C|} Lee_C(x+y, x-y)$.

Proof (*i*) Let $w_L(c) = n_1(c) + 2n_2(c) + 3n_3(c)$ where $n_i(c)$ denote the number of elements of c with Lee weight $0, 1, 2, 3$, respectively. Since $n = n_0(c) + n_1(c) + n_2(c) + n_3(c)$, $3n - w_L(c) = 3n_0(c) + 2n_1(c) + n_2(c)$. From the definition,

$$\begin{aligned} Lee_C(x, y) &= \sum_{c \in C} x^{3n - w_L(c)} y^{w_L(c)} = \sum_{c \in C} x^{3n_0(c) + 2n_1(c) + n_2(c)} y^{n_1(c) + 2n_2(c) + 3n_3(c)} \\ &= \sum_{c \in C} x^{3n_0(c)} (x^2y)^{n_1(c)} (y^2x)^{n_2(c)} y^{3n_3(c)} = Swe_C(x^3, x^2y, y^2x, y^3) \end{aligned}$$

(ii) From Theorems 5.7 and 5.8,

$$\begin{aligned} Lee_{C^\perp}(x,y) &= \frac{1}{|C|}Swe_C(x^3+3x^2y+3y^2x+y^3, x^3-y^3-3x^2y+3xy^2,\\ &\quad x^3-y^3+x^2y-xy^2, x^3+y^3-x^2y-xy^2)\\ &= \frac{1}{|C|}Swe_C((x+y)^3,(x+y)^2(x-y),(x-y)^2(x+y),(x-y)^3)\\ &= \frac{1}{|C|}Lee_C(x+y,x-y). \qquad \square \end{aligned}$$

Theorem 5.10 *Let C be a linear code of length n over S. Then we have*

(i) $Ham_{C^\perp}(x,y) = \frac{1}{|C|}Ham_C(x+7y, x-y)$;
(ii) $Ham_C(x,y) = Swe_C(x,y,y,y)$.

Proof (i) It is straightforward from [13].
(ii) The Hamming weight $w_H(c)$ is defined as $w_H(c) = n_0(c)+n_1(c)+n_2(c)+n_3(c)$.

$$\begin{aligned} Ham_C(x,y) &= \sum_{c\in C} x^{n-w_H(c)}y^{w_H(c)} = \sum_{c\in C} x^{n_0(c)}y^{n_1(c)+n_2(c)+n_3(c)}\\ &= Swe_C(x,y,y,y). \qquad \square \end{aligned}$$

Example 5.11 Let $C = \{(0,0),(v,v)\}$ be a linear code of length 2 over S. The Lee weight enumerator is $Lee_C(x,y) = x^6+x^4y^2$; the Hamming enumerator is $Ham_C(x,y) = x^2+y^2$. Lee weight enumerator of $C^\perp$ is $Lee_{C^\perp}(x,y) = x^6+4x^5y+7x^4y^2+8x^3y^3+7x^2y^4+5xy^5+y^6$; Hamming weight enumerator of $C^\perp$ is

$$Ham_{C^\perp}(x,y) = x^2+6xy+25y^2.$$

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International J.Math. Combin. Vol.2(2016), 109-120

Nonsplit Geodetic Number of a Graph

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Abstract: Let G be a graph. If $u, v \in V(G)$, a $u - v$ geodesic of G is the shortest path between u and v. The closed interval I[u, v] consists of all vertices lying in some u - v geodesic of G. For $S \subseteq V(G)$ the set I[S] is the union of all sets I[u, v] for $u, v \in S$. A set S is a geodetic set of G if $I(S) = V(G)$. The cardinality of a minimum geodetic set of G is the geodetic number of G, denoted by g(G). In this paper, we study the nonsplit geodetic number of a graph $g_{ns}(G)$. The set $S \subseteq V(G)$ is a nonsplit geodetic set in G if S is a geodetic set and $\langle V(G) - S \rangle$ is connected, nonsplit geodetic number $g_{ns}(G)$ of G is the minimum cardinality of a nonsplit geodetic set of G. We investigate the relationship between nonsplit geodetic number and geodetic number. We also obtain the nonsplit geodetic number in the cartesian product of graphs.

Key Words: Cartesian products, distance, edge covering number, Smarandachely k-geodetic set, geodetic number, vertex covering number.

AMS(2010): 05C05, 05C12.

§1. Introduction

As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively. The graphs considered here are finite, undirected,simple and connected. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G and the maximum eccentricity is the diameter, diam G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. We define I[u, v] to the set (interval) of all vertices lying on some $u - v$ geodesic of G and for a nonempty subset S of V (G), $I[S] = \cup_{u,v \in S} I[u, v]$. A set S of vertices of G is called a geodetic set in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set, and generally, if there is a k-subset T of $V(G)$ such that $I(S) \bigcup T = V(G)$, where $0 \le k < |G| - |S|$, then S is called a *Smarandachely k-geodetic set* of G. The cardinality of a minimum geodetic set in G is called

[1]Received December 3, 2014, Accepted December 6, 2015.

the geodetic number and is denoted by $g(G)$. The concept of geodetic number of a graph was introduced in [1, 4, 7], further studied in [2, 3], and the split geodetic number of a graph was introduced in [10]. It was shown in [7] that determining the geodetic number of a graph is an NP-hard problem.

A set of vertices S in a graph G is a nonsplit geodetic set if S is a geodetic set and the subgraph $G[V-S]$ induced by $\langle V(G)-S\rangle$ is connected. The minimum cardinality of a nonsplit geodetic set, denoted $g_{ns}(G)$, is called the nonsplit geodetic number of G.

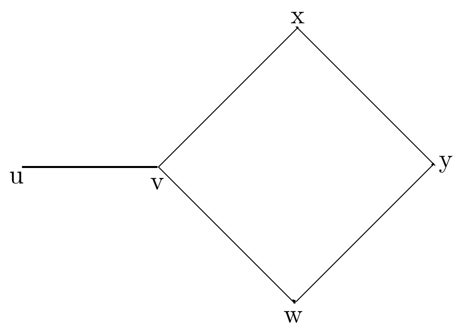

Figure 1.1

Consider the graph G of Figure 1.1. For the vertices u and y in G $d(u,y)=3$ and every vertex of G lies on an $u-y$ geodesic in G. Thus $S=\{u,y\}$ is the geodetic set of G and so $g(G)$. Here the induced subgraph $\langle V(G)-S\rangle$ is connected. So that S is a minimum nonsplit geodetic set of G. Therefore nonsplit geodetic number $g_{ns}(G)=2$.

A vertex v is an extreme vertex in a graph G, if the subgraph induced by its neighbours is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G. For any undefined term in this paper, see [1, 6]

§2. Preliminary Notes

We need the following results to prove our results.

Theorem 2.1 *Every geodetic set of a graph contains its extreme vertices.*

Theorem 2.2 *For any tree T with k pendant vertices, $g(T)=k$.*

Theorem 2.3 *For any graph G of order n, $\alpha_1(G)+\beta_1(G)=n$.*

Theorem 2.4 *For cycle C_n of order $n \geq 3$,*

$$g(C_n) = \begin{cases} 2 & \text{if } n \text{ even}, \\ 3 & \text{if } n \text{ odd}. \end{cases}$$

Theorem 2.5 *If G is a nontrivial connected graph, then $g(G) \leq g(G \times K_2)$.*

§3. Nonsplit Geodetic Number

Theorem 3.1 *For cycle C_n of order $n \geq 3$,*

$$g_{ns}(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even}, \\ \lfloor \frac{n}{2} \rfloor + 2 & \text{if } n \text{ is odd}. \end{cases}$$

Proof Suppose C_n be cycle with $n \geq 3$, we have the following

Case 1. Let n be even. Consider $C_{2p} = \{v_1, v_2, \cdots, v_{2p}, v_1\}$ be a cycle with 2p vertices. Then v_{p+1} is the antipodal vertex of v_1. Suppose $S = \{v_1, v_{p+1}\}$ be the geodetic set of G. It is clear that $\langle V(G) - S \rangle$ is not connected. Thus S is not a nonsplit geodetic set. But $S' = \{v_1, v_2, \cdots, v_{p+1}\}$ is a nonsplit geodetic set of G. So that $g_{ns}(G) \leq (p + 1)$. If S_1 is any set of vertices of G with $|S_1| < |S'|$ then S_1 contains at most p-elements. Hence $V(G) - S_1$ is not connected. This follows that $g_{ns}(G) = p + 1 = \frac{n}{2} + 1$.

Case 2. Let n be odd. Consider $C_{2p+1} = \{v_1, v_2, \cdots, v_{2p+1}, v_1\}$ be a cycle with 2p+1 vertices. Then v_{p+1} and v_{p+2} are the antipodal vertices of v_1. Now consider $S = \{v_1, v_{p+1}, v_{p+2}\}$ be the geodetic set of G and it is clear that $\langle V(G) - S \rangle$ is not connected. Thus S is not a nonsplit geodetic set. But $S' = \{v_1, v_2, \cdots, v_{p+1}, v_{p+2}\}$ is a nonsplit geodetic set of G so that $g_{ns}(G) \leq p + 2$. If S_1 is any set of vertices of G with $|S_1| < |S'|$ then S_1 contains at most p-elements. Hence $\langle V(G) - S_1 \rangle$ is not connected. This follows that

$$g_{ns}(G) = p + 2 = \left\lfloor \frac{n}{2} \right\rfloor + 2. \qquad \square$$

Theorem 3.2 *For any nontrivial tree T with k-pendant-vertices, then $g_{ns}(T) = k$.*

Proof Let $S = \{v_1, v_2, \cdots, v_k\}$ be the set containing pendant vertices of a tree T. By Theorem 2.2, $g(T) \geq |S|$. On the other hand, for an internal vertex v of T there exist pendant vertices x,y of T such that v lies on the unique x-y geodesic in T. Thus, $v \in I[S]$ and $I[S] = V(T)$.Then $g(T) \leq |S|$. Thus S itself a minimum geodetic set of T.Therefore $g(T) = |S| = k$ and $\langle V - S \rangle$ is connected. Hence $g_{ns}(T) = k$. □

Theorem 3.3 *For any integers $r, s \geq 2$,$g_{ns}(K_{r,s}) = r + s - 1$.*

Proof Let G =$K_{r,s}$, such that U = $\{u_1, u_2, \cdots, u_r\}$, W = $\{w_1, w_2, \cdots, w_s\}$ are the partite sets of G, where $r \leq s$ and also $V = U \cup W$.

Consider $S = U \cup W - x$ for any $x \in W$. Every $w_k \in W$, $1 \leq k \leq s-1$ lies on $u_i - u_j$ geodesic for $1 \leq i \neq j \leq r$, so that S is a geodetic set of G. Since $\langle V(G) - S \rangle$ is connected and hence S itself a nonsplit geodetic set of G . Let S' be any set of vertices such that $|S'| < |S|$. If S' is not a subset of U then $\langle V(G) - S' \rangle$ is not connected and so S' is not a nonsplit geodetic set of G. If S' is not a subset of $W - x$, again S' is not a nonsplit geodetic set of G, by a similar argument.If $S' =$ U then S' is a geodetic set but $\langle V(G) - S' \rangle$ is not connected, so S' is not nonsplit geodetic set. If $S' = W - x$ then S' is not a nonsplit geodetic set of G. From the above argument, it is clear that S is a minimum nonsplit geodetic set of G. Hence $g_{ns}(Kr, s) = |S| = r + s - 1$. □

Theorem 3.4 *If G is a star then* $g_{ns}(G) = n - 1$.

Proof Let $V(G) = \{v_1, v_2, \cdots, v_{n-1}, v_n\}$ and let $S = \{v_1, v_2, \cdots, v_{n-1}\}$ be the set of pendant vertices of G and is the geodetic set of G. Clearly, the subgraph induced by $\langle V(G) - S = v_n \rangle$ is connected. Hence $S = \{v_1, v_2, \cdots, v_{n-1}\}$ is a minimum nonsplit geodetic set of G. Therefore $g_{ns}(G) = n - 1$. □

Theorem 3.5 *For any nontrivial connected graph G different from star of order n and diameter d,* $g_{ns}(G) \leq n - d + 1$.

Proof Let u and v be the vertices of G for which $d(u, v) = d$ and let $u = v_0, v_1, \cdots, v_d = v$ be a $u - v$ path of length d. Now $S = V(G) - \{v_1, v_2, \cdots, v_{d-1}\}$ then $I[S] = V[G]$ and consequently $g_{ns}(G) \leq |S| \leq n - d + 1$. □

Theorem 3.6 *For any tree* T, $g_{ns}(T) + g(T) < 2m$.

Proof Suppose $S = \{v_1, v_2, v_3, \cdots, v_k\}$ be the set of all pendant vertices in T , forms a minimal geodetic set of $I[S] = V(T)$. Further $\{u_1, u_2, u_3, \cdots, u_l\} \subset V(G) - S$ is the set of internal vertices in T. Then $\langle V(G) - S \rangle$ forms a minimal non split geodetic set of T, it follows that $|S| + |S| < 2m$. Hence $g_{ns}(T) + g(T) < 2m$. □

Theorem 3.7 *For any graph G of order n,* $g_{ns}(G) \leq g_s(G)$*, where G is not a cycle..*

Proof Let G be any graph with n vertices. Consider a nonsplit geodetic set $S = \{v_1, v_2, \cdots, v_k\}$ of a graph G. Since $\langle V(G) - S \rangle$ is connected, the set S is not a split geodetic set of G. Now, we consider a set $S' = S \cup \{a, b\}$ for any $a, b \in V(G)$ such that $\langle V(G) - S' \rangle$ is disconnected. Therefore S' is the split geodetic set of G with minimum cardinality. Thus $|S| < |S'|$. Clearly $g_{ns}(G) \leq g_s(G)$. □

Theorem 3.8 *Let G be a cycle of order n then* $g_s(G) \leq g_{ns}(G)$.

Proof Let G be a cycle of order n, we discuss the following cases.

Case 1. Suppose n is even. Let $S = \{v_i, v_j\}$ be the split geodetic set of G where v_i, v_j are the two antipodal vertices of G. The $v_i - v_j$ geodesic includes all the vertices of G and $\langle V(G) - S \rangle$

is disconnected. But $S' = \{v_i, v_{i+1}, \cdots, v_j\}$ is a nonsplit geodetic set of G and the induced subgraph $\langle V(G) - S' \rangle$ is connected. Thus $|S| \leq |S'|$. Clearly $g_s(G) \leq g_{ns}(G)$.

Case 2. Suppose n is odd. Let $S = \{v_i, v_j, v_k\}$ be the split geodetic set of G. By the Theorem 2.4, no two vertices of S form a non split geodetic set and $\langle V(G) - S \rangle$ is disconnected. But $S' = \{v_i, v_{i+1}, \cdots, v_j, v_k\}$ is a nonsplit geodetic set of G and the induced subgraph $\langle V(G) - S' \rangle$ is connected. Thus $|S| \leq |S'|$. Clearly $g_s(G) \leq g_{ns}(G)$. □

Theorem 3.9 *For the wheel $W_n = k_1 + C_{n-1}$ $(n \geq 5)$,*

$$g_{ns}(W_n) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{n-1}{2} & \text{if n is odd} \end{cases}$$

Proof Let $W_n = K_1 + C_{n-1}$ and let $V(W_n)$=$\{x, u_1, u_2, \cdots, u_{n-1}\}$, where $deg(x) = n-1 > 3$ and $deg(u_i) = 3$ for each $i \in \{1, 2, \cdots, n-1\}$. We discuss the following cases.

Case 1. Let n be even. Consider geodesic

$P: \{u_1, u_2, u_3\}, Q: \{u_3, u_4, u_5\}, \cdots, R: \{u_{2n-1}, u_{2n}, u_{2n+1},\}$.

It is clear that the vertices $u_2, u_4 \cdots, u_{2n}$ lies on the geodesic P, Q and R. Also $u_1, u_3, u_5, \cdots, u_{2n-1}, u_{2n+1}$ is a minimum nonsplit geodetic set such that $\langle V(G) - S \rangle$ is connected and it has $\frac{n}{2}$ vertices. Hence $g_{ns}(W_n) = \frac{n}{2}$.

Case 2. Let n be odd. Consider geodesic

$P: \{u_1, u_2, u_3\}, Q: \{u_3, u_4, u_5\}, \cdots, R: \{u_{2n-1}, u_{2n}, u_{2n+1},\}$.

