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Numeralogy (I)
or
Properties of Numbers

In Florentin Smarandache: “Collected Papers”, vol. II. Chisinau (Moldova): Universitatea de Stat din Moldova, 1997.

1) Reverse sequence:

1, 21, 321, 4321, 54321,654321,7654321, 87654321,987654321,10987654321, 1110987654321, 121110987654321, ...

2) Multiplicative sequence:

2, 3 ,6, 12, 18, 24, 36, 48, 54, ...

General definition: if m_1, m_2, are the first two terms of the sequence, then m_k, for $k \geq 3$, is the smallest number equal to the product of two previous distinct terms.

All terms of rank ≥ 3 are divisible by m_1 and m_2.

In our case the first two terms are 2, respectively 3.

3) Wrong numbers:

(A number $n = \overline{a_1 a_2 \ldots a_k}$, of at least two digits, with the following property:

the sequence $a_1, a_2, \ldots a_k, b_{k+1}, b_{k+2}, \ldots$ (where b_{k+i} is the product of the previous k terms, for any $i \geq 1$) contains n as its term.)

The author conjectures that there is no wrong number (!)

Therefore, this sequance is empty.

4) Impotent numbers:

2, 3, 4, 5, 7, 9, 11, 13, 17, 19, 23, 25, 29, 31, 41, 43, 47, 49, 53, 59, 61, ...

(A number n those proper divisors product is less than n.)

Remark: this sequence is $\{p, p^2$; where p is a positive prime$\}$.

5) Random sieve:

1, 5, 6, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 53, 59, ...

General definition:

- choose a positive number u_1 at random;
- delete all multiples of all its divisors, exept this number;
- chose another number u_2 greater than u_1 among those remaining;
- delete all multiples of all its divisors, ecxept this second number;

... so on.

The remaining numbers are all coprime two by two.

The sequence obtained $u_k, k \geq 1$, is less dense than the prime number sequence, but it tends to the prime number sequence as k tends to infinite. That's why this sequence may be important.

In our case, $u_1 = 6, u_2 = 19, u_3 = 35, \ldots$.

6) Cubic base:

0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 22, 23, 24, 25, 26, 27, 30, 31, 32, 100, 101, 102, 103, 104, 105, 106, 107, 110, 111, 112, 113, 114, 115, 116, 117, 120, 121, 122, 123, ...

(Each number n written in the cubic base.)

(One defines over the set of natural numbers the following infinte base: for $k \geq 1$ $s_k = k\hat{}3$.)

We prove that every positive integer A may be uniquely written in the cubic base as:

$A = (\overline{a_n \ldots a_2 a_1})_{(C3)} \stackrel{def}{=} \sum_{i=1}^{n} a_i c_i$, with $0 \leq a_1 \leq 7, 0 \leq a_2 \leq 3, 0 \leq a_3 \leq 2$ and $0 \leq a_i \leq 1$ for $i \geq 4$, and of course $a_n = 1$, in the following way:

- if $c_n \leq A < c_{n+1}$ then $A = c_n + r_1$;

- if $c_m \leq r_1 < c_{m+1}$ then $r_1 = c_m + r_2, m < n$;

and so on untill one obtains a rest $r_j = 0$.

Therefore, any number may be written as a sum of cubes (1 not counted as cube - being obvious)+e, where $e = 0, 1, \ldots,$ or 7.

If we denote by $c(A)$ the superior square part of A (i.e.. the largest cube less than or equal to A), then A is written in the cube base as:

$$A = c(A) + c(A - c(A)) + c(A - c(A) - c(A - c(A))) + \ldots.$$

This base may be important for partitions with cubes.

7) Anti-symmetric sequence:

11, 1212, 123123, 12341234, 1234512345, 123456123456, 12345671234567, 1234567812345678, 123456789123456789, 12345678910123456789 10, 1234567891011123456789 1011, 123456789101112123456789101112, ...

8-16) Recurence type sequences:

A. $1, 2, 5, 26, 29, 677, 680, 701, 842, 845, 866, 1517, 458330, 458333, 458354, \ldots$

($ss2(n)$ is the smallest number, strictly greater than the previous one, which is the squares sum of two previous distinct terms of the sequence;
in our particular case the first two terms are 1 and 2.)

Recurrence definition: 1) The number $a \leq b$ belong to SS2;

2) If b, c belong to SS2, then $b^2 + c^2$ belong to SS2 too;

3) Only numbers, obtained by rules 1) and/or 2) applied a finite number of times, belongs to SS2.

The sequence (set) SS2 is increasingly ordered.

[Rule 1) may be changed by: the given numbers $a_1, a_2, \ldots, a_k$, where $k \geq 2$, belongs to SS2.]

B. 1, 1, 2, 4, 5, 6, 16, 17, 18, 20, 21, 22, 25, 26, 27, 29, 30, 31, 36, 37, 38, 40, 41, 42, 43, 45, 46, ...

($ss1(n)$ is the smallest number, strictly greater than the previous one (for $n \geq 3$), which is the squares sum of one or more previous distinct terms of the sequence;
in our particular case the first term is 1.)

Recurrence definition:

1) The number a belongs to SS1;

2) If $b_1, b_2, \ldots, b_k$ belongs to SS1, where $k \geq 1$, then $b_1\hat{}2 + b_2\hat{}2 + \ldots + b_k\hat{}2$ belongs to SS1 too;

3) Only numbers, obtained by reles 1) and/or 2) applied a finite number of times, belong to SS1.

The sequence (set) SS1 is increasingly ordered.

[Rule 1) may be changed by: the given numbers $a_1, a_2, \ldots, a_k$, where $k \geq 1$, belong to SS1.]

C. 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, ...

($nss2(n)$ is the smallest number, strictly greater than the previous one, which is NOT the squares sum of two previous distinct terms of the sequence;
in our particular case the first two terms are 1 and 2.)

Recurrence definition:

1) The numbers $a \leq b$ belong to NSS2;

2) If b, c belomg to NSS2, then $b\hat{}2 + c\hat{}2$ DOES NOT belong to NSS2; any other numbers belong to NSS2;

3) Only numbers, obtained by rules 1) and/or 2) applied a finite number of times, belong to NSS2.

The sequence (set) NSS2 is increasingly ordered.

[Rule 1) may be changed by: the given numbers $a_1, a_2, \ldots, a_k$, where $k \geq 2$, belong to NSS2.]

D. 1, 2, 3, 6, 7, 8, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 42, 43, 44, 47, ...

($nss1(n)$ is the smallest number, strictly greater than the previous one, which is NOT the squares sum of the one or more previous distinct terms of the sequence;
in our particular case the first term is 1.)

Recurrence definition:

1) The number a belongs to NSS1;

2) If $b_1, b_2, \ldots, b_k$ belongs to NSS1, where $k \geq 1$, then $b_1\hat{}2 + b_2\hat{}2 + \ldots + b_k\hat{}2$ DO NOT belong to NSS1; any other numbers belong to NSS1;

3) Only numbers, obtained by rules 1) and/or 2) applied a finite number of times, belong to NSS1.

The sequence (set) NSS1 is increasingly ordered.

[Rule 1) may change by: the given numbers $a_1, a_2, \ldots, a_k$, where $k \geq 1$, belong to NSS1.]

E. 1, 2, 9, 730, 737, 389017001, 389017008, 389017729, ...

($cs2(n)$ is the smallest number, strictly greater than the previous one, which is the cubes sum of two previous distinct terms of the sequence;
in our particular case the first two terms are 1 and 2.)

Recurrence definition:

1) The numbers $a \leq b$ belong to CS2;

2) If c, d belong to CS2, then $c\hat{}3 + d\hat{}3$ belongs to CS2 too;

3) Only numbers, obtained by rules 1) and/or 2) applied a finite number of times, belong to CS2.

The sequence (set) CS2 is increasingly ordered.

[Rule 1) may be changed by: the given numbers $a_1, a_2, \ldots a_k$, where $k \geq 2$, belong to CS2.]

F. 1, 1, 2, 8, 9, 10, 512, 513, 514, 520, 521, 522, 729, 730, 731, 737, 738, 739, 1241, ...

($cs1(n)$ is the smallest number, strictly greater than the previous one (for $n \geq 3$), which is the cubes sum of one or more previous distinct terms of the sequence;
in our particular case the first term is 1.)

Recurrence definition:

1) The numbers $a \leq b$ belong to CS1;

2) If $b_1, b_2, \ldots, b_k$ belongs to CS1, where $k \geq 1$, then $b_1\hat{}3 + b_2\hat{}3 + \ldots + b_k\hat{}3$ belong to CS1 too;

3) Only numbers, obtained by rules 1) and/or 2) applied a finite number of times, belong to CS1.

The sequence (set) CS1 is increasingly ordered.