It is clear that the vertices $u_2, u_4, \cdots, u_{2n}$ lies on the geodesic P, Q and R. Also $u_1, u_3, u_5, \cdots, u_{2n-1}, u_{2n+1}$ is a minimum nonsplit geodetic set such that $\langle V(G) - S \rangle$ is connected and it has $\frac{n-1}{2}$ vertices. Hence $g_{ns}(W_n) = \frac{n-1}{2}$. □

Theorem 3.10 *Let G be a graph such that both G and $\overline{G}$ are connected then $g_{ns}(G) + g_{ns}(\overline{G}) \leq n(n-3) + 2$.*

Proof Since both G and $\overline{G}$ are connected , we have $\Delta(G) \cdot \Delta\overline{(G)} < n-1$.Thus $\beta_0(G), \beta_0\overline{(G)} \geq 2$. Hence,

$$g_{ns} \leq n - 1 \Rightarrow g_{ns}(G) \leq 2(n-1) - n + 1 \Rightarrow g_{ns}(G) \leq 2m - n + 1.$$

Similarly, $g_{ns}(\overline{G}) \leq 2\overline{m} - n + 1$. Thus,

$$\begin{aligned} g_{ns}(G) + g_{ns}(\overline{G}) \leq 2(m + (\overline{m})) - 2n + 2 \quad &\Rightarrow \quad g_{ns}(G) + g_{ns}(\overline{G}) \leq n(n-1) - 2n + 2 \\ &\Rightarrow \quad g_{ns}(G) + g_{ns}(\overline{G}) \leq n^2 - n - 2n + 2 \\ &\Rightarrow \quad g_{ns}(G) + g_{ns}(\overline{G}) \leq n^2 - 3n + 2 \\ &\Rightarrow \quad g_{ns}(G) + g_{ns}(\overline{G}) \leq n(n-3) + 2. \end{aligned}$$ □

Theorem 3.11 *For any nontrivial tree T, $g_{ns}(T) \geq \alpha_0(T)$.*

Proof Let S be a minimum cover set of vertices in T. Then S has at least one vertex and every vertex in S is adjacent to some vertices in $\langle V(G)-S\rangle$. This implies that S is a nonsplit geodetic set of G. Thus $g_{ns}(T) \geq \alpha_0(T)$. □

Theorem 3.12 *For any nontrivial tree T with m edges, $g_{ns}(T) \leq m - \lceil \frac{\alpha_1(T)}{2} \rceil + 2$, where $\alpha_1(T)$ is an edge covering number.*

Proof Suppose $S' = \{e_1, e_2, \cdots, e_i\}$ be the set of all end edges in T and $J \subseteq E(T) - S'$ be the minimal set of edges such that $|S' \cup J| = \alpha_1(T)$. By the Theorem 2.2 S' is the minimal geodetic set of G. Also it follows that $\langle V(G) - S'\rangle$ is connected. Clearly,

$$g_{ns}(T) \leq |E(T)| - \left|\left\lceil \frac{S' \cup J}{2} \right\rceil\right| + 2 \Rightarrow g_{ns}(T) \leq m - \left\lceil \frac{\alpha_1(T)}{2} \right\rceil + 2.$$ □

Theorem 3.13 *For a cycle C_n of order n, $g_{ns}(G) = \alpha_0(C_n) + 1$.*

Proof Consider a cycle C_n of order n. We discuss the following cases.

Case 1. Suppose that n is even and $\alpha_0(C_n)$ is the vertex covering number of C_n. We have by Theorem 3.1, $g_{ns}(G) = \frac{n}{2} + 1$ and also for an even cycle, vertex covering number $\alpha_0(C_n) = \frac{n}{2}$. Hence,

$$g_{ns}(G) = \frac{n}{2} + 1 = \alpha_0(C_n) + 1.$$

Case 2. Suppose that n is odd and $\alpha_0(C_n)$ is the vertex covering number of C_n. We have by Theorem 3.1, $g_{ns}(G) = \lfloor \frac{n}{2} \rfloor + 2$ and also for an odd cycle, vertex covering number $\alpha_0(C_n) = \lfloor \frac{n}{2} \rfloor + 1$. Hence,

$$g_{ns}(G) = \lfloor \frac{n}{2} \rfloor + 2 \Rightarrow g_{ns}(G) = \alpha_0(C_n) + 1.$$ □

Theorem 3.14 *If is a connected noncomplete graph G of order n,$g_{ns} \leq (n - \kappa(G)) + 1$, where $\kappa(G)$ is vertex connectivity.*

Proof Let $\kappa(G) = k$. Since G is connected but not complete $1 \leq \kappa(G) \leq n-2$. Let $U = \{u_1, u_2, \cdots, u_k\}$ be a minimum cut set of G, let $G_1, G_2, \cdots, G_r (r \geq 2)$ be the components of $G - U$ and let $W = V(G) - (U - 1)$ then every vertex $u_i (1 \leq i \leq k)$ is adjacent to at least one vertex of G_j for every $(i \leq j \leq r)$. Therefore, every vertex u_i belongs to a W geodesic path. Thus

$$g_{ns}(G) = |W| \leq (V(G) - U) + 1 \leq (n - \kappa(G)) + 1.$$ □

§4. Corona Graph

Let G and H be two graphs and let n be the order of G. The corona product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and then joining by an edge, all the vertices from the i^{th}-copy of H with the i^{th}-vertex of G.

Theorem 4.1 *Let G be a connected graph of order n and H be any graph of order m then $g_{ns}(G \circ H) = nm$.*

Proof Let S be a nonsplit geodetic set in $G \circ H$, $v_i \in V(G), 1 \leq i \leq n$ and $u_j \in V(H), 1 \leq j \leq m$. For each v_i there is a copy Hv_i which contains u_j vertices. Clearly $V(Hu_j) \cap S$ is a geodetic set of $G \circ H$ and $\langle V(G) - S \rangle$ is connected. Further every $w_k \in (G \circ H)$ lies on the geodesic path in S. Therefore S is the minimum nonsplit geodetic set. Thus, $|S| = g_{ns}(G \circ H) = nm$. □

§5. Adding a Pendant Vertex

An edge $e = (u, v)$ of a graph G with $deg(u) = 1$ and $deg(v) > 1$ is called an *pendant edge* and u an pendant vertex.

Theorem 5.1 *Let G' be the graph obtained by adding an pendant edge (u, v) to a cycle $G = C_n$ of order $n > 3$, with $u \in G$ and $v \notin G$, then*

$$g_{ns}(G') = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd} \end{cases}$$

Proof Let $\{u_1, u_2, u_3, \cdots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph obtained from $G = C_n$ by adding an pendant edge (u, v) such that $u \in G$ and $v \notin G$. We discuss the following cases.

Case 1. For $G = C_{2n}$, let $S = \{v, u_i\}$ be a non split geodetic set of G' , where v is the pendant vertex of G' and $diam(G') = v - u_i$ path , clearly $I[S] = V[G']$. Also for all $x, y \in V(G') - S$, $\langle V(G') - S \rangle$ is connected. Hence, $g_{ns}(G') = 2$.

Case 2. For $G = C_{2n+1}$, let $S = \{v, a, b\}$ be a non split geodetic set of G' , where v is the pendant vertex of G' and $a, b \in G$ such that $d(v, a) = d(v, b)$. Thus $I[S] = V[G']$ and $\langle V(G') - S \rangle$ is connected. Hence, $g_{ns}(G') = 3$. □

Theorem 5.2 *Let G' be the graph obtained by adding a pendant vertex $(u_i, v_i)$for $i = 1, 2, 3, \cdots, n$ to each vertex of $G = C_n$ such that $u \in G$,$v_i \notin G$, then $g_{ns}(G') = k$.*

Proof Let $G = C_n = \{u_1, u_2, u_3, \cdots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph obtained by adding an pendant vertex $\{u_i, v_i\}, i = 1, 2, 3, \cdots, n$ to each vertex of G such that $u_i \in G$ and $v_i \notin G$. Let $S = \{v_1, v_2, v_3, \cdots, v_k\}$ be a non split geodetic set of G'. Clearly $I[X] \neq V(G')$. Also, $x, y \in V(G') - S$ with $V(G') - S$ connected. Thus, $g_{ns}(G') = k$. □

Theorem 5.3 *Let G' be the graph obtained by adding k pendant vertices $\{(u, v_1), \cdots, (u, v_k)\}$ to a cycle $G = C_n$ of order $n > 3$, with $u \in G$ and $\{v_1, v_2, \cdots, v_k\} \notin G$. Then*

$$g_{ns}(G') = \begin{cases} k+1 & \textit{if n is even} \\ k+2 & \textit{if n is odd} \end{cases}$$

Proof Consider $\{u_1, u_2, u_3, \cdots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph obtained from $G = C_n$ by adding k pendant edges $\{u_iv_1, u_iv_2, \cdots, u_iv_k\}$ such that u_i a single vertex of G and $\{v_1, v_2, v_3, \cdots, v_k\}$ does not belongs to G. We discuss the following cases.

Case 1. Let $G = C_{2n}$. Consider $X = \{v_1, v_2, v_3, \cdots, v_k\} \cup u_i$, for any vertex u_i of G . Now $S = \{X\}$ be a non split geodetic set , such that $\{v_1, v_2, v_3, \cdots, v_k\}$ are the pendant vertices of G' and u_j is the antipodal vertex of u_i in G. Thus $I[X] = V[G']$. Consider $P = \{v_1, v_2, v_3, \cdots, v_k\}$ as a set of pendant vertices such that $|P| < |S|$ is not a non split geodetic set i.e for some vertex $u_j \in V_{G'}$,$u_j \notin I[P]$. If $P = X$, then P is not nonsplit geodetic set. Thus S is a minimum non split geodetic set of G' and $\langle V(G') - S\rangle$ is connected. Thus, $g_{ns}(G') = k + 1$.

Case 2. Let $G = C_{2n+1}$. Consider $S = \{v_1, v_2, v_3, \cdots, v_k, a, b\}$ be a non split geodetic set, where $\{v_1, v_2, \cdots, v_k\} \notin G$ are k pendant vertices of G' not in G and $a, b \in G$ such that $d(u, a) = d(u, b)$. Thus $I[S] = V[G']$. Also $x, y \in V(G') - S$ it follows that $\langle V(G') - S\rangle$ is connected. Therefore, $g_{ns}(G') = k + 2$. □

§6. Cartesian Products

The cartesian product of the graphs H_1 and H_2 written as $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices u_1, u_2 and v_1, v_2 being adjacent in $H_1 \times H_2$ iff either $u_1 = v_1$ and $(u_2, v_2) \in E(H_{2)}$, or $u_2 = v_2$ and $(u_1, v_1) \in E(H_{1)}$.

Theorem 6.1 *Let K_2 and $G = C_n$ be the graphs then*

$$g_{ns}(K_2 \times G) = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if } n > 5 \text{ is odd} \\ 4 & \text{if n=3} \end{cases}$$

Proof Consider $G = C_n$, let $K_2 \times G$ be graphs formed from two copies G_1 and G_2 of G. Let $V = \{v_1, v_2, \cdots, v_n\}$ be the vertices of G_1 and $W = \{w_1, w_2, \cdots, w_n\}$ be the vertices of G_2 and $U = V \cup W$. We consider the following cases.

Case 1. Let n be even. Consider $S = \{v_i, w_j\}$ be the non split geodetic of $K_2 \times G$ such that v_i to w_j path is equal to $diam(K_2 \times G)$ which includes all the vertices of $K_2 \times G$. Hence $\langle U - S\rangle$ is connected. Therefore, $g_{ns}[K_2 \times G] = 2$.

Case 2. Let n be odd. Consider $S = \{v_i, w_j, v_k\}$ be the non split geodetic set of $K_2 \times G$ such that v_i to w_j path is equal to $diam(K_2 \times G)$ and is equal to $w_j - v_k path$ and also $v_i - w_j \cup w_j - v_k$ path includes all the vertices of $K_2 \times G$. Hence $\langle U - S\rangle$ is connected. Therefore, $g_{ns}[K_2 \times G] = 3$.

Case 3. For $n = 3$, let $S = \{v_i, w_j, v_k\}$ be the geodetic set of $K_2 \times G$, that is $v_i - w_j$ is equal to $diam(K_2 \times G)$ and is equal to $w_j - v_k$ and also $I[S] = U(K_2 \times G)$. But $\langle U - S\rangle$ is not connected. Let $S' = S \cup \{v_n\} = \{v_i, w_j, v_k, v_n\}$ be the non split geodetic set of $K_2 \times G$. Hence, $\langle U - S'\rangle$ is connected. Therefore, $g_{ns}[K_2 \times G] = 4$. □

Theorem 6.2 *For any complete graph K_n of order n, $g_{ns}[K_2 \times K_n] = n + 1$.*

Proof Consider $K_2 \times K_n$ be graph formed from two copies of G_1 and G_2 of G. Now, let us prove the result by mathematical induction,

For $n = 2$, $g_{ns}[k_2 \times K_2] = 3$, since $K_2 \times K_2 = C_4$ by Theorem 3.1 we have $g_{ns}[C_4] = 3$ the result is true.

Let us assume that the result is true for n=m,that is $g_{ns}[K_2 \times K_m] = m + 1$.

Now, we shall prove the result for $n = m+1$. Let $S = \{v_1, v_2, v_3, \cdots, v_{m+2}\}$ be the nonsplit geodetic set formed from some elements in G_1 and the elements which are not corresponds to elements in G_1 of $K_2 \times K_{m+1}$. Clearly $I[S] = V(K_2 \times K_n)$. Let P be any set of vertices such that $|P| < |S|$. Suppose $P = \{v_1, v_2, v_3, \cdots, v_m\}$ which is not non split geodetic set, because $I[P] \neq V[K_2 \times K_{m+1}]$. So S itself a minimum geodetic set of $K_2 \times K_{m+1}$. Hence, $g_{ns}[K_2 \times K_{m+1}] = m + 1 + 1$. Thus, $g_{ns}(K_2 \times K_n) = n + 1$. □

Theorem 6.3 *For any complete graph of order* $n \geq 3$, $g_{ns}(K_n \times K_n) = n$.

Proof We shall prove the result by mathematical induction, For $n \geq 3$, let us assume that the result is true for $n = m$, that is $g_{ns}(K_m \times K_m) = m$.

Now, we shall prove the result for $n = m + 1$. Let $S = \{v_1, v_2, v_3, \cdots, v_{m+1}\}$ be the non split geodetic set formed from some elements in G_1 and the elements which are not corresponds to elements in G_1 of $K_{m+1} \times K_{m+1}$. Clearly $I[S] = V(K_2 \times K_n)$. Now, consider P be any set of vertices such that $|P| < |S|$. Suppose $P = \{v_1, v_2, v_3, \cdots, v_m\}$ which is not non split geodetic set, because $I[P] \neq V(K_{m+1} \times K_{m+1})$. So S itself a minimum geodetic set of $K_{m+1} \times K_{m+1}$. Hence, $g_{ns}(K_{m+1} \times K_{m+1}) = m + 1$. Thus $g_{ns}(K_n \times K_n) = n$. □

Theorem 6.4 *Let G and H be graphs then* $g_{ns}(G \times H) \geq max\{g(G), g(H)\}$.*Equality holds when G,H are complete graphs and* $n \geq 3$.

Proof If S is a minimum geodetic set in $G \times H$ then we have $I[S] = \cup_{a,b \in S} I[a, b] = \cup_{a,b \in S} I[a_1, b_1] \times I[a_2, b_2] \subseteq (\cup_{a_1,b_1 \in S} I[a_1, b_1]) \times (\cup_{a_2,b_2 \in S} I[a_2, b_2]) = I[S_1] \times I[S_2]$,$V(G \times H) = I[S] \subseteq I[S_1] \times I[S_2]$. Therefore S_1 and S_2 geodetic set in G, H respectively, so $g_{ns}(G \times H) = |S| \geq max\{|s_1|, |s_2|\} \geq max\{g(G), g(H)\}$, proving the lower bound.

Consider complete graphs G, H with vertex sets $V(G) = \{u_1, u_2, \cdots, u_p\}$ and $V(H) = \{v_1, v_2, \cdots, v_q\}$ where without loss of generality $p \geq q$. Then $g(G) = p$ and $g(H) = q$. Let $S = \{(u_1, v_2), (u_2, v_2), \cdots, (u_q, v_q), (u_{q+1}, v_q), (u_{q+2}, v_q), \cdots, (u_p, v_q)\}$.

It is straight forward to verify that S is a non split geodetic set for $G \times H$. Hence, $g_{ns}(G \times H) \leq |S| \leq p = max\{g(G), g(H)\} \leq g_{ns}(G \times H)$, so equality holds. □

Theorem 6.5 *Let* $G = T$ *and* $H = K_2$ *be the graphs with* $g(G) = p \geq g(H) = q \geq 2$ *then* $g_{ns}(G \times H) \leq pq - q$.

Proof Set $X = G \times H$. Let $S = \{g_1, g_2, ..., g_p\}$ and $T = \{h_1, h_2, \cdots, h_q\}$ be the geodetic sets of G and H respectively, and $U = \{(S \times T) / \cup_{i,j=1}^{p,q} \{(g_i, h_j)\}\}$.

We claim that $I_X[U] = V(X)$. Let (g, h) be an arbitrary vertex of X.Then there exists indices i and i' such that $g \in I_G[g_i, g_{i'}]$ and there are indices j and j' such that $h \in I_H[h_j, h_{j'}]$. Since $p, q \geq 2$ we may assume that $i = i'$ and $j = j'$. Indeed ,if say $g = g_i$ then i' to be an arbitrary index from $\{1, 2, \cdots, p\}$ different from i. Set $B = \{(g_i, h_j), (g_i, h_{j'}), (g_{i'}, h_j), (g_{i'}, h_{j'})\}$.

Suppose that one of the vertices from B is not in U. We may without loss of generality assume $(g_i, h_j) \not\in U$. This means that $i = j$. Therefore $i' \neq j$ and $i \neq j'$. Then we infer that $(g, h) \in I_X[(g_i, h_{j'}), (g_{i'}, h_j)]$. Otherwise, all vertices from B are in U, then $(g, h) \in I_X[(g_i, h_j), (g_{i'}, h_{j'})]$. Hence, $g_{ns}[G \times H] \leq pq - q$. □

Theorem 6.6 *Let K_2 and T be the graphs then $g_{ns}(K_2 \times T) = g_{ns}(T)$.*

Proof Consider a tree T. Let $K_2 \times T$ be a graph formed from two copies T_1 and T_2 of T and S be a minimum non split geodetic set of $K_2 \times T$. Now, we define S' to be the union of those vertices of S in T_1 and the vertices of T_1 corresponding to vertices of T_2 belonging to S. Let $v \in V(T_1)$ lies on some $x - y$ geodesic,where $x, y \in S$.Since S is a non split geodetic set by Theorem 3.2, i.e., $g_{ns}(T) = k$ at least one of x and y belongs to V_1. If both $x, y \in V_1$ then $x, y \in S'$. Hence, we may assume that $x \in V_1, y \in V_2$. If y corresponds to x then $v = x \in S'$. Hence, we assume that y corresponds to $y' \in S'$,where $y' \neq x$. Since $d(x, y) = d(x, y') + 1$ and the vertex v lies on an $x - y$ geodesic in $K_2 \times G$. Hence, v lies on $x - y$ geodesic in G that is $g_{ns}(G) \leq g_{ns}(K_2 \times G)$.