[Rule 1) may be changed by: the given numbers $a_1, a_2, \ldots a_k$, where $k \geq 2$, belong to CS1.]

G. 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 36, 37, 38, ...

($ncs2(n)$ is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of two previous distinct terms of the sequence; in our particular case the first term is 1 and 2.)

Recurrence definition:

1) The numbers $a \leq b$ belong to NCS2;

2) If c, d belong to NCS2, then $c\hat{}3+d\hat{}3$ DOES NOT belong to NCS2; any other numbers do belong to NCS2;

3) Only numbers, obtained by rules 1) and/or 2) applied a finite number of times, belong to NCS2.

The sequence (set) NCS2 is increasingly ordered.

[Rule 1) may be changed by: the given numbers $a_1, a_2, \ldots a_k$, where $k \geq 2$, belong to NCS2.]

H. 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 32, 33, 34, 37, 38, 39, ...

($ncs1(n)$ is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of the one or more previous distinct terms of the sequence;
in our particular case the first term is 1.)

Recurrence definition:

1) The number a belongs to NCS1;

2) If $b_1, b_2, \ldots, b_k$ belongs to NCS1, where $k \geq 1$, then $b_1\hat{}2 + b_2\hat{}2 + \ldots + b_k\hat{}2$ DO NOT belong to NCS1; any other numbers belong to NCS1;

3) Only numbers, obtained by rules 1) and/or 2) applied a finite number of times, belong to NCS1.

The sequence (set) NCS1 is increasingly ordered.

[Rule 1) may change by: the given numbers $a_1, a_2, \ldots, a_k$, where $k \geq 1$, belong to NCS1.]

I. General-recurrence type sequence:

General recurrence definition:

Let $k \geq j$ be natural numbers, $a_1, a_2, \ldots, a_k$ given elements, and R a j-relationship (relation among j elements).

Then:

1) The elements $a_1, a_2, \ldots, a_k$ belong to SGR.

2) If $m_1, m_2, \ldots, m_j$ belong to SGR, then $R(m_1, m_2, \ldots, m_j)$ belongs to SGR too.

3) only elements, obtained by rules 1) and/or 2) applied a finite number of times, belong to SGR.

The sequence (set) SGR is increasingly ordered.

Method of consttruction of the general recurrence sequence:

- level 1: the given elements $a_1, a_2, \ldots, a_k$ belong to SGR;
- level 2: apply the relationship R for all combinations of j elements among $a_1, a_2, \ldots, a_k$; the results belong to SGR too;

order all elements of level 1 and 2 together;

....

- level $i+1$:

if $b_1, b_2, \ldots, b_m$ are all elements of levels $1, 2, \ldots, i-1$, and $c_1, c_2, \ldots, c_n$ are all elements of level i, then apply the relationship R for all combinations of j elements among $b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_n$ such that at least an element is from the level i;

the results belong to SGR too;

order all elements of levels i and $i+1$ together;

and so on...

17)-19) Partition type sequences:

A. 1, 1, 1, 2, 2, 2, 2, 3, 4, 4, ...

(How many times is n written as sum of non-nul squares, desregarding the terms order; for example:

$$9 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$$
$$= 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 2^2$$
$$= 1^2 + 2^2 + 2^2$$
$$= 3^2,$$

therefore $ns(9) = 4$.)

B. 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, ...

(How many times is n written as a sum of non-null cubes, desregarding the terms order; for example:

$9 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$

$= 1^3 + 2^3$,

therefore $nc(9) = 2$.)

C. General-partition type sequence:

Let f be an arithmetic function, and R a relation among numbers.

{ How many times can n be written under the form:

$$n = R(f(n_1), f(n_2), \ldots, f(n_k))$$

for some k and $n_1, n_2, \ldots, n_k$ such that

$$n_1 + n_2 + \ldots + n_k = n?\}$$

20) Concatenate sequence:

1, 22, 333, 4444, 55555, 666666, 7777777, 88888888, 999999999, 10101010101010101010, 1111111111111111111111, 121212121212121212121212, 13131313131313131313131313, 1414141414141414141414141414, 151515151515151515151515151515,...

21) Triangular base:

1, 2, 10, 11, 12, 100, 101, 102, 110, 1000, 1001, 1002, 1010, 1011, 10000, 10001, 10002, 10010, 10011, 10012, 100000, 100001, 100002, 100010, 100011, 100012, 100100, 1000000, 1000001, 1000002, 1000010, 1000011, 1000012, 1000100,...