Conversely, let S contains a vertex with the property that every vertex of T_1 lies on $x - w$ geodesic T_1 for some $w \in S$. Let S' consists of x together with those vertices of T_2 corresponding to those $S - \{x\}$. Thus, $|S'| = |S|$. We show that S' is a non split geodetic set of $K_2 \times T$. Hence $g_{ns}(K_2 \times T) \leq g_{ns}(T)$. Thus, $g_{ns}(K_2 \times T) = g_{ns}(T)$. □

Theorem 6.7 *Let K_2 and $G = P_n$ be the two graphs,*

$$g_{ns}(K_2 \times G) = \begin{cases} 2 & if n \geq 3 \\ 3 & if n = 2 \end{cases}$$

Proof Consider a trivial graph K_1 as a connected graph. Let G_1 and G_2 be the two copies of G and also $V(G_1) = \{a_1, a_2, \cdots, a_n\}$, $V(G_2) = \{b_1, b_2, \cdots, b_n\}$. Let $S = \{a_1, b_n\}$ be the non split geodetic set of $K_2 \times G$ and also $d(a_1, b_n) = diam(a_1, b_n)$. Thus, $V - S = \{a_2, a_3, \cdots, a_n, b_1, b_2, \cdots, b_{n-1}\}$ is the induced subgraph and it is connected. Hence $g_{ns}[K_2 \times G] = 2$.

Similarly,the result is obvious for $n = 2$ that is $g_{ns}[K_2 \times G] = 3$. □

§7. Block Graphs

A block graph has a subgraph G_1 of G(not a null graph) such that G_1 is non separable and if G_2 is any other graph of G, then $G_1 \cup G_2 = G_1$ or $G_1 \cup G_2$ is separable. For any graph G a complete subgraph of G is called clique of G. The number of vertices in a largest clique of G is called the clique number of G and denoted by $\omega(G)$.

Theorem 7.1 *For any block graph G, $g_{ns}(G) = n - c_i$ where n be the number of vertices and c_i be the number of cut vertices.*

Proof Let $V = \{v_1, v_2, \cdots, v_n\}$ be the number of vertices of G. Consider S be the geodetic set of G and $\langle V(G) - S\rangle$ is connected. Thus S itself a nonsplit geodetic set of G. Since every geodetic set does not contain any cut vertices. Hence, $g_{ns}(G) = n - c_i$. □

Theorem 7.2 *For any block graph G, $g_{ns}(G) \leq \omega(G) + 2c_i$ where $\omega(G)$ be the clique number and c_i be the number of cut vertices.*

Proof Let $V = \{v_1, v_2, \cdots, v_n\}$ be the number of vertices of G. In a block graph, every geodetic set is a nonsplit geodetic set. Consider S be the geodetic set of G and $\langle V(G) - S\rangle$ is connected. Thus S itself a nonsplit geodetic set of G. By the definition, the number of vertices in a largest clique of G is $\omega(G)$ and also every geodetic set does not contain any cut vertices of G. It follows that $g_{ns}(G) \leq \omega(G) + 2c_i$. □

Theorem 7.3 *For any block graph G, $g_{ns}(G) = \alpha_0(G) + 1$ where $\alpha_0(G)$ be the vertex covering number.*

Proof Let G be a block graph of order n. Now, we prove the result by mathematical induction.

For $c_i = 1$, the vertex covering number of G is

$$\alpha_0(G) = n - c_i - 1 \Rightarrow \alpha_0(G) = n - 1 - 1 \Rightarrow \alpha_0(G) + 1 = n - 1,$$

by Theorem 7.1, we have

$$g_{ns}(G) = n - c_i \Rightarrow g_{ns}(G) = n - 1.$$

Therefore, $g_{ns}(G) = \alpha_0(G) + 1$. Thus the result is result is true for $c_i = 1$. Let us assume that the result is true for $c_i = m$ that is $g_{ns}(G) = \alpha_0(G) + 1$.

Now, we shall prove the result for $c_i = m + 1$, where m+1 is the number of cut vertices. Let $S = \{v_1, v_2, \cdots, v_n\}$ be the minimum nonsplit geodetic set of G. Since every geodetic set does not contain any cut vertex, by Theorem 7.1 we have $g_{ns}(G) = n - m - 1$. Therefore,

$$\alpha_0(G) = n - c_i - 1 \Rightarrow \alpha_0(G) = (n - m - 1) - 1 \Rightarrow \alpha_0(G) + 1 = n - m - 1.$$

Thus, $g_{ns}(G) = \alpha_0(G) + 1$. □

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International J.Math. Combin. Vol.2(2016), 121-131

k-Difference cordial labeling of graphs

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Abstract: In this paper we introduce new graph labeling called k-difference cordial labeling. Let G be a (p,q) graph and k be an integer, $2 \leq k \leq |V(G)|$. Let $f : V(G) \to \{1, 2, \cdots, k\}$ be a map. For each edge uv, assign the label $|f(u) - f(v)|$. f is called a k-difference cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denote the number of vertices labelled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labelled with 1 and not labelled with 1. A graph with a k-difference cordial labeling is called a k-difference cordial graph. In this paper we investigate k-difference cordial labeling behavior of star, m copies of star and we prove that every graph is a subgraph of a connected k-difference cordial graph. Also we investigate 3-difference cordial labeling behavior of some graphs.

Key Words: Path, complete graph, complete bipartite graph, star, k-difference cordial labeling, Smarandachely k-difference cordial labeling.

AMS(2010): 05C78.

§1. Introduction

All graphs in this paper are finite and simple. The graph labeling is applied in several areas of sciences and few of them are coding theory, astronomy, circuit design etc. For more details refer Gallian [2]. Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uwv. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. In [1], Cahit introduced the concept of cordial labeling of graphs. Recently Ponraj et al. [4], introduced difference cordial labeling of graphs. In this way we introduce k-difference cordial labeling of graphs. Also in this paper we investigate the k-difference cordial labeling behavior of star, m copies of star etc. $\lfloor x \rfloor$ denote the smallest integer less than or equal to x. Terms and results not here follows from Harary [3].

[1]Received June 4, 2015, Accepted May 26, 2016.

§2. k-Difference Cordial Labeling

Definition 2.1 *Let G be a (p,q) graph and k be an integer $2 \leq k \leq |V(G)|$. Let $f : V(G) \rightarrow \{1, 2, \cdots, k\}$ be a function. For each edge uv, assign the label $|f(u) - f(v)|$. f is called a k-difference cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, and Smarandachely k-difference cordial labeling if $|v_f(i) - v_f(j)| > 1$ or $|e_f(0) - e_f(1)| > 1$, where $v_f(x)$ denote the number of vertices labelled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labelled with 1 and not labelled with 1. A graph with a k-difference cordial labeling or Smarandachely k-difference cordial labeling is called a k-difference cordial graph or Smarandachely k-difference cordial graph, respectively.*

Remark 2.2 (1) p-difference cordial labeling is simply a difference cordial labeling;

(2) 2-difference cordial labeling is a cordial labeling.

Theorem 2.3 *Every graph is a subgraph of a connected k-difference cordial graph.*

Proof Let G be (p,q) graph. Take k copies of graph K_p. Let G_i be the i^{th} copy of K_p. Take k copies of the $\overline{K}_{\binom{p}{2}}$ and the i^{th} copies of the $\overline{K}_{\binom{p}{2}}$ is denoted by G_i'. Let $V(G_i) = \{u_i^j : 1 \leq j \leq k, 1 \leq i \leq p\}$. Let $V(G_i') = \{v_i^j : 1 \leq j \leq k, 1 \leq i \leq p\}$. The vertex and edge set of super graph G^* of G is as follows:

Let $V(G^*) = \bigcup\limits_{i=1}^{k} V(G_i) \cup \bigcup\limits_{i=1}^{k} V(G_i') \cup \{w_i : 1 \leq i \leq k\} \cup \{w\}$.

$E(G^*) = \bigcup\limits_{i=1}^{k} E(G_i) \cup \{u_1^j v_i^j : 1 \leq i \leq \binom{p}{2}, 1 \leq j \leq k-1\} \cup \{u_1^k w, wv_i^k : 1 \leq i \leq \binom{p}{2}\} \cup \{u_p^j w_j : 1 \leq j \leq k\} \cup \{u_2^j u_2^{j+1} : 1 \leq j \leq k-1\} \cup \{w_1 w_2\}$.

Assign the label i to the vertices of G_i, $1 \leq i \leq k$. Then assign the label $i+1$ to the vertices of G_i', $1 \leq i \leq k-1$. Assign the label 1 to the vertices of G_k'. Then assign 2 to the vertex w. Finally assign the label i to the vertex w_i, $1 \leq i \leq k$. Clearly $v_f(i) = p + \binom{p}{2} + 1$, $i = 1, 3, \ldots, k$, $v_f(2) = p + \binom{p}{2} + 2$ and $e_f(1) = k\binom{p}{2} + k$, $e_f(0) = k\binom{p}{2} + k + 1$. Therefore G^* is a k-difference cordial graph. □

Theorem 2.4 *If k is even, then k-copies of star $K_{1,p}$ is k-difference cordial.*

Proof Let G_i be the i^{th} copy of the star $K_{1,p}$. Let $V(G_i) = \{u_j, v_i^j : 1 \leq j \leq k, 1 \leq i \leq p\}$ and $E(G_i) = \{u_j v_i^j : 1 \leq j \leq k, 1 \leq i \leq p\}$. Assign the label i to the vertex u_j, $1 \leq j \leq k$. Assign the label $i+1$ to the pendent vertices of G_i, $1 \leq i \leq \frac{k}{2}$. Assign the label $k - i + 1$ to the pendent vertices of $G_{\frac{k}{2}+i}$, $1 \leq i \leq \frac{k}{2} - 1$. Finally assign the label 1 to all the pendent vertices of the star G_k. Clearly, $v_f(i) = p + 1$, $1 \leq i \leq k$, $e_f(0) = e_f(1) = \dfrac{kp}{2}$. Therefore f is a k-difference cordial labeling of k-copies of the star $K_{1,p}$. □

Theorem 2.5 *If $n \equiv 0 \pmod{k}$ and $k \geq 6$, then the star $K_{1,n}$ is not k-difference cordial.*

Proof Let $n = kt$. Suppose f is a k-difference cordial labeling of $K_{1,n}$. Without loss of generality, we assume that the label of central vertex is r, $1 \leq r \leq k$. Clearly $v_f(i) = t$,

$1 \leq i \leq n$ and $i \neq r$, $v_f(r) = t+1$. Then $e_f(1) \leq 2t$ and $e_f(0) \geq (k-2)t$. Now $e_f(0) \geq (k-2)t - 2t \geq (k-4)t \geq 2$, which is a contradiction. Thus f is not a k-difference cordial. □

Next we investigate 3-difference cordial behavior of some graph.

§3. 3-Difference Cordial Graphs

First we investigate the path.

Theorem 3.1 *Any path is* 3*-difference cordial.*

Proof Let $u_1u_2\ldots u_n$ be the path P_n. The proof is divided into cases following.

Case 1. $n \equiv 0 \pmod 6$.

Let $n = 6t$. Assign the label 1, 3, 2, 1, 3, 2 to the first consecutive 6 vertices of the path P_n. Then assign the label 2, 3, 1, 2, 3, 1 to the next 6 consecutive vertices. Then assign the label 1, 3, 2, 1, 3, 2 to the next six vertices and assign the label 2, 3, 1, 2, 3, 1 to the next six vertices. Then continue this process until we reach the vertex u_n.

Case 2. $n \equiv 1 \pmod 6$.

This implies $n-1 \equiv 0 \pmod 6$. Assign the label to the vertices of u_i, $1 \leq i \leq n-1$ as in case 1. If u_{n-1} receive the label 2, then assign the label 2 to the vertex u_n; if u_{n-1} receive the label 1, then assign the label 1 to the vertex u_n.

Case 3. $n \equiv 2 \pmod 6$.

Therefore $n-1 \equiv 1 \pmod 6$. As in case 2, assign the label to the vertices u_i, $1 \leq i \leq n-1$. Next assign the label 3 to u_n.

Case 4. $n \equiv 3 \pmod 6$.

This forces $n-1 \equiv 2 \pmod 6$. Assign the label to the vertices u_1, u_2, $\ldots u_{n-1}$ as in case 3. Assign the label 1 or 2 to u_n according as the vertex u_{n-2} receive the label 2 or 1.

Case 5. $n \equiv 4 \pmod 6$.

This implies $n-1 \equiv 3 \pmod 6$. As in case 4, assign the label to the vertices u_1, u_2, $\cdots$, u_{n-1}. Assign the label 2 or 1 to the vertex u_n according as the vertex u_{n-1} receive the label 1 to 2.

Case 6. $n \equiv 5 \pmod 6$.

This implies $n-1 \equiv 4 \pmod 6$. Assign the label to the vertices u_1, u_2, $\cdots$, u_{n-1} as in Case 5. Next assign the label 3 to u_n. □

Example 3.2 A 3-difference cordial labeling of the path P_9 is given in Figure 1.

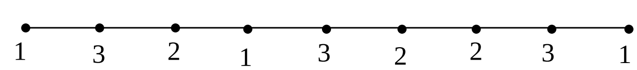

Figure 1

Corollary 3.3 *If* $n \equiv 0, 3 \pmod 4$, *then the cycle* C_n *is* 3-*difference cordial.*

Proof The vertex labeling of the path given in Theorem 3.1 is also a 3-difference cordial labeling of the cycle C_n. □

Theorem 3.4 *The star* $K_{1,n}$ *is* 3-*difference cordial iff* $n \in \{1, 2, 3, 4, 5, 6, 7, 9\}$.

Proof Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Our proof is divided into cases following.

Case 1. $n \in \{1, 2, 3, 4, 5, 6, 7, 9\}$.

Assign the label 1 to u. The label of u_i is given in Table 1.

$n\backslash u_i$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
1	2								
2	2	3							
3	2	3	1						
4	2	3	1	2					
5	2	3	1	2	3				
6	2	3	1	2	3	2			
7	2	3	1	2	3	2	3		
9	2	3	1	2	3	2	3	1	2

Table 1

Case 2. $n \notin \{1, 2, 3, 4, 5, 6, 7, 9\}$.

Let $f(u) = x$ where $x \in \{1, 2, 3\}$. To get the edge label 1, the pendent vertices receive the label either $x - 1$ or $x + 1$.

Subcase 1. $n = 3t$.

Subcase 1a. $x = 1$ or $x = 3$.

When $x = 1$, $e_f(1) = t$ or $t + 1$ according as the pendent vertices receives t's 2 or (t+1)'s 2. Therefore $e_f(0) = 2t$ or $2t - 1$. Thus $e_f(0) - e_f(1) = t - 2 > 1$, $t > 4$ a contradiction. When $x = 3$, $e_f(1) = t$ or $t + 1$ according as the pendent vertices receives t's 2 or (t+1)'s 2. Therefore $e_f(0) = 2t$ or $2t - 1$. Thus $e_f(0) - e_f(1) = t$ or $t - 2$. Therefore, $e_f(0) - e_f(1) > 1$, a contradiction.

Subcase 1b. $x = 2$.

In this case, $e_f(1) = 2t$ or $2t + 1$ according as pendent vertices receives t's 2 or (t-2)'s 2. Therefore $e_f(0) = t$ or $t - 1$. $e_f(1) - e_f(0) = t$ or $t + 2$ as $t > 3$. Therefore, $e_f(0) - e_f(1) > 1$, a contradiction.

Subcase 2. $n = 3t + 1$.

Subcase 2a. $x = 1$ or 3.

Then $e_f(1) = t$ or $t+1$ according as pendent vertices receives t's 2 or (t+1)'s 2. Therefore $e_f(0) = 2t+1$ or $2t$. $e_f(0) - e_f(1) = t+1$ or $t-1$ as $t > 3$. Therefore, $e_f(0) - e_f(1) > 3$, a contradiction.

Subcase 2b. $x = 2$.

In this case $e_f(1) = 2t$ or $2t+1$ according as pendent vertices receives t's 1 and t's 3 and t's 1 and (t+3)'s 3. Therefore $e_f(0) = t+1$ or t. $e_f(1) - e_f(0) = t-1$ or t as $t > 3$. Therefore, $e_f(0) - e_f(1) > 1$, a contradiction.

Subcase 3. $n = 3t+2$.

Subcase 3a. $x = 1$ or 3.

This implies $e_f(1) = t+1$ and $e_f(0) = 2t+1$. $e_f(0) - e_f(1) = t$ as $t > 3$. Therefore, $e_f(0) - e_f(1) > 1$, a contradiction.

Subcase 3b. $x = 2$.

This implies $e_f(1) = 2t+2$ and $e_f(0) = t$. $e_f(1) - e_f(0) = t+2$ as $t > 1$. Therefore, $e_f(1) - e_f(0) > 1$, a contradiction. Thus $K_{1,n}$ is 3-difference cordial iff $n \in \{1, 2, 3, 4, 5, 6, 7, 9\}$.
□

Next, we research the complete graph.

Theorem 3.5 *The complete graph K_n is 3-difference cordial if and only if $n \in \{1, 2, 3, 4, 6, 7, 9, 10\}$.*

Proof Let u_i, $1 \le i \le n$ be the vertices of K_n. The 3-difference cordial labeling of K_n, $n \in \{1, 2, 3, 4, 6, 7, 9, 10\}$ is given in Table 2.

$n\backslash u_i$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
1	1									
2	1	2								
3	1	2	3							
4	1	1	2	3						
6	1	1	2	2	3	3				
7	1	1	1	2	2	3	3			
9	1	1	1	2	2	2	3	3	3	
10	2	2	2	2	1	1	1	3	3	3

Table 2

Assume $n \notin \{1, 2, 3, 4, 6, 7, 9, 10\}$. Suppose f is a 3-difference cordial labeling of K_n.

Case 1. $n \equiv 0 \pmod 3$.

Let $n = 3t$, $t > 3$. Then $v_f(0) = v_f(1) = v_f(2) = t$. This implies $e_f(0) = \binom{t}{2} + \binom{t}{2} + \binom{t}{2} +$

$t^2 = \frac{5t^2-3t}{2}$. Therefore $e_f(1) = t^2 + t^2 = 2t^2$. $e_f(0) - e_f(1) = \frac{5t^2-3t}{2} - 2t^2 > 1$ as $t > 3$, a contradiction.

Case 2. $n \equiv 1 \pmod 3$.

Let $n = 3t + 1$, $t > 3$.

Subcase 1. $v_f(1) = t + 1$.

Therefore $v_f(2) = v_f(3) = t$. This forces $e_f(0) = \binom{t+1}{2} + \binom{t}{2} + \binom{t}{2} + t(t+1) = \frac{1}{2}(5t^2 + t)$. $e_f(1) = t(t+1) + t^2 = 2t^2 + t$. Then $e_f(0) - e_f(1) = \frac{1}{2}(5t^2 + t) - (2t^2 + t) > 1$ as $t > 3$, a contradiction.

Subcase 2. $v_f(3) = t + 1$.

Similar to Subcase 1.

Subcase 3. $v_f(2) = t + 1$.

Therefore $v_f(1) = v_f(3) = t$. In this case $e_f(0) = \frac{5t^2+t}{2}$ and $e_f(1) = t(t+1) + t(t+1) = 2t^2 + 2t$. This implies $e_f(0) - e_f(1) = \frac{5t^2+t}{2} - (2t^2 + 2t) > 1$ as $t > 3$, a contradiction.

Case 3. $n \equiv 2 \pmod 3$.

Let $n = 3t + 2$, $t \geq 1$.

Subcase 1. $v_f(1) = t$.

Therefore $v_f(2) = v_f(3) = t + 1$. This gives $e_f(0) = \binom{t}{2} + \binom{t+1}{2} + \binom{t+1}{2} + t(t+1) = \frac{5t^2+3t}{2}$ and $e_f(1) = t(t+1) + (t+1)^2 = 2t^2 + 3t + 1$. This implies $e_f(0) - e_f(1) = \frac{5t^2+3t}{2} - (2t^2 + 3t + 1) > 1$ as $t \geq 1$, a contradiction.

Subcase 2. $v_f(3) = t$.

Similar to Subcase 1.

Subcase 3. $v_f(2) = t$.

Therefore $v_f(1) = v_f(3) = t + 1$. In this case $e_f(0) = \binom{t+1}{2} + \binom{t+1}{2} + \binom{t}{2} + (t+1)(t+1) = \frac{5t^2+5t+2}{2}$ and $e_f(1) = t(t+1) + t(t+1) = 2t^2 + 2t$. This implies $e_f(0) - e_f(1) = \frac{5t^2+5t+2}{2} - (2t^2 + 2t) > 1$ as $t \geq 1$, a contradiction. □

Theorem 3.6 *If m is even, the complete bipartite graph $K_{m,n}$ $(m \leq n)$ is 3-difference cordial.*

Proof Let $V(K_{m,n}) = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(K_{m,n}) = \{u_i v_j : 1 \leq i \leq$

$m, 1 \leq j \leq n\}$. Define a map $f : V(K_{m,n}) \to \{1, 2, 3\}$ by

$$\begin{array}{rlll} f(u_i) & = 1, & 1 \leq i \leq \frac{m}{2} & \\ f(u_{\frac{m}{2}+i}) & = 2, & 1 \leq i \leq \frac{m}{2} & \\ f(v_i) & = 3, & 1 \leq i \leq \lceil \frac{m+n}{3} \rceil & \\ f(v_{\lceil \frac{m+n}{3} \rceil + i}) & = 1, & 1 \leq i \leq \lceil \frac{m+n}{3} \rceil - \frac{m}{2} - 1 & if \quad m+n \equiv 1, 2 \pmod 3 \\ & & 1 \leq i \leq \lceil \frac{m+n}{3} \rceil - \frac{m}{2} & if \quad m+n \equiv 0 \pmod 3 \\ f(v_{2\lceil \frac{m+n}{3} \rceil - \frac{m}{2} - 1 + i}) & = 2, & 1 \leq i \leq n - 2\lceil \frac{m+n}{3} \rceil + \frac{m}{2} + 1 & if \quad m+n \equiv 1, 2 \pmod 3 \\ f(v_{2\lceil \frac{m+n}{3} \rceil - \frac{m}{2} + i}) & = 2, & 1 \leq i \leq n - 2\lceil \frac{m+n}{3} \rceil + \frac{m}{2} & if \quad m+n \equiv 0 \pmod 3 \end{array}$$

Since $e_f(0) = e_f(1) = \frac{mn}{2}$, f is a 3-difference cordial labeling of $K_{m,n}$. □

Example 3.7 A 3-difference cordial labeling of $K_{5,8}$ is given in Figure 2.

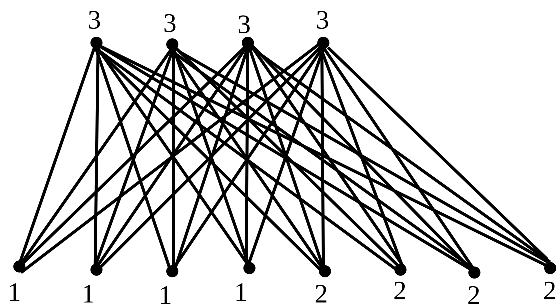

Figure 2

Next, we research some corona of graphs.

Theorem 3.8 *The comb $P_n \odot K_1$ is 3-difference cordial.*

Proof Let P_n be the path $u_1u_2 \ldots u_n$. Let $V(P_n \odot K_1) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = E(P_n) \cup \{u_iv_i : 1 \leq i \leq n\}$.

Case 1. $n \equiv 0 \pmod 6$.

Define a map $f : V(G) \to \{1, 2, 3\}$ by

$$\begin{array}{lllll} f(u_{6i-5}) & = & f(u_{6i}) & = & 1, \quad 1 \leq i \leq \frac{n}{6} \\ f(u_{6i-4}) & = & f(u_{6i-1}) & = & 3, \quad 1 \leq i \leq \frac{n}{6} \\ f(u_{6i-3}) & = & f(u_{6i-2}) & = & 2, \quad 1 \leq i \leq \frac{n}{6}. \end{array}$$

In this case, $e_f(0) = n - 1$ and $e_f(1) = n$.

Case 2. $n \equiv 1 \pmod 6$.

Assign the label to the vertices u_i, v_i $(1 \leq i \leq n-1)$ as in case 1. Then assign the labels 1, 2 to the vertices u_n, v_n respectively. In this case, $e_f(0) = n - 1$, $e_f(1) = n$.

Case 3. $n \equiv 2 \pmod 6$.

As in Case 2, assign the label to the vertices u_i, v_i $(1 \le i \le n-1)$. Then assign the labels 3, 3 to the vertices u_n, v_n respectively. In this case, $e_f(0) = n$, $e_f(1) = n-1$.

Case 4. $n \equiv 3 \pmod 6$.

Assign the label to the vertices u_i, v_i $(1 \le i \le n-1)$ as in case 3. Then assign the labels 2, 1 to the vertices u_n, v_n respectively. In this case, $e_f(0) = n-1$, $e_f(1) = n$.

Case 5. $n \equiv 4 \pmod 6$.

As in Case 4, assign the label to the vertices u_i, v_i $(1 \le i \le n-1)$. Then assign the labels 2, 3 to the vertices u_n, v_n respectively. In this case, $e_f(0) = n-1$, $e_f(1) = n$.

Case 6. $n \equiv 5 \pmod 6$.

Assign the label to the vertices u_i, v_i $(1 \le i \le n-1)$ as in case 5. Then assign the labels 3, 1 to the vertices u_n, v_n respectively. In this case, $e_f(0) = n-1$, $e_f(1) = n$. Therefore $P_n \odot K_1$ is 3-difference cordial. □

Theorem 3.9 *$P_n \odot 2K_1$ is 3-difference cordial.*

Proof Let P_n be the path $u_1u_2\cdots u_n$. Let $V(P_n \odot 2K_1) = V(P_n) \cup \{v_i, w_i : 1 \le i \le n\}$ and $E(P_n \odot 2K_1) = E(P_n) \cup \{u_iv_i, u_iw_i : 1 \le i \le n\}$.

Case 1. n is even.

Define a map $f : V(P_n \odot 2K_1) \to \{1, 2, 3\}$ as follows:

$$\begin{aligned} f(u_{2i-1}) &= 1, \quad 1 \le i \le \tfrac{n}{2} \\ f(u_{2i}) &= 2, \quad 1 \le i \le \tfrac{n}{2} \\ f(v_{2i-1}) &= 1, \quad 1 \le i \le \tfrac{n}{2} \\ f(v_{2i}) &= 2, \quad 1 \le i \le \tfrac{n}{2} \\ f(w_i) &= 3, \quad 1 \le i \le \tfrac{n}{2}. \end{aligned}$$

In this case, $v_f(1) = v_f(2) = v_f(3) = n$, $e_f(0) = \frac{3n}{2}$ and $e_f(1) = \frac{3n}{2} - 1$.

Case 2. n is odd.

Define a map $f : V(P_n \odot 2K_1) \to \{1, 2, 3\}$ by $f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 3$, $f(v_1) = f(v_3) = 1$, $f(w_1) = f(w_2) = 3$, $f(v_2) = f(w_3) = 2$,

$$\begin{aligned} f(u_{2i+2}) &= 2, \quad 1 \le i \le \tfrac{n-3}{2} \\ f(u_{2i+3}) &= 1, \quad 1 \le i \le \tfrac{n-3}{2} \\ f(v_{2i+2}) &= 2, \quad 1 \le i \le \tfrac{n-3}{2} \\ f(v_{2i+3}) &= 1, \quad 1 \le i \le \tfrac{n-3}{2} \\ f(w_{i+3}) &= 3, \quad 1 \le i \le n-3. \end{aligned}$$

Clearly, $v_f(1) = v_f(2) = v_f(3) = n$, $e_f(0) = e_f(1) = \frac{3n-1}{2}$. □

Next we research on quadrilateral snakes.

Theorem 3.10 *The quadrilateral snakes Q_n is 3-difference cordial.*

Proof Let P_n be the path $u_1u_2\cdots u_n$. Let $V(Q_n) = V(P_n) \cup \{v_i, w_i : 1 \le i \le n-1\}$ and $E(Q_n) = E(P_n) \cup \{u_iv_i, v_iw_i, w_iu_{i+1} : 1 \le i \le n-1\}$. Note that $|V(Q_n)| = 3n-2$ and $|E(Q_n)| = 4n-4$. Assign the label 1 to the path vertices u_i, $1 \le i \le n$. Then assign the labels 2, 3 to the vertices v_i, w_i $1 \le i \le n-1$ respectively. Since $v_f(1) = n$, $v_f(2) = v_f(3) = n-1$, $e_f(0) = e_f(1) = 2n-2$, f is a 3-difference cordial labeling. □

The next investigation is about graphs $B_{n,n}$, $S(K_{1,n})$, $S(B_{n,n})$.

Theorem 3.11 *The bistar $B_{n,n}$ is 3-difference cordial.*

Proof Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \le i \le n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \le i \le n\}$. Clearly $B_{n,n}$ has $2n+2$ vertices and $2n+1$ edges.

Case 1. $n \equiv 0 \pmod 3$.

Assign the label 1, 2 to the vertices u and v respectively. Then assign the label 1 to the vertices u_i, v_i $(1 \le i \le \frac{n}{3})$. Assign the label 2 to the vertices $u_{\frac{n}{3}+i}$, $v_{\frac{n}{3}+i}$ $(1 \le i \le \frac{n}{3})$. Finally assign the label 3 to the vertices $u_{\frac{2n}{3}+i}$, $v_{\frac{2n}{3}+i}$ $(1 \le i \le \frac{n}{3})$. In this case $e_f(1) = n+1$ and $e_f(0) = n$.

Case 2. $n \equiv 1 \pmod 3$.

Assign the labels to the vertices u, v, u_i, v_i $(1 \le i \le n-1)$ as in Case 1. Then assign the label 3, 2 to the vertices u_n, v_n respectively. In this case $e_f(1) = n$ and $e_f(0) = n+1$.

Case 3. $n \equiv 2 \pmod 3$.

As in Case 2, assign the label to the vertices u, v, u_i, v_i $(1 \le i \le n-1)$. Finally assign 1, 3 to the vertices u_n, v_n respectively. In this case $e_f(1) = n$ and $e_f(0) = n+1$. Hence the star $B_{n,n}$ is 3-difference cordial. □

Theorem 3.12 *The graph $S(K_{1,n})$ is 3-difference cordial.*

Proof Let $V(S(K_{1,n})) = \{u, u_i, v_i : 1 \le i \le n\}$ and $E(S(K_{1,n})) = \{uu_i, u_iv_i : 1 \le i \le n\}$. Clearly $S(K_{1,n})$ has $2n+1$ vertices and $2n$ edges.

Case 1. $n \equiv 0 \pmod 3$.

Define a map $f : V(S(K_{1,n})) \to \{1, 2, 3\}$ as follows: $f(u) = 2$,

$$\begin{aligned} f(u_i) &= 1, \quad 1 \le i \le t \\ f(u_{t+i}) &= 2, \quad 1 \le i \le 2t \\ f(v_i) &= 3, \quad 1 \le i \le 2t \\ f(v_{2t+i}) &= 1, \quad 1 \le i \le t. \end{aligned}$$

Case 2. $n \equiv 1 \pmod 3$.

As in Case 1, assign the label to the vertices u, u_i, v_i $(1 \leq i \leq n-1)$. Then assign the label 1, 3 to the vertices u_n, v_n respectively.

Case 3. $n \equiv 2 \pmod 3$.

As in Case 2, assign the label to the vertices u, u_i, v_i $(1 \leq i \leq n-1)$. Then assign the label 2, 1 to the vertices u_n, v_n respectively. f is a 3-difference cordial labeling follows from the following Table 3.

Values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$e_f(0)$	$e_f(1)$
$n = 3t$	$2t$	$2t+1$	$2t$	$3t$	$3t$
$n = 3t+1$	$2t+1$	$2t+1$	$2t+1$	$3t+1$	$3t+1$
$n = 3t+2$	$2t+2$	$2t+2$	$2t+1$	$3t+2$	$3t+2$

Table 3

Theorem 3.13 *$S(B_{n,n})$ is 3-difference cordial.*

Proof Let $V(S(B_{n,n})) = \{u, w, v, u_i, w_i, v_i, z_i : 1 \leq i \leq n\}$ and $E(S(B_{n,n})) = \{uw, wv, uu_i, u_iw_i, vv_i, v_iz_i : 1 \leq i \leq n\}$. Clearly $S(B_{n,n})$ has $4n+3$ vertices and $4n+2$ edges.

Case 1. $n \equiv 0 \pmod 3$.

Define a map $f : V(S(B_{n,n})) \to \{1, 2, 3\}$ by $f(u) = 1$, $f(w) = 3$, $f(v) = 2$,

$$\begin{aligned} f(w_i) &= 2, \quad 1 \leq i \leq n \\ f(v_i) &= 1, \quad 1 \leq i \leq n \\ f(z_i) &= 3, \quad 1 \leq i \leq n \\ f(u_i) &= 1, \quad 1 \leq i \leq \tfrac{n}{3} \\ f(u_{\frac{n}{3}+i}) &= 2, \quad 1 \leq i \leq \tfrac{n}{3} \\ f(u_{\frac{2n}{3}+i}) &= 3, \quad 1 \leq i \leq \tfrac{n}{3}. \end{aligned}$$

Case 2. $n \equiv 1 \pmod 3$.

As in Case 1, assign the label to the vertices u, w, v, u_i, v_i, w_i, z_i $(1 \leq i \leq n-1)$. Then assign the label 1, 2, 1, 3 to the vertices u_n, w_n, v_n, z_n respectively.

Case 3. $n \equiv 2 \pmod 3$.

As in Case 2, assign the label to the vertices u, w, v, u_i, v_i, w_i, z_i $(1 \leq i \leq n-1)$. Then assign the label 2, 2, 1, 3 to the vertices u_n, w_n, v_n, z_n respectively. f is a 3-difference cordial labeling follows from the following Table 4.