(Numbers written in the triangular base, defined as follows: $t(n) = n(n+1)/2$, for $n \geq 1$.)

22) Double factorial base:

1, 10, 100, 101, 110, 200, 201, 1000, 1001, 1010, 1100, 1101, 1110, 1200, 10000, 1001, 10010, 10100, 10101, 10110, 10200, 10201, 11000, 11001, 11010, 11100, 11101, 11110, 11200, 11201, 12000, ...

(Numbers written in the double factorial base, defined as follows: $df(n) = n!!$)

23) Non-multiplicative sequence:

General definition: let $m_1, m_2, \ldots, m_k$ be the first k given terms of the sequence, where $k \geq 2$;

then m_i, for $i \geq k+1$, is the smallest number not equal to the product of k previous distinct terms.

24) Non-arithmetic progression:

1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, 64, ...

General definition: if m_1, m_2, are the first two terms of the sequence, then m_k, for $k \geq 3$, is the smallest number such that no 3-term arithmetic progression is in the sequence.

in our case the first two terms are 1, respectively 2.

Generalization: same initial conditions, but no i-term arithmetic progression in the sequence (for a given $i \geq 3$).

25) Prime product secuence:

2, 7, 31, 211, 2311, 30031, 510511, 9699691, 223092871, 6469693231, 200560490131, 7420738134811, 304250263527211, ...

$P_n = 1 + p_1 p_2 \ldots p_n$, where p_k is the k-th prime.

Question: How many of them are prime?

26) Square product sequence:

2, 5, 37, 577, 14401, 518401, 25401601, 1625702401, 131681894401, 13168189440001, 1593350922240001, ...

$S_n = 1 + s_1 s_2 \ldots s_n$, where s_k is the k-th square number.

Question: How many of them are prime?

27) Cubic product sequence:

2, 9, 217, 13825, 1728001, 373248001, 128024064001, 65548320768001, ...

$C_n = 1 + c_1 c_2 \ldots c_n$, where c_k is the k-th cubic number.

Question: How many of them are prime?

28) Factorial product sequence:

2, 3, 13, 289, 34561, 24883201, 125411328001, 5056584744960001, ...

$F_n = 1 + f_1 f_2 \ldots f_n$, where f_k is the k-th factorial number.

Question: How many of them are prime?

29) U-product sequence {generalization}:

Let $u_n, n \geq 1$, be a positive integer sequence. Then we define a U-sequence as follows:

$U_n = 1 + u_1 u_2 \ldots u_n$.

30) Non-geometric progression:

1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 24, 26, 27, 29, 30, 31, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 50, 51, 53, ...

General definition: if m_1, m_2, are the first two terms of the sequence, then m_k, for $k \geq 3$, is the smallest number such that no 3-term geometric progression is in the sequence.

In our case the first two terms are 1, respectively 2.

31) Unary sequence:

11, 111, 11111, 1111111, 11111111111, 1111111111111, 11111111111111111, 1111111111111111111, 11111111111111111111111, 11111111111111111111111111111, 1111111111111111111111111111111, ...

$u(n) = \overline{11\ldots1}, p_n$ digits of "1", where p_n is the n-th prime.

The old quenstion: are there an infinite number of primes belomging to the sequence?

32) No prime digits sequence:

1, 4, 6, 8, 9, 10, 11, 1, 1, 14, 1, 16, 1, 18, 19, 0, 1, 4, 6, 8, 9, 0, 1, 4, 6, 8, 9, 40, 41, 42, 4, 44, 4, 46, 4, 48, 49, 0, ...

(Take out all prime digits of n.)

33) No square digits sequence:

2, 3, 5, 6, 7, 8, 2, 3, 5, 6, 7, 8, 2, 2, 22, 23, 2, 25, 26, 27, 28, 2, 3, 3, 32, 33, 3, 35, 36, 37, 38, 3, 2, 3, 5, 6, 7, 8, 5, 5, 52, 52, 5, 55, 56, 57, 58, 5, 6, 6, 62, ...

(Take out all square degits of n.)

34) Concatenated prime sequence:

2, 23, 235, 2357, 235711, 23571113, 2357111317, 235711131719, 23571113171923, ...

Conjecture: there are infinetely many primes among these numbers!

35) Concatenated odd sequence:

1, 13, 135, 1357, 13579, 1357911, 135791113, 13579111315, 1357911131517, ...

Conjecture: there are infinetely many primes among these numbers!

36) Concatenated even sequence:

2, 24, 246, 2468, 246810, 24681012, 2468101214, 246810121416, ...

Conjecture: none of them is a perfect power!

37) Concatenated S-sequence {generalization}:

Let $s_1, s_2, s_3, s_4, \ldots, s_n, \ldots$ be an infinite sequence (noted by S.)

Then:

$s_1, \overline{s_1, s_2}, \overline{s_1 s_2 s_3}, \overline{s_1 s_2 s_3 s_4}, \overline{s_1 s_2 s_3 s_4 \ldots s_n}, \ldots$ is called the Concatenated S-sequence.

Question:

a) How many terms of the Concatenated S-sequence belong to the initial S-sequence?

b) Or, how many terms of the Concatenated S-sequence verify the realtion of other given sequences?

The first three cases are particular.

Look now at some other examples, when S is the sequence of squares, cubes, Fibonacci respectively (and one can go so on):

Concatenated Square sequence:

1, 14, 149, 14916, 1491625, 149162536, 14916253649, 1491625364964, ...

How many of them are perfect squares?

Concatenated Cubic sequence:

1, 18, 1827, 182764, 182764125, 182764125216, 1827631252166343, ...

How many of them are perfect cubes?

Concatenated Fibonacci sequence:

1, 11, 112, 1123, 11235, 112358, 11235813, 1123581321, 112358132134, ...

Does any of these numbers is a Fibonacci number?

References

[1] F.Smarandache, "Properties of Numbers", University of Craiova Archives, 1975; [see also Arzona State University Special Collections, Tempe, Arizona, USA].

38) Teh Smallest Power Function:

$SP(n)$ is the smallest number m such that $m\hat{\ }m$ is divisible by n.

The following sequence $SP(n)$ is generated:

1, 2, 3, 2, 5, 6, 7, 4, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, 21, 22, 23, 6, 5, 26, 3, 14, 29, 30, 31, 4, 33, 34, 35, 6, 37, 38, 39, 20, 41, 42, ...

Remark:

If p is prime, then $SP(n) = p$.

If r is square free, then $SP(r) = r$.

If $n = (p_1\hat{\ }s_1) \cdot \ldots \cdot (p_k\hat{\ }s_k)$ and all $s_i \leq p_i$, then $SP(n) = n$.

If $n = p\hat{\ }s$, where p is prime, then:

$$SP(n) = \begin{cases} p, \text{ if } 1 \le s \le p; \\ p\hat{\ }2, \text{ if } p+1 \le s \le 2p\hat{\ }2; \\ p\hat{\ }3, \text{ if } 2p\hat{\ }2+1 \le s \le 3p\hat{\ }3; \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ p\hat{\ }t, \text{ if } (t-1)p\hat{\ }(t-1)+1 \le s \le tp\hat{\ }t. \end{cases}$$

Generally, if $n = (p_1\hat{\ }s_1) \cdot \ldots \cdot (p_k\hat{\ }s_k)$, with all p_i prime, then:

$SP(n) = (p_1\hat{\ }t_1) \cdot \ldots \cdot (p_k\hat{\ }t_k)$, where $t_i = u_i$ if $(u_i - 1)p\hat{\ }(u_i - 1)+ \le s_i \le u_i p_i\hat{\ }u_i$ for $1 \le i \le k$.

39) A $3n$-digital subsequence:

13, 26, 39, 412, 515, 618, 721, 824, 927, 1030, 1133, 1236, ...

(numbers that can be partitioned into two groups such that the second is three times biger than the first)

40) A $4n$-digital subsequence:

14, 28, 312, 416, 520, 624, 728, 832, 936, 1040, 1144, 1248, ...

(numbers that can be partitioned into two grpoups such that the second is four times biger than the first)

41) A $5n$-digital subsequence:

15, 210, 315, 420, 525, 630, 735, 840, 945, 1050, 1155, 1260, ... (numbers that can be partitioned into two groups such that the second is five times biger than the first)

42) A second function (numbers):

1, 2, 3, 2, 5, 6, 7, 4, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, 21, 22, 23, 12, 5, 26, 9, 14, 29, 30, 31, 8, 33, ...

($S2(n)$ is tha smallest integer m such that m^2 is divisible by n)

43) A third function (numbers):

1, 2, 3, 2, 5, 6, 7, 8, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, 21, 22, 23, 6, 5, 26, 3, 14, 29, 30, 31, 4, 33, ...

($S3(n)$ is the smallest integer m such that m^3 is divisible by n)