Values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 3$	$\frac{4n+3}{3}$	$\frac{4n+3}{3}$	$\frac{4n+3}{3}$	$\frac{4n+2}{2}$	$\frac{4n+2}{2}$
$n \equiv 1 \pmod 3$	$\frac{4n+5}{3}$	$\frac{4n+2}{3}$	$\frac{4n+2}{3}$	$\frac{4n+2}{2}$	$\frac{4n+2}{2}$
$n \equiv 2 \pmod 3$	$\frac{4n+4}{3}$	$\frac{4n+4}{3}$	$\frac{4n+1}{3}$	$\frac{4n+2}{2}$	$\frac{4n+2}{2}$

Table 4

Finally we investigate cycles $C_4^{(t)}$.

Theorem 3.14 *$C_4^{(t)}$ is 3-difference cordial.*

Proof Let u be the vertices of $C_4^{(t)}$ and i^{th} cycle of $C_4^{(t)}$ be $uu_1^iu_2^iu_3^iu$. Define a map f from the vertex set of $C_4^{(t)}$ to the set $\{1,2,3\}$ by $f(u)=1$, $f(u_2^i)=3$, $1 \le i \le t$, $f(u_1^i)=1$, $1 \le i \le t$, $f(u_3^i)=2$, $1 \le i \le t$. Clearly $v_f(1)=t+1$, $v_f(2)=v_f(3)=t$ and $e_f(0)=e_f(1)=2t$. Hence f is 3-difference cordial. □

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International J.Math. Combin. Vol.2(2016), 132-138

Traversability and Covering Invariants of Token Graphs

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Abstract: Let $F_k(G), k \geq 1$, G be the token graph of a connected graph G. In this paper, we investigate the Eulerian and Hamiltonian property of token graphs and obtain the covering invariants for complete graph of token graph.

Key Words: Token graph, α_0, α_1, β_0, β_1, symmetric difference.

AMS(2010): 05C45, 05C70.

§1. Introduction

All graphs considered here are simple, connected,undirected graphs. Let G be a graph with vertex set V(G)=$\{v_1, v_2, v_3, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, e_3, \cdots, e_m\}$. We refer the reader to Harary [3].

R.Fabila-Monroy and et.al. introduced a model in which, k indistinguished tokens move from vertex to vertex along the edges of a graph. This idea is formalized as follows, for a graph G and integer $k \geq 1$, we define $F_k(G)$ to be the graph with vertex set $\binom{V(G)}{k}$, where two vertices A and B of $F_k(G)$ are adjacent whenever their symmetric difference $A \triangle B$ is a pair $\{a, b\}$ such that $a \in A$, $b \in B$ and $ab \in E(G)$. Thus the vertices of $F_k(G)$ correspond to configurations of k-indistinguishable tokens placed at distinct vertices of G, where two configurations are adjacent whenever one configuration can be reached from the other by moving one token along an edge from its current position to an unoccupied vertex. The $F_k(G)$ is called the *k-token graph* of G.

Many problems in mathematics and computer science are modeled by moving objects on the vertices of a graph according to certain prescribed rules. In *graph pebbeling*, a pebbling step consists of removing two pebbels from a vertex and placing one pebble on an adjacent vertex; [4] and [5] for surveys. Related pebbling games have been used to study rigidity [6,7], motion planning[1,9], and as models of computation[10]. In the "chip firing game", a vertex v fires by distributing one chip to each of its neighbors(assuming the number of chips at v is at least its degree). This model has connections with matroid, the Tutte polynomial, and mathematical physics [8].

Inspired by this we investigate the some more properties like traversability and covering invariants of token graphs.

[1]Received September 26, 2015, Accepted May 28, 2016.

Remark 1.1([2]) Let G be a graph and $F_k(G)$ be the token graph of G with $k \geq n-1$, $|V(F_k(G))|=\binom{n}{k}$, $|E(F_k(G))|=\binom{n-2}{k-1}|E(G)|$.

Remark 1.2([2]) Two vertices A and B are adjacent in $F_k(G)$ if and only if $V(G) \setminus A$ and $V(G) \setminus B$ are adjacent in $F_{n-1}(G)$, $F_k(G) \cong F_{n-k}(G)$, with only one token, the token graph is isomorphic to G. Thus, $F_1(G) \cong G$.

Remark 1.3 Degree of vertices in $F_k(G)$ is

$$\begin{aligned} deg(V_{F_k}(G)) &= \sum_{i=1}^{n-1} deg_G(V_i) - 2 \\ &\quad \text{(sum of pairs of vertices } v_i; i \in k \text{ of } G \text{ which are the elements of } V) \end{aligned}$$

if v_i and v_j are two vertices in G, then in $F_k(G)$

$$\begin{aligned} v_i v_j &= 1, if\ \ v_i\ is\ adjacent\ \ to\ \ v_j\ \ in\ G. \\ &= 0,\ \ if\ v_i\ is\ not\ adjacent\ to\ v_j\ in\ G. \end{aligned}$$

Remark 1.4 If degree of all the vertices in a graph G is even or even regular then by the Remark 3 degree of all the vertices in $F_k(G)$ is even, irrespective of tokens being odd or even.

Remark 1.5 If degree of all the vertices in a graph G is odd or odd regular then by Remark 3, degree of all the vertices in $F_k(G)$ is even, only when k is even token.

Remark 1.6 If G contains both even and odd degree vertices then the vertices in $F_k(G)$ are also of odd and even degree, irrespective of tokens being odd or even.

§2. **Traversability of Token Graphs**

In this section we obtain the traversability properties of token graphs.

Theorem 2.1 *Let G be a connected graph. Then $F_k(G)$ is Eulerian if and only if it satisfies either of the following conditions.*

(i) *Every vertex in G is of even degree;*
(ii) *Every vertex in G is of odd degree and k-is even.*

Proof Let $F_k(G)$ be a token graph of graph G. Assume $F_k(G)$ is Eulerian, that is each vertices in $F_k(G)$ is of even degree. By the Remark 1.3, we have, $d(V_{F_k(G)}) = deg(u) + deg(v)$-2(sum of pair of adjacent elements of G in V of $F_k(G)$).

Depending upon the degree, we consider the following cases.

Case 1. Suppose $deg(u)+deg(v)$ is odd, then by Remark 3, $d(V_{F_k(G)})$ is odd, a contradiction. Thus condition (i) is satisfied.

Case 2. Suppose $deg(u)+deg(v)$ is even, where u and v are odd or odd regular with odd tokens then by Remarks 1.3 and 1.5. $F_k(G)$ is non-Eulerian, a contradiction. Thus the condition (ii) is satisfied.

If G is Eulerian, that is it contains even degree of vertices. Then by the Remark 4, $F_k(G)$ is Eulerian. That is it contains even degree vertices.

The converse follows from Remarks 1.4 and 1.5. □

Corollary 2.2 *If G be Eulerian graph, then $F_{n-1}(G)$ is also Eulerian.*

Proof Let G be Eulerian graph. Then by the Remark 1.2, $F_{n-1}(G)$ is Eulerian. □

Lemma 2.3 *If G is hamiltonian, then $F_{n-1}(G)$ is also hamiltonian.*

Proof Suppose G is hamiltonian, by the Remark 1.2, we know that

$$G \cong F_1(G) \ \text{ and } \ F_k(G) \cong F_{n-1}(G).$$

If $k = 1$ then,

$$F_1(G) \cong F_{n-1}(G)$$

Therefore,

$$G \cong F_1(G), \quad G \cong F_{n-1}(G).$$

Thus, $F_{n-1}(G)$ is also hamiltonian. □

Theorem 2.4 *$F_k(G)$ is hamiltonian if and only if G is complete graph.*

Proof Let G be complete graph and let $\{v_1, v_2, v_3, \cdots, v_n\}$ be the vertices in a graph G. In complete graph all vertices are mutually adjacent and G is hamiltonian.

By the definition of token graph, $F_k(G)$ contains $\binom{n}{k}$ number of vertices and by the Lemma 2.3, $F_1(G)$ and $F_{n-1}(G)$ are hamiltonian.

Now, we have to prove for $k = 2, 3, 4, \cdots, n-2$ tokens. We prove this by induction method, here $(k+1)^{th}$ term is n-2 token.

If k=2 token then,

$$\begin{aligned} V(F_2(G)) &= \{(v_1v_2), (v_1v_3), (v_1v_4), \cdots, (v_1v_n) \cup (v_2v3), (v_2v_4), (v_2v_5), \\ &\quad \cdots, (v_2v_n) \cup, \cdots, \cup (v_{n-1}v_n)\}. \end{aligned}$$

Here we consider two vertices $A = \{v_1v_n\}$ and $B = \{v_2v_n\}$. By the symmetric difference we get v_1v_2. That is,$|A \triangle B| = (A \cup B) - (A \cap B)$ $=v_1v_2v_n - v_n$= v_1v_2. Therefore v_1v_2 are adjacent in G then A and B are also adjacent in $F_2(G)$.

Similarly v_2v_n is adjacent with v_3v_n and the same follows for all vertices.

Now if k=3 token, then

$$V(F_3(G)) = \{(v_1v_2v_3),(v_1v_2v_4),\cdots,(v_1v_2v_3),(v_1v_3v_4),(v_1v_3v_5),\cdots,(v_1v_3v_n),\cdots,(v_{n-2}v_{n-1}v_n)\}.$$

Here also $(v_1v_2v_n)$ is adjacent with $(v_1v_3v_n)$,$(v_1v_3v_n)$ with $(v_1v_4v_n)$,$\cdots$ and $(v_{n-3}v_{n-2}v_{n-1})$ with $(v_{n-2}v_{n-1}v_n)$. Hence, we get spanning cycle in $F_3(G)$ as $\{v_1v_{n-1}v_n, v_2v_{n-1}v_n, v_3v_{n-1}v_n,\cdots, v_{n-2}v_{n-1}v_n, v_1v_{n-1}v_n\}$. Therefore, $F_3(G)$ is hamiltonian graph. Thus the result is true for all k=n.

Similarly, If k=n-2 token, then $V(F_{n-2}(G)) = V(F_2(G))$. By the Lemma 2.3 and Remark 1.2, $F_{n-1}(G) \cong F_1(G)$. That is,

$$F_k(G) \cong F_{n-k}(G), \qquad (1)$$

$$F_2(G) \cong F_{n-k}(G), \qquad (2)$$

and if $k = 1$, then

$$F_1(G) \cong F_{n-1}(G). \qquad (3)$$

Then $G \cong F_1(G) \cong F_{n-1}(G)$. Thus $F_{n-1}(G)$ is hamiltonian.

For the converse, assume $F_k(G)$ is hamiltonian, we have to prove G is complete. Suppose G is not complete graph then by the symmetric difference the vertices in $F_k(G)$; k= 2,3,...,n-2, form a sub graph homiomorphic to $K_{2,3}$ a contradiction.

Theorem 2.5 *If G is wheel, then $F_k(G)$ is hamiltonian graph.*

Proof Let G be wheel, hence it contains spanning cycle and let $\{v_1, v_2, v_3,\cdots, v_n\}$ be the vertex of graph G. Here v_n is a vertices of maximum degree in G. Let $V_1, V_2, V_3,\cdots, V_{\binom{n}{k}}$ be the vertices in graph $F_k(G)$. By the lemma 2.3, we know that

$$G \cong F_1(G) \cong F_{n-1}(G).$$

Then G is hamiltonian then $F_1(G)$ and $F_{n-1}(G)$ are also hamiltonian.

Now we prove for $k = 2, 3, 4,\cdots, n-2$ tokens. We know that $F_k(G) \cong F_{n-1}(G)$. If k=2 then,

$$\begin{aligned} V(F_2(G)) &= \{V_1, V_2, V_3,\cdots, V_{\binom{n}{k}}\} \\ &= \{(v_1v_2),(v_1v_3),\cdots,(v_1v_n),(v_2v_3),\cdots,(v_2v_n),(v_3v_4),\\ &\quad (v_3v_5),...,(v_3v_n),...,(v_{n-1}v_n)\}.\end{aligned}$$

In graph G, the n^{th} vertex is adjacent with remaining all the vertices. Therefore by the symmetric difference we get spanning cycle as $\{v_1v_n, v_1v_{n-1}, v_1v_{n-2},\cdots, v_1v_2, v_2v_n, v_2v_{n-1},\cdots, v_2v_3, v_3v_n,\cdots, v_3v_4, v_4v_n,\cdots, v_{n-2}v_{n-1}, v_{n-1}v_n, v_1v_n\}$. Thus $F_2(G)$ contains spanning cycle then $F_{n-2}(G)$ also contains spanning cycle. Clearly, by Remark 2.2, $F_2(G)$ and $F_{n-2}(G)$ are hamiltonian.

Similarly for all tokens we get spanning cycle. Hence $F_k(G)$ is hamiltonian. □

§3. Covering Invariants Of Token Graphs

In the following section, we determine the point covering number $\alpha_0(G)$,line covering number $\alpha_1(G)$, point independence number $\beta_0(G)$ and line independence number $\beta_1(G)$ of token graph of complete graph.

Theorem 3.1 *For any complete graph $K_n; n > 1$,*

$$\alpha_1(F_k(K_n)) = \left\lceil \frac{\binom{n}{k}}{2} \right\rceil, \quad \beta_1(F_k(K_n)) = \left\lfloor \frac{\binom{n}{k}}{2} \right\rfloor.$$

Proof Let K_n be the complete graph with n-vertices and $F_k(K_n)$ be the token graph of complete graph with $\binom{n}{k}$ number of vertices. $\left\lceil \frac{\binom{n}{k}}{2} \right\rceil$ lines are required cover all the points in $F_k(K_n)$. By Remark 1.2, $F_1(K_n) \cong K_n$ and $F_{n-1}(K_n) \cong K_n$, i.e., $F_k(K_n) \cong K_n$ when $k = 1$ or $n - 1$.

For $k = 2, 3, 4, \cdots, n-2$, the vertices $F_k(K_n)$ are adjacent but not mutually and by Remark 1.1, it contains $\binom{n}{k}$ number of vertices. Hence $\left\lceil \frac{\binom{n}{k}}{2} \right\rceil$ number of lines are require to cover all the points. $\alpha_1(F_k(K_n)) = \left\lceil \frac{\binom{n}{k}}{2} \right\rceil$.

From the Gallai result, we know that

$$\alpha_1(G) + \beta_1(G) = |G|$$

In $F_k(K_n)$,

$$\begin{aligned}
\alpha_1(F_k(K_n)) + \beta_1(F_k(K_n)) &= \binom{n}{k} \Rightarrow \lceil \frac{\binom{n}{k}}{2} \rceil + \beta_1(F_k(K_n)) = \binom{n}{k}, \\
\beta_1(F_k(K_n)) &= \binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{2} \right\rceil \\
&= \binom{n}{k} - \frac{\binom{n}{k}}{2} \quad (\text{Note that } \frac{\binom{n}{k}}{2} \cong \left\lceil \frac{\binom{n}{k}}{2} \right\rceil.) \\
&= \frac{\binom{n}{k}}{2}.
\end{aligned}$$

But $F_k(G)$ contains odd number of vertices then,

$$\beta_1(K_n) = \frac{\binom{n}{k}}{2} \cong \lfloor \frac{\binom{n}{k}}{2} \rfloor, \quad \beta_1(K_n) = \lfloor \frac{\binom{n}{k}}{2} \rfloor. \qquad \square$$

Theorem 3.2 *For any complete graph $K_n; n > 1$,*

$$\beta_0(F_k(K_n)) = \begin{cases} \left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil & if \ \binom{n}{k} \ is \ even, \\ \left\lfloor \frac{\binom{n}{k}}{\triangle(K_n)} \right\rfloor + 1 & if \ \binom{n}{k} \ is \ odd \end{cases}$$

and $\alpha_0(F_k(K_n)) = \binom{n}{k} - \beta_0(F_k(K_n))$.

Proof Let K_n be the complete graph with n-vertices and $F_k(K_n)$ be the token graph of complete graph with $\binom{n}{k}$ number of vertices. By the definition of complete graph, $\triangle(K_n) = n-1$ and $\alpha_0(K_n) = n-1$, $\beta_0(K_n) = 1$. Therefore by the Remark 1.1, $\alpha_0(F_k(K_n)) = n-1$ and $\beta_0(F_k(K_n)) = 1$ when $k = 1$ or $n-1$.

Now we have to prove for $k = 2, 3, 4, \cdots, n-2$ tokens. $F_k(K_n)$ contains $\binom{n}{k}$ number of vertices, and $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil$ are required to cover the vertices in $F_k(K_n)$. $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil$ number of vertices are non-adjacent to each other and adjacent with remaining $\binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil$ number of vertices in $F_k(K_n)$, when $\binom{n}{k}$ is even. if $\binom{n}{k}$ is odd, then $\binom{n}{k}$ is covered by $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil + 1$ vertices, which are non-adjacent to each other and are adjacent with remaining $\binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil + 1$ vertices in $F_k(K_n)$. Thus independence number in $F_k(K_n)$ is $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil$ or $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil + 1$. So,

$$\beta_0(F_k(K_n)) = \begin{cases} \left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil & \text{if } \binom{n}{k} \text{ is even,} \\ \left\lfloor \frac{\binom{n}{k}}{\triangle(K_n)} \right\rfloor + 1 & \text{if } \binom{n}{k} \text{ is odd.} \end{cases}$$

From the Gallai result, we know that

$$\alpha_1(G) + \beta_1(G) = |G|.$$

In $F_k(K_n)$, when $\binom{n}{k}$ is even then,

$$\begin{aligned} \alpha_0(F_k(K_n)) + \beta_0(F_k(K_n)) &= \binom{n}{k} \\ \Rightarrow \alpha_0(F_k(K_n)) + \left\lceil \frac{\binom{n}{k}}{\triangle(G)} \right\rceil &= \binom{n}{k} \\ \alpha_0(F_k(K_n)) &= \binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{\triangle(G)} \right\rceil \\ \alpha_0(F_k(K_n)) &= \binom{n}{k} - \beta_0(F_k(K_n)). \end{aligned}$$

If $\binom{n}{k}$ is odd then,

$$\begin{aligned} \alpha_0(F_k(K_n)) + \beta_0(F_k(K_n)) &= \binom{n}{k} \\ \Rightarrow \alpha_0(F_k(K_n)) + \left\lceil \frac{\binom{n}{k}}{\triangle(G)} \right\rceil + 1 &= \binom{n}{k} \end{aligned}$$

$$\alpha_0(F_k(K_n)) = \binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{\triangle(G)} \right\rceil + 1$$

$$\alpha_0(F_k(K_n)) = \binom{n}{k} - \beta_0(F_k(K_n)).$$

Acknowledgement The authors are thankful to University Grants Commission (UGC), Govt.of India for financial support through research grant under UGC-RGNF-2014-15-SC-KAR-75098/(SA-III/www.UGC-RGNF.nic.in).

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Different Labelings on Parallel Transformations of a Class of Trees

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Abstract: A graph $G = (V, E)$ with p vertices and q edges is said to be a mean graph if there exists an injective function $f : V \to \{0, 1, \cdots, q\}$ that induces an edge labeling $f^* : E \to \{1, 2, \cdots, q\}$ defined by

$$\begin{aligned} f^*(uv) &= \frac{f(u)+f(v)}{2} \text{ if } f(u)+f(v) \text{ is even} \\ &= \frac{f(u)+f(v)+1}{2} \text{ if } f(u)+f(v) \text{ is odd} \end{aligned}$$

for every edge uv of G.Further f is called a super-mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, \cdots, p+q\}$. If the vertex labels are all even numbers in $\{2, 4, \cdots, 2q\}$ so that $f^*(e) = \frac{f(u)+f(v)}{2}$ then f is an even mean labeling of G and if the vertex labels are in $\{1, 3, \cdots, 2q-1\}$ so that $f^*(e) = \frac{f(u)+f(v)+1}{2}$, then G is an odd-mean graph. In this paper, we investigate a typical class of trees based on this definition.

Key Words: Mean labeling, super-mean labeling, even-mean labeling, odd-mean labeling, parallel transformation of trees.

AMS(2010): 05C78.

§1. Introduction

Throughout this paper ,by a graph we mean a simple finite undirected graph without isolated vertices.For basic notations and terminology in graph theory we follow [1]. The concept of mean labeling was introduced in [5], super-mean labeling in [4] and odd-mean labeling in [2].

§2. T_n Class of Trees

In [3], T_n class of trees are defined as follows.

Definition 2.1 *Let T be a tree and x and y be two adjacent vertices in T.Let there be two end vertices(non-adjacent vertices of degree 1)x' and y' in T such that the length of $x - x'$ is equal to the length of the path $y - y'$. If the edge xy is deleted and x' , y' are joined by an edge $x'y'$,*

[1]Received April 24, 2015, Accepted May 31, 2016.

then such a transformation of edges from xy to $x'y'$ is called a parallel transformation of an edge in T.

Definition 2.2 *A tree is said to be a T_n tree if and only if a resultant parallel transformation of edges reduce T into a Hamiltonian path. Such Hamiltonian path is denoted as P_T.*

T_{27} is given below in figure 2.1(a). Here e_1, e_2, e_3, e_4 and e_5 are the edges to be deleted and e_i' ; $i = 1, \cdots, 5$ (shown in broken lines) the corresponding edges to be added to generate P_T from T_{27} (Figure 2.1(b)).

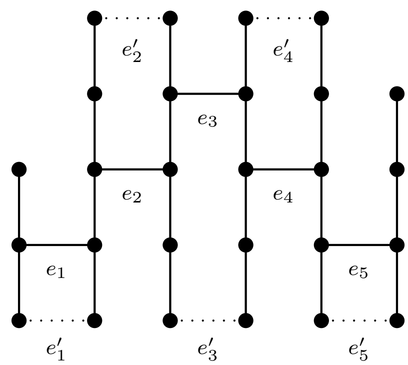

Fig 2.1 (a)

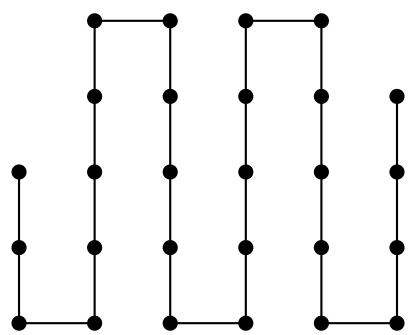

Fig 2.1 (b)

Theorem 2.1 *T_n is a mean graph.*

Proof Let T_n be a tree on n vertices and by definition there exist a path P_T corresponding to T_n. Let $E = \{e_1, e_2, \cdots, e_{n-1}\}$ be the edges of T_n. Let $E_S = \{e_1, e_2, \cdots, e_s\}$ be the set of edges to be deleted from and $E_s' = \{e_1', e_2', \cdots, e_s'\}$ be the edges to be added to T_n so as to obtain a Hamiltonian path P_T with $V(P_T) = V(T_n)$ and $E(P_T) = \{E(T_n) - E_S\} \cup E_S'$. Label the vertices of P_T as $x_1, x_2, \cdots, x_n$ starting from the initial pendant vertex.

Define an injective mapping $f : V(P_T) \to \{0, 1, \cdots, n-1\}$, as $f(x_i) = i - 1$ for all i. Now f induces edge labeling f^* on $E(P_T)$ as

$$\begin{aligned} f^*(xy) &= \frac{f(x)+f(y)}{2} \text{ if } f(x) \text{ and } f(y) \text{ are of same parity} \\ &= \frac{f(x)+f(y)+1}{2} \text{ otherwise} \end{aligned}$$

where $xy \in E(P_T)$.

Since P_T is a path, every edge of P_T is of the form $x_i x_{i+1}$.

$$\begin{aligned} f^*(x_i x_{i+1}) &= \frac{f(x_i)+f(x_{i+1})+1}{2}, \text{ since } f(x_i) \text{ and } f(x_{i+1}) \text{ are of different parity} \\ &= \frac{i-1+i+1}{2} = i \text{ for } i = 1, 2, \cdots, n-1 \end{aligned}$$

Obviously f is injective and $f^*(G) = \{1, 2, \cdots, n-1\}$. So it is proved that f is a mean labeling on P_T. We have to prove that f is a mean labeling on T_n.
For this, it is enough to prove that $f^*(e_s) = f^*(e'_s)$ where $e_s = x_i x_j \in E(T_n)$ and $e'_s = x_{i+r} x_{j-r} \in E(T_n)$.

Now, e'_s must be of the form $x_{i+r} x_{i+r+1}$, since it is an edge of a path P_T. So

$$\begin{aligned} (x_{i+r}, x_{j-r}) &= (x_{i+r}, x_{i+r+1}) \\ \frac{f(x_{i+r})+f(x_{j-r})+1}{2} &= \frac{f(x_{i+r})+f(x_{i+r+1})+1}{2} \end{aligned}$$

Therefore

$$j = i + 2r + 1$$

So

$$f^*(e_s) = \frac{f(x_i)+f(x_{i+2r+1})+1}{2} = i + r$$

and

$$\begin{aligned} f^*(e'_s) &= f^*(x_{i+r}, x_{i+r+1}) \\ &= \frac{f(x_{i+r})+f(x_{i+r+1})+1}{2} = i + r \end{aligned}$$

Therefore,

$$f^*(e_s) = f^*(e'_s).$$

Thus, f admits mean labeling on T_n. Hence we get the theorem. □

Definition 2.3 *A graph with p vertices and q edges is said to be odd mean if there exists a function $f : V(G) \to \{0, 1, \cdots, 2q-1\}$ which is one-one and the induced map $f^* : E(G) \to \{1, 3, \cdots, 2q-1\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$, if $f(u)+f(v)$ is even or $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd, is a bijection. If a graph has an odd mean labeling, then we say that G is an odd mean graph.*

Definition 2.4 *A function f is called an even-mean labeling of a graph G with p vertices and q edges if f is an injection from the vertices of G to $\{2, 4, \cdots, 2q\}$ such that when each edge uv is assigned the label $\frac{f(u)+f(v)}{2}$, then the resulting edge labels are distinct.A graph which admits an even mean labeling is said to be an even-mean graph.*

Theorem 2.2 *T_n satisfies both even and odd mean labeling.*

Proof To prove T_n is an even-mean graph,we consider $f_e : V(G) \to \{2, 4, \cdots, 2q\}$ such that $f_e(x_i) = 2i$ for $i = 1, 2, \cdots, n$.

Now, to show T_n is odd-mean, we take another injective mapping $f_o : V(G) \to \{1, 3, \cdots, 2q+1\}$ as $f_o(x_i) = 2i - 1$ for $i = 1, 2, \cdots, n$. □

Theorem 2.3 *Parallel transformation of trees generate a class of super-mean graphs.*

Proof Consider a T_n tree on n vertices. By definition there exist a P_T corresponding to T_n. Let $E = \{e_1, \cdots, e_{n-1}\}$ be the edges of T_n. Let $E_r = \{e_1, e_2, \cdots, e_r\}$ be the edges to be deleted from $T_n, E_r \subset E$ and $E'_r = \{e'_1, \cdots, e'_r\}$ be the set of edges to be added to T_n to make a path P_T, such that if e_n is the deleted edge, e'_n is the corresponding edge added at a distance d_n by parallel transformation. Now we have $V(P_T) = V(T_n)$ and $E(P_T) = \{E(T_n) - E_r\} \cup E'_r$.

Now we label the vertices of P_T by $x_1, x_2, \cdots, x_n$ successively starting at one end vertex of the path P_T. Define a mapping $f : V(P_T) \to \{1, 2, \cdots, 2n-1\}$ such that $f(x_i) = 2i - 1$ for all $i = 1, \ldots, n$. Now, by the definition itself, f is one-one. Let f^* be the induced mapping defined on the edge set of P_T such that
$f \cup f^* = \{1, 2, \cdots, 2n-1\}$ as

$$\begin{aligned} f^*(xy) &= \frac{f(x)+f(y)}{2} \text{ if } f(x)+f(y) \text{ is even} \\ &= \frac{f(x)+f(y)+1}{2} \quad \text{ if } f(x)+f(y) \text{ is odd} \end{aligned}$$

where $xy \in E(P_T)$.

Since P_T is a path, every edge of P_T is of the form $x_i x_{i+1}$ for $i = 1, 2, \cdots, n-1$

$$\begin{aligned} f^*(x_i x_{i+1}) &= \frac{f(x_i)+f(x_{i+1})}{2} \\ &= 2i \quad ; \quad i = 1, 2, \cdots, n-1 \end{aligned}$$

Hence it is clear that f^* is one one and $f(G) \cup f^*(G) = \{1, 2, \cdots, 2n-1\}$. Hence f is a super mean labeling on P_T. Now it is to show f is super mean on T_n. It is enough to show that $f^*(e_k) = f^*(e'_k)$.

Let $e_k = x_i x_j$ where $x_i x_j \in E(T_n)$. To get P_T, we have to delete e_k and adjoin e'_k at a distance d from x_i such that $e'_k = x_{i+r} x_{j-r}$. Since e'_k is an edge of P_T, it must be of the form $e'_k = x_{i+r} x_{i+r+1}$.

Hence

$$\begin{aligned}
(x_{i+r}, x_{j-r}) &= (x_{i+r}, x_{i+r+1}) \\
\frac{f(x_{i+r}) + f(x_{j-r})}{2} &= \frac{f(x_{i+r}) + f(x_{i+r+1})}{2} \\
&\Longrightarrow j = i + 2r + 1 \\
f^*(e_k) &= f^*(x_i x_j) \\
&= \frac{f(x_i) + f(x_j)}{2} \\
&= \frac{f(x_i) + f(x_{i+2r+1})}{2} = 2(i+r) \\
f^*(e'_k) &= f^*(x_{i+r}, x_{i+r+1}) \\
&= \frac{f(x_{i+r}) + f(x_{i+r+1})}{2} = 2(i+r)
\end{aligned}$$

Therefore,

$$f^*(e_k) = f^*(e'_k).$$

Thus, f is super mean on T_n also. Hence, T_n is a super mean graph.

Example 2.1 In Figure 2.2, we show a super-mean labeling on tree T_{20}.

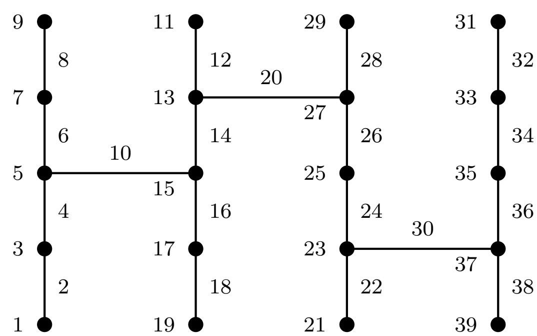

Fig 2.2

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Cycle and Armed Cap Cordial Graphs

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Abstract: Let G = (V,E) be a graph with p vertices and q edges. A *Cap (∧) cordial labeling* of a Graph G with vertex set V is a bijection from V to 0,1 such that if each edge uv is assigned the label

$$f(uv) = \begin{cases} 1, & \text{if } f(u)=f(v)=1, \\ 0, & \text{otherwise.} \end{cases}$$

with the condition that the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. Otherwise, it is called a *Smarandache ∧ cordial labeling* of G. A graph that admits a ∧ cordial labeling is called a ∧ cordial graph (CCG). In this paper, we proved that cycle C_n (n is even), bistar $B_{m,n}$, $P_m \odot P_n$ and Helm are ∧ cordial graphs.

Key Words: Cap cordial labeling, Smarandache ∧ cordial labeling, Cap cordial graph.

AMS(2010): 05C78.

§1. Introduction

A graph G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G which is called edges. Each pair $e = \{uv\}$ of vertices in E is called an edge or a line of G. In this paper, we proved that Cycle C_n (n : even), Bi-star $B_{m,n}$, $P_m \odot P_n$ and Helm are ∧ cordial graphs.

§2. Preliminaries

Let G = (V,E) be a graph with p vertices and q edges. A ∧ (cap) cordial labeling of a Graph G

[1]Received February 10, 2015, Accepted December 15, 2015.

with vertex set V is a bijection from V to (0, 1) such that if each edge uv is assigned the label

$$f(uv) = \begin{cases} 1, & \text{if } f(u) = f(v) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

with the condition that the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. Otherwise, it is called a *Smarandache* $\wedge$ *cordial labeling* of G.

The graph that admits a $\wedge$ cordial labeling is called a $\wedge$ cordial graph (CCG). we proved that cycle C_n (n is even), bistar $B_{m,n}$, $P_m \odot P_n$ and Helm are $\wedge$ cordial graphs

Definition 2.1 *A graph with sequence of vertices $u_1, u_2, \cdots, u_n$ such that successive vertices are joined with an edge, P_n is a path of length $n-1$.*

The closed path of length n is Cycle C_n.

Definition 2.2 *A $P_m \odot P_n$ graph is a graph obtained from a path P_m by joining a path of length P_n at each vertex of P_m.*

Definition 2.3 *A bistar is a graph obtained from a path P_2 by joining the root of stars S_m and S_n at the terminal vertices of P_2. It is denoted by B_m, n.*

Definition 2.4 *A Helm graph is a graph obtained from a Cycle C_n by joining a pendent vertex at each vertex of on C_n. It is denoted by $C_n \odot K_1$.*

§3. Main Results

Theorem 3.1 *A cycle C_n (n : odd) is a $\wedge$ cordial graph*

Proof Let $V(C_n) = \{u_i : 1 \leq i \leq n\}$, $E(C_n) = \{[(u_i u_{i+1}) : 1 \leq i \leq n-1] \bigcup (u_1 u_n)\}$. A vertex labeling $f : V(C_n) \to \{0, 1\}$ is defined by

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2}, \\ 1, & \frac{n+1}{2} \leq i \leq n \end{cases}$$

with an induced edge labeling $f^*(u_1 u_n) = 0$,

$$f^*(u_i u_{i+1}) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2}, \\ 1, & \frac{n+1}{2} \leq i \leq n-1, \end{cases}$$

Here $V_0(f) + 1 = V_1(f)$ and $E_0(f) = E_1(f) + 1$. It satisfies the condition

$$|V_0(f) - V_1(f)| \leq 1, \quad |E_0(f) - E_1(f)| \leq 1.$$

Hence, C_n is $\wedge$ cordial graph. □

For example, C_7 is $\wedge$ cordial graph as shown in the Figure 1.

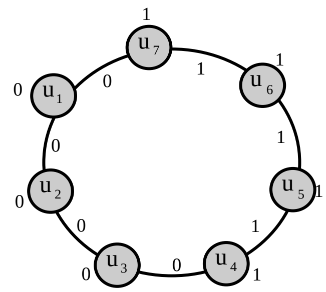

Figure 1 Graph C_7

Theorem 3.2 *A star S_n is a $\wedge$ cordial graph.*

Proof Let $V(S_n) = \{u, u_i : 1 \leq i \leq n\}$ and $E(S_n) = \{(uu_i) : 1 \leq i \leq n\}$. Define $f : V(S_n) \to 0, 1$ with vertex labeling as follows:

Case 1. If n is even, then $f(u) = 1$,

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{2}, \\ 1, & \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

and an induced edge labeling

$$f^*(uu_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{2}, \\ 1, & \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$

Here $V_0(f) + 1 = V_1(f)$ and $E_0(f) = E_1(f)$. It satisfies the condition

$$|V_0(f) - V_1(f)| \leq 1 \text{ and } |E_0(f) - E_1(f)| \leq 1.$$

Case 2. If n is odd, then $f(u) = 1$,

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n+1}{2}, \\ 1, & \frac{n+3}{2} \leq i \leq n \end{cases}$$

and with an induced edge labeling

$$f^*(uu_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n+1}{2}, \\ 1, & \frac{n+3}{2} \leq i \leq n. \end{cases}$$

Here $V_0(f) = V_1(f)$ and $E_0(f) = E_1(f) + 1$. It satisfies the condition

$$|V_0(f) - V_1(f)| \leq 1 \text{ and } |E_0(f) - E_1(f)| \leq 1.$$

Hence, S_n is $\wedge$ cordial graph. □

For example, S_5 and S_6 are cordial graphs as shown in the Figures 2 and 3.

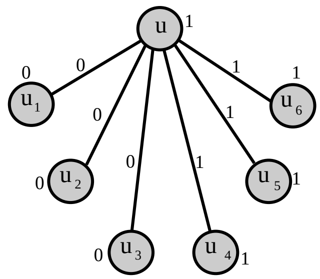

Figure 2 Graph S_6

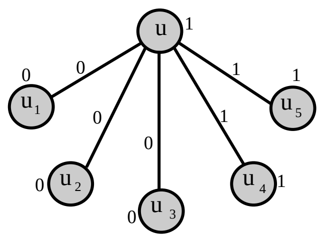

Figure 3 Graph S_5

Theorem 3.3 *A bistar $B_{m,n}$ is a $\wedge$ cordial graph.*

Proof Let $V(B_{m,n}) = \{(u,v),(u_i : 1 \leq i \leq m),(v_j : 1 \leq j \leq n)\}$ and $E(B_{m,n}) = \{[(uu_i) : 1 \leq i \leq m] \bigcup [(vv_i) : 1 \leq i \leq m] \bigcup [(uv)]\}$. Define $f : V(B_{m,n}) \to \{0,1\}$ by two cases.

Case 1. If $m = n$, the vertex labeling is defined by $f(u) = \{0\}$, $f(v) = \{1\}$, $f(u_i) = \{0, 1 \leq i \leq m\}$, $f(v_i) = \{1, 1 \leq i \leq m\}$ with an induced edge labeling $f^*(uu_i) = \{0, 1 \leq i \leq m\}$, $f^*(vv_i) = \{1, 1 \leq i \leq m\}$ and $f^*(uv) = 0$. Here $V_0(f) = V_1(f)$ and $E_0(f) = E_1(f) + 1$. It satisfies the condition

$$|V_0(f) - V_1(f)| \leq 1 \text{ and } |E_0(f) - E_1(f)| \leq 1.$$

Case 2. If $m < n$, the vertex labeling is defined by $f(u) = \{0\}$, $f(v) = \{1\}$, $f(u_i) = \{0, 1 \leq i \leq m\}$, $f(v_i) = \{1, 1 \leq i \leq m\}$,

$$f(v_{m+i}) = \begin{cases} 1, & i \equiv 1 \, mod \, 2, \\ 0, & i \equiv 0 \, mod \, 2, \qquad 1 \leq i \leq n - m, \end{cases}$$

with an induced edge labeling $f^*(uu_i) = \{0, 1 \leq i \leq m\}$, $f^*(vv_j) = \{1, 1 \leq j \leq m\}$, $f^*(uv) = 0$,

$$f^*(vvm+i) = \begin{cases} 1, & i \equiv 1 \, mod \, 2, \\ 0, & i \equiv 0 \, mod \, 2, \quad 1 \leq i \leq n-m. \end{cases}$$

Here, if $n-m$ is odd, then $V_0(f)+1 = V_1(f)$ and $E_0(f) = E_1(f)$; if $n-m$ is even, then $V_0(f) = V_1(f)$ and $E_0(f) = E_1(f)+1$. It satisfies the condition

$$|V_0(f) - V_1(f)| \leq 1 \text{ and } |E_0(f) - E_1(f)| \leq 1.$$

Case 3. If $n < m$, by substituting m by n and n by m in Case 2 the result follows.

Hence, $B_{m,n}$ is a $\wedge$ cordial graph. □

For example $B_{3,3}$, $B_{2,6}$ and $B_{6,2}$ are cordial graphs as shown in the Figures 4, 5 and 6.

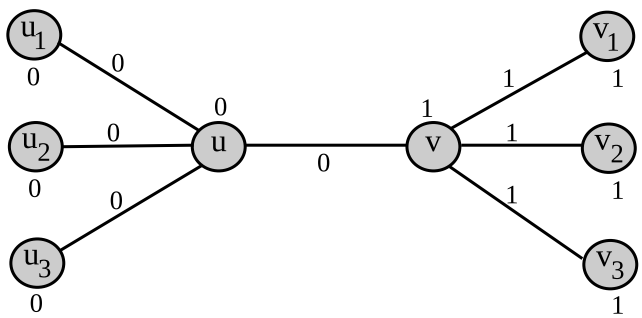

Figure 4 Graph $B_{3,3}$

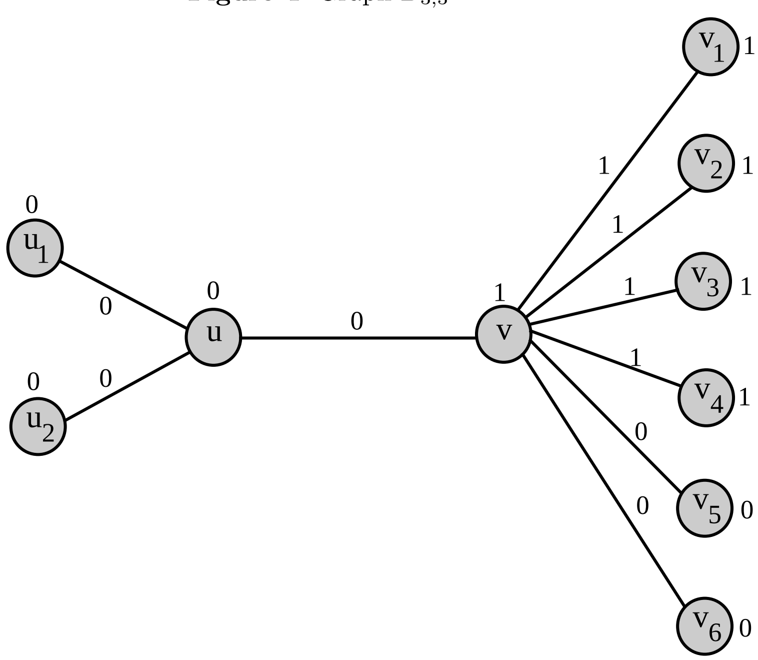

Figure 5 Graph $B_{2,6}$

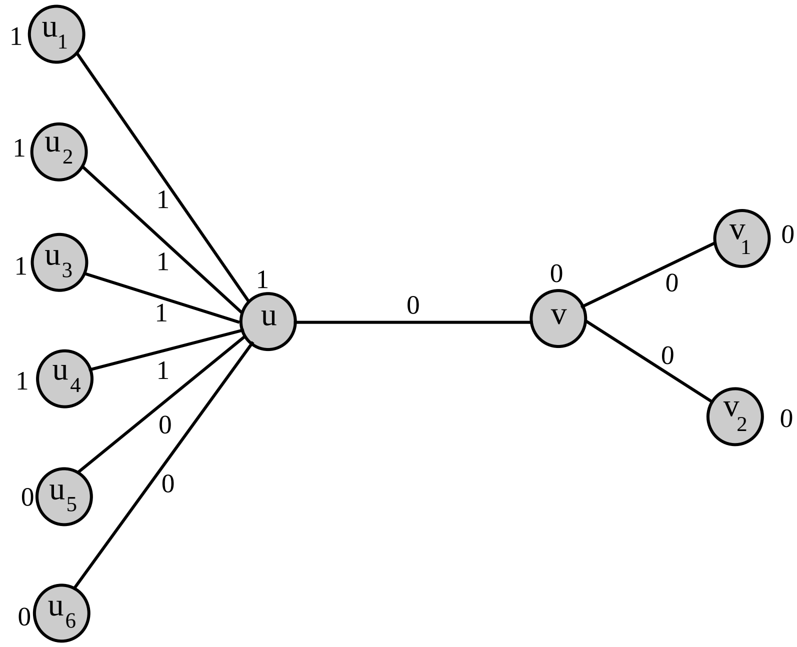

Figure 6 Graph $B_{6,2}$

Theorem 3.4 *A graph $P_m \ominus P_n$ is $\wedge$ cordial.*

Proof Let G be the graph $P_m \ominus P_n$ with $V(G) = \{[u_i : 1 \leqslant i \leqslant m], [v_{i_j} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n-1]\}$ and $E(G) = \{[(u_i u_{i+1}) : 1 \leqslant i \leqslant m-1] \bigcup [(u_i v_{i1}) : 1 \leqslant i \leqslant m] \bigcup [(v_{ij} v_{ij+1}) : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n-2]\}$. Define $f : V(G) \to \{0,1\}$ by cases following.

Case 1. If m is even, then the vertex labeling is defined by

$$f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{m}{2}, \\ 1, & \frac{m}{2}+1 \le i \le m, \end{cases} \qquad f(v_{ij}) = \begin{cases} 0, & 1 \le i \le \frac{m}{2}, 1 \leqslant j \leqslant n-1, \\ 1, & \frac{m}{2}+1 \le i \le m, 1 \leqslant j \leqslant n-1 \end{cases}$$

with an induced edge labeling

$$f^*(u_i u_{i+1}) = \begin{cases} 0, & 1 \le i \le \frac{m}{2}, \\ 1, & \frac{m}{2}+1 \le i \le m-1, \end{cases} \qquad f^*(u_i v_{i1}) = \begin{cases} 0, & 1 \le i \le \frac{m}{2}, \\ 1, & \frac{m}{2}+1 \le i \le m, \end{cases}$$

$$f^*(v_{ij} v_{ij+1}) = \begin{cases} 0, & 1 \le i \le \frac{m}{2}, 1 \leqslant j \leqslant n-2, \\ 1, & \frac{m}{2}+1 \le i \le m, 1 \leqslant j \leqslant n-2. \end{cases}$$

Here $V_0(f) = V_1(f)$ and $E_0(f) = E_1(f) + 1$. It satisfies the condition

$$|V_0(f) - V_1(f)| \le 1 \text{ and } |E_0(f) - E_1(f)| \le 1.$$

Case 2. If m is odd and n is odd, the vertex labeling is defined by

$$f(u_i)=\begin{cases}0, & 1\le i\le \frac{m-1}{2},\\ 1, & \frac{m+1}{2}\le i\le m,\end{cases}\qquad f(v_{ij})=\begin{cases}0, & t1\le i\le \frac{m-1}{2}, 1\leqslant j\leqslant n-1,\\ 1, & \frac{m+1}{2}\le i\le m, 1\leqslant j\leqslant n-1,\end{cases}$$

$$f(v_{\frac{m+1}{2}j})=\begin{cases}1, & 1\le j\le \frac{n}{2},\\ 0, & \frac{n}{2}+1\leqslant j\leqslant n-1\end{cases}$$

with an induced edge labeling

$$f^*(u_iu_{i+1})=\begin{cases}0, & 1\le i\le \frac{m-1}{2},\\ 1, & \frac{m+1}{2}+1\le i\le m-1,\end{cases}\qquad f^*(u_iv_{i1})=\begin{cases}0, & 1\le i\le \frac{m-1}{2},\\ 1, & \frac{m+1}{2}+1\le i\le m,\end{cases}$$

$$f^*(v_{ij}v_{ij+1})=\begin{cases}0, & 1\le i\le \frac{m-1}{2}, 1\leqslant j\leqslant n-2,\\ 1, & \frac{m+3}{2}\le i\le m, 1\leqslant j\leqslant n-2,\end{cases}$$

$$f^*(v_{\frac{m+1}{2}j}v_{\frac{m+1}{2}j+1})=\begin{cases}1, & 1\le j\le \frac{n-3}{2},\\ 0, & \frac{n-1}{2}\le j\leqslant n-2.\end{cases}$$

Here $V_0(f)+1=V_1(f)$ and $E_0(f)=E_1(f)$. It satisfies the condition

$$|V_0(f)-V_1(f)|\le 1 \text{ and } |E_0(f)-E_1(f)|\le 1.$$

Case 3. If m is odd and n is even, the vertex labeling is defined by

$$f(u_i)=\begin{cases}0, & 1\le i\le \frac{m-1}{2},\\ 1, & \frac{m+1}{2}\le i\le m,\end{cases}\qquad f(v_{ij})=\begin{cases}0, & 1\le i\le \frac{m-1}{2}, 1\leqslant j\leqslant n-1,\\ 1, & \frac{m+1}{2}\le i\le m, 1\leqslant j\leqslant n-1\end{cases}$$

with an induced edge labeling

$$f^*(u_iu_{i+1})=\begin{cases}0, & 1\le i\le \frac{m-1}{2},\\ 1, & \frac{m+1}{2}+1\le i\le m-1,\end{cases}\qquad f^*(u_iv_{i1})=\begin{cases}0, & 1\le i\le \frac{m-1}{2},\\ 1, & \frac{m+1}{2}+1\le i\le m,\end{cases}$$

$$f^*(v_{ij}v_{ij+1})=\begin{cases}0, & 1\le i\le \frac{m-1}{2}, 1\leqslant j\leqslant n-2,\\ 1, & \frac{m+1}{2}\le i\le m, 1\leqslant j\leqslant n-2,\end{cases}$$

$$f^*(v_{\frac{m+1}{2}j}v_{\frac{m+1}{2}j+1})=\begin{cases}1, & 1\le j\le \frac{n-4}{2},\\ 0, & \frac{n-2}{2}\le j\leqslant n-2.\end{cases}$$

Here $V_0(f)=V_1(f)$ and $E_0(f)=E_1(f)+1$. It satisfies the condition

$$|V_0(f)-V_1(f)|\le 1 \text{ and } |E_0(f)-E_1(f)|\le 1.$$

Hence, the graph $P_m \ominus P_n$ is $\wedge$ cordial. □

For example, $P_4 \ominus P_5$, $P_5 \ominus P_5$ and $P_5 \ominus P_6$ are $\wedge$ cordial as shown in Figures 7, 8 and 9.

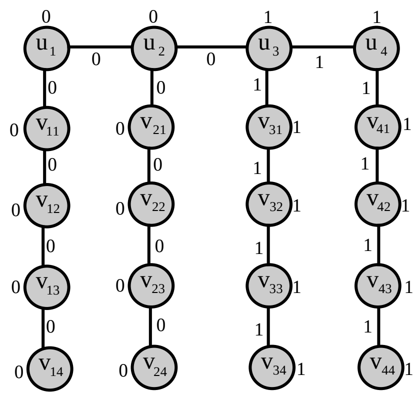

Figure 7 Graph $P_4 \ominus P_5$

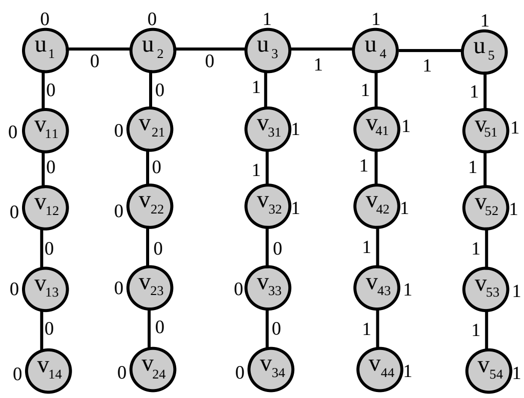

Figure 8 Graph $P_5 \ominus P_5$

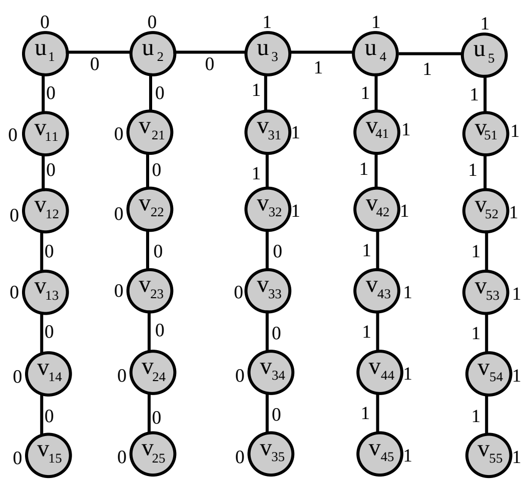

Figure 9 Graph $P_5 \ominus P_6$

Theorem 3.5 *A Helm* $(C_n \odot K_1)$ *is* $\wedge$ *cordial.*

Proof Let G be the graph $(C_n \odot K_1)$ with $V(G) = \{u_i, v_i : 1 \leqslant i \leqslant m\}$ and $E(G) = \{(u_i v_i) : 1 \leqslant i \leqslant m\}$. A vertex labeling on G is defined by $f(u_i) = \{1, 1 \leqslant i \leqslant m\}$, $f(v_i) = \{0, 1 \leqslant i \leqslant m\}$ with an induced edge labeling $f^*(u_i u_{i+1}) = \{1, 1 \leqslant i \leqslant m-1\}$, $f^*(u_m u_1) = 1$, $f^*(u_i v_i) = \{0, 1 \leqslant i \leqslant m\}$. Here $V_0(f) = V_1(f)$ and $E_0(f) = E_1(f)$. It satisfies the condition

$$|V_0(f) - V_1(f)| \leq 1 \text{ and } |E_0(f) - E_1(f)| \leq 1.$$

Hence, A Helm is $\wedge$ cordial. □

For example, a Helm $(C_6 \odot K_1)$ is $\wedge$ cordial as shown in the Figure 10.

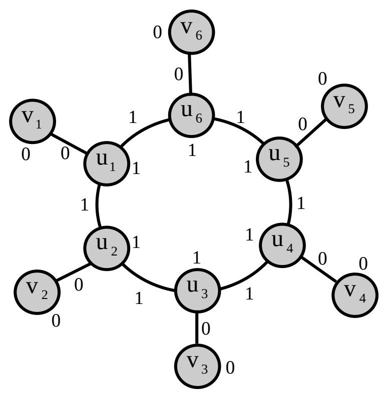

Figure 10 Graph $(C_6 \odot K_1)$

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International J.Math. Combin. Vol.2(2016), 153-160

Traffic Congestion – A Challenging Problem in the World

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Abstract: The purpose of this paper is to describe the problems which involves in the reduction of traffic congestion. In particular we use graph theoretical approach which is quite appropriate. We use crossing number technique to reduce traffic congestion. The minimum number of crossing points in a complete graph is given by $Cr(Kn) \leq \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]$ where[ ] represents greatest integer function. And we illustrate the result with counter examples.

Key Words: crossing number, complete graph, traffic control, edge connectivity, vertex connectivity.

AMS(2010): 05C90.

§1. Introduction

The crossing number (sometimes denoted as $C(G)$) of a graph G is the smallest number of pair wise crossings of edges among all drawings of G in the plane. In the last decade, there has been significant progress on a true theory of crossing numbers. There are now many theorems on the crossing number of a general graph and the structure of crossing critical graphs, whereas in the past, most results were about the crossing numbers of either individual graphs or the members of special families of graphs. The study of crossing numbers began during the Second World War with Paul Turan. In [1], he tells the story of working in a brickyard and wondering about how to design an efficient rail system from the kilns to the storage yards. For each kiln and each storage yard, there was a track directly connecting them. The problem he Consider was how to lay the rails to reduce the number of crossings, where the cars tended to fall off the tracks, requiring the workers to reload the bricks onto the cars. This is the problem of finding the crossing number of the complete bipartite graph. It is also natural to try to compute the crossing number of the complete graph. To date, there are only conjectures for the crossing numbers of these graphs Called Guys conjecture which suggest that crossing number of complete

[1]Received October 2, 2015, Accepted June 6, 2016.

graph Kn is given by $V(K_n) = Z(n)[2][3]$ where [ ] represents greatest integer function.

$$z(n) = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]$$

which can also be written as

$$z(n) = \begin{cases} \frac{1}{64}n(n-2)^2(n-4) & n \text{ even} \\ \frac{1}{64}(n-1)^2(n-3)^2 & n \text{ odd} \end{cases}$$

Guy prove it for $n \leq 10$ in 1972 in 2007 Richter prove it for $n \leq 12$ For any graph G, we say that the crossing number $c(G)$ is the minimum number of crossings with which it is possible to draw G in the plane. We note that the edges of G need not be straight line segments, and also that the result is the same whether G is drawn in the plane or on the surface of a sphere. Another invariant of G is the rectilinear crossing number, $c(G)$, which is the minimum number of crossings when G is drawn in the plane in such a way that every edge is a straight line segment. We will find by an example that this is not the same number obtained by drawing G on a sphere with the edges as arcs of great circles. In drawing G in the plane, we may locate its vertices wherever it is most convenient. A plane graph is one which is already drawn in the plane in such a way that no two of its edges intersect. A planar graph is one which can be drawn as a plane graph [6]. In terms of the notation introduced above, a graph G is planar if and only if $c(G) = 0$. The earliest result concerning the drawing of graphs in the plane is due to Fary [4] [7], who showed that any planar graph (without loops or multiple edges) can be drawn in the plane in such a way that every Edge is straight. Thus Farys result may be rephrased: if $c(G) = 0$, then $c(G) = 0$. In a drawing, the nodes of the graph are mapped into points of a plane, and the arcs into continuous curves of the plane, no three having a point in common. A minimal drawing does not contain an arc which crosses itself, nor two arcs with more than one point in common, [5],[8]. In general for a set of n line segments, there can be up to $\bigcirc(n^2)$ intersection points, since if every segment intersects every other segment, there would be

$$\frac{n(n-1)}{2} = \bigcirc(n^2)$$

Crossing points to compute them all we require $\bigcirc(n^2)$ algorithm.

The *traffic theory* is a physical phenomenon that aims at understanding and improving automobile traffic, and the problem associated with it such as traffic congestion [9]. The traffic control problem is to minimize the waiting time of the public transportation while maintaining the individual traffic flow optimally [10]. Significant development of traffic control systems using traffic lights have been achieved since the first traffic controller was installed in London in 1868. The first green wave was realized in Salt Lake City (U.K.) in 1918, and the first area traffic controller was introduced in Toronto in 1960. At the beginning, electromechanical devices were used to perform traffic control. Then Intelligent Transportation System (ITS) is used extensively in urban areas to control traffic at an intersection [11]. The traffic data in a particular region can be used to direct the traffic flow to improve traffic output without adding new roads. In order to collect accurate traffic data semi conductor-based controllers known as sensors were placed in different places to collect traffic information are used in traffic control system [11], [12], [13]. Nowadays, microprocessor based controller are used in Traffic Control Systems. The combinatorial approach to the optimal traffic control problem was founded by Stoffers [14] in 1968 by introducing the Compatibility Graph of traffic streams. One of the main uses of traffic theory is the development of traffic models which can be used for estimation, prediction, and control related tasks for the automobile traffic process. The term Intelligent Transportation System (ITS) refers to information and communication technology applied to transport infrastructure and vehicles, that improves transport outcomes such as transport safety, transport productivity, transport reliability, informed traveler choice, environmental performance etc. [15] , [16]. ITS mainly comes from the problems caused by traffic congestion and synergy of new information technology for simulation, real time control and communication networks. Traffic congestion has been increased world wide as a result of increased motorization, urbanization, population growth and changes in population density. Congestion reduces efficiency of transportation infrastructure and increases travel time, air pollution and fuel consumption. At the beginning of 1920, in United States large increase in both motorization and urbanization led to the migration of the population from sparsely populated rural areas and densely packed urban areas into suburbs (sub urban areas). Intelligent Transport Systems vary in technologies applied, from basic management system such as car navigation; traffic signal control systems; container management system; variable message sign; automatic number plate recognition or speed cameras to monitor applications; such as security CCTV systems; and to more advanced applications that integrate live data and feedback from a number of other sources, such as parking guidance and information systems; weather information etc. Additional predictive techniques are being developed to allow advanced modeling and comparison with historical data. The traffic flow predictions will be delivered to the drivers via different channels such as roadside billboards, radio stations, internet, and on vehicle GPS (Global Positioning Systems) systems. One of the components of an ITS is the live traffic data collection. To collect accurate traffic data sensors have to be placed on the roads and streets to measure the flow of traffic. Some of the constituent technologies implemented in ITS are namely, Wireless Communication, Computational technologies, Sensing technologies, Video Vehicle Detection etc. Urban traffic congestion is a significant and growing problem in many parts of the world. Moreover, as congestion continues to increase, the conventional approach of "building more roads" doesn't always work for a variety of political, financial, and environmental reasons. In fact, building new roads can actually compound congestion, in some cases, by inducing greater demands for vehicle travel that quickly eat away the additional capacity? Against this backdrop of serious existing and growing congestion traffic Control techniques and information systems are needed that can substantially increase capacity and Improve traffic flow efficiency. Application of ITS technologies in areas such as road user information and navigation systems, improved traffic control systems and vehicle guidance and control systems has significant potential for relieving traffic congestions.

Theorem 1.1 *The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.*

Theorem 1.2 *The vertex connectivity of any graph G can never exceed the edge connectivity of G.*

Theorem 1.3 *The maximum vertex connectivity one can achieve in a graph of n vertices and e edges is $e \geq n-1$ Thus we conclude that* $\text{vertexconnectivity} \leq \text{edgeconnectivity} \leq \frac{2e}{n}$.

Definition 1.4 *A graph $G(v,e)$ where v is the set of vertices and e the set of edges is said to be complete if degree of each vertex is $v-1$.*

Definition 1.5 *The number of edges incident on a vertex is said to be degree of the vertex.*

Proposed Solution 1.6 As congestion continues to increase, the conventional approach of *building more roads* doesn't always work for a variety of political, financial, and environmental reasons. In fact, building new roads can actually compound congestion. There is no particular technique which reduces traffic congestion. Numbers of techniques are simultaneously required to curb this problem. Traffic congestion is one of the challenging problem in the world the aim of this research paper is that how to curb this problem. Before giving the solution to the problem we would like to introduce you the graph theoretical approach of the problem, using underlying graphs. We represent various cities by vertices and roads connected them by edges. Since every city must be connected with all other cities in particular geographical area so first of all we are dealing with complete graphs then we shall remove all the edges in the graph in such a way that maximum crossing pointes will be removed and there is no effect in the connectivity. Following are the techniques require curbing traffic congestion.

Definition 1.7 *Let $G(v,e)$ be a complete graph v the set of vertices and e the set of edges. the crossing number $Cr(G)$ of a complete graph $G(v,e)$ is the least number of crossings, common points of two arcs other than a vertex, in any drawing of graph in a plane (or on a sphere) in a drawing the vertex of the graph are mapped into points of a plane and the arcs into continue curves of the plane no three having a point in common, unless it be an end point (vertex) of the arc. A drawing which exhibits a crossing number is called minimal a minimal drawing does not contain an arc which crosses itself nor two arcs with more than one point in common. For any complete graph K_n it has been shown that the minimum number of crossing points is given by $Cr(Kn) \leq \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]$, where[ ] represents greatest integer function.*

Since minimum number of crossing points means minimum number of interruptions on roads which minimize the waiting time of the traffic participants so traffic congestion is reduces. After drawing the graph with minimum crossings we remove all those edges in a graph which does not effect the connectivity of the graph but reduce more crossings so the graph becomes more efficient as shown in Figure 1 here red dots represent vertices and yellow dots represent crossing points.

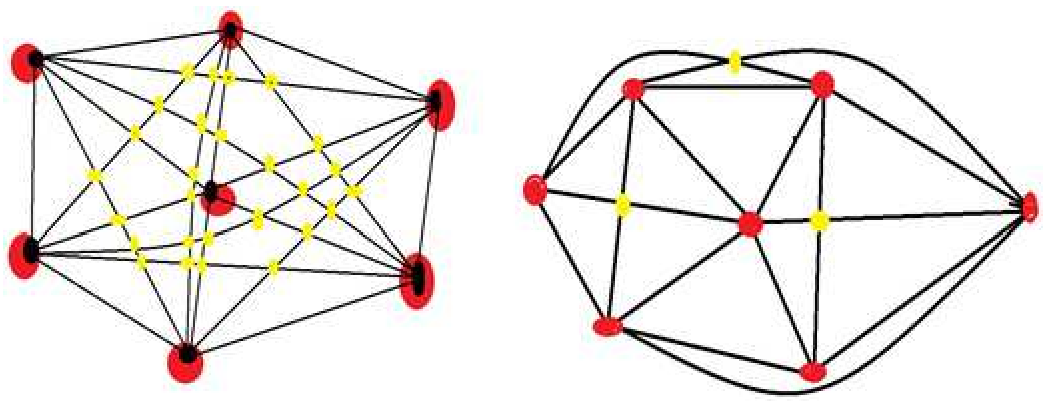

Figure 1

We shall keep this point in mind that no three edges has a point in common if it is necessary then we have to keep other crossing point at least one kilometer away from one another as shown in Figure 2

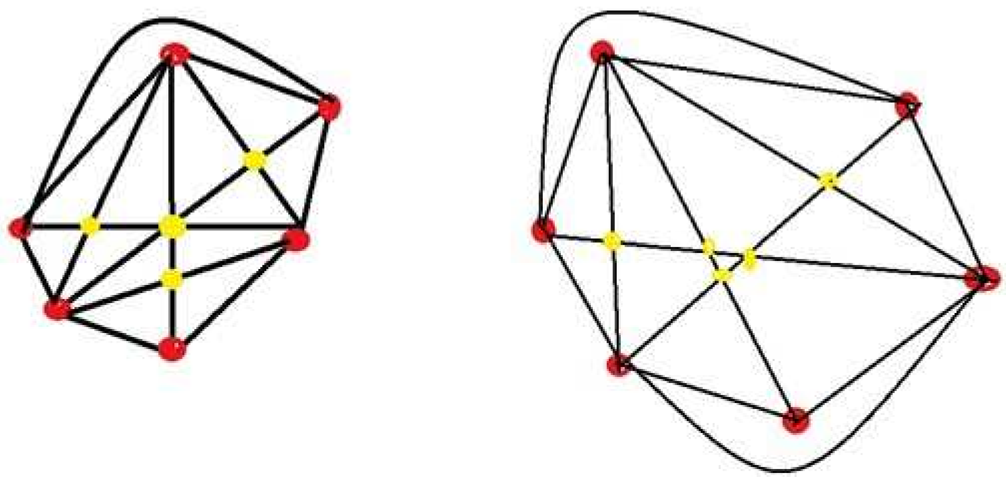

Figure 2

Because more than one crossing at point will increase interruption on traffic flow, first of all we have to try our best to reduce the intensity of crossing points, if it is not possible then flyovers should be constructed at every crossing so that there is no interruption on traffic flow, in this case only slow moments are possible not traffic jam. The above crossing point technique will reduce traffic congestion in a large extent. if there are maximum crossing points on the roads then maximum traffic congestion is possible so minimum crossing means minimum traffic congestion so we have to reduce the crossing points then traffic congestion is reduced. The aim to reduce traffic congestion is to reduce the crossing points. There are certain crossing points where more than two roads cross each other and traffic lights are imposed to allow the traffic flow alternatively one by one. so traffic congestion is increased because more and more vehicles have to be stopped on the roads First of all we shall try over best to reduce those crossing points where more than two crossing points exist which is quite possible, if it is necessary then, on these intersection pointes where more than two crossings points exist we have to make flyovers (as shown in fig 1.3 below these flyovers have already designed in certain parts of the world for this purpose) in such a way that there should not be any crossing point and traffic flow should be in continuous manner may be some times there are slow moments but still it will reduce traffic congestion.

Figure 3

§3. Methods

Method 3.1 There should be exclusive lanes for public transport so that private transport system does not affect the moment of the public transport.

Method 3.2 A turning restriction is imposed on the traffic, and vehicles are allow to turn on certain places turning points are at least one kilometer away from the crossing points if the turning points are at crossing points it will definitely increase traffic congestion, and proper fly-overs system is imposed as shown in Figure 4, which will automatically reduce traffic congestion in a large extent.

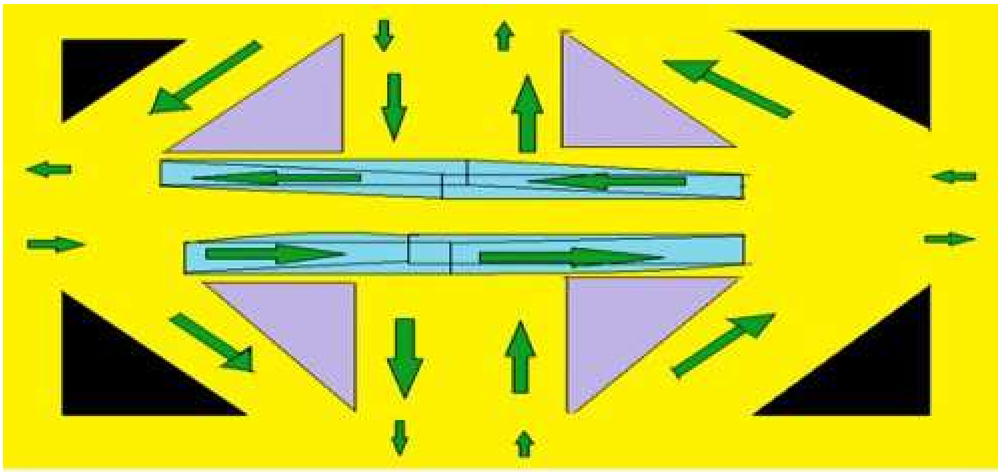

Figure 4

Method 3.3 A double win method has already imposed in certain cities of the world and has been proved to be very happy one. A congestion charge is essentially an economic method of regulating traffic by imposing fees on vehicle users that travels a city more crowed roads, but charge vary by city to city depending up on crowed on the city.
Method 3.4 The Parking restrictions on road side should be banned to reduce congestion.
Method 3.5 Remove some link roads at high efficiency points. Then we have to connect them other side so that minimum crossings are possible.

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The people who get on in this world are the people who get up and look for circumstances they want, and if they cannot find them. they make them.

By George Bernard Shaw, a British dramatist .

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June 2016

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An International Journal on Mathematical Combinatorics

ISSN 1937-1055