Number π , Numbers $h_{n,m}, g_{n,m}, f_{n,m}$

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Abstract
We give some formulas related with the numbers $\pi, h_{n,m}, g_{n,m}, f_{n,m}$

Resumen
Se muestran fórmulas que involucran el número π y los números $h_{n,m}, g_{n,m}, f_{n,m}$

1. Introducción

Para $n,m \in \mathbb{N}$, los números $h_{n,m}, g_{n,m}, f_{n,m}$, se definen por:

$$h_{n,m} = (n!)^m \sum_{k=1}^{n} \frac{1}{k^m} = \sum_{k=1}^{n} \left(\frac{n!}{k} \right)^m \tag{1.1}$$

$$g_{n,m} = \left(\frac{(2n)!}{2^n n!} \right)^m \sum_{k=1}^{n} \frac{1}{(2k-1)^m} = \sum_{k=1}^{n} \left(\frac{(2n)!}{2^n n! (2k-1)} \right)^m \tag{1.2}$$

$$f_{n,m} = (n!)^m \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k^m} = \sum_{k=1}^{n} (-1)^{k-1} \left(\frac{n!}{k} \right)^m \tag{1.3}$$

de (1.1),(1.2),(1.3), se ve que: $h_{n,m}, g_{n,m}, f_{n,m} \in \mathbb{N}$.

2. Recurrencias para $h_{n,m}, g_{n,m}, f_{n,m}$.

$$h_{n+1,m} = (n+1)^m h_{n,m} + (n!)^m \quad , n,m \in \mathbb{N} \tag{2.1}$$

$$h_{1,m} = 1$$

$$g_{n+1,m} = (2n+1)^m g_{n,m} + \left(\frac{(2n)!}{2^n n!} \right)^m \quad , n,m \in \mathbb{N} \tag{2.2}$$

$$g_{1,m} = 1$$

$$f_{n+1,m} = (n+1)^m \, f_{n,m} + (-1)^{n-1} (n!)^m \; , n,m \in \mathbb{N} \qquad (2.3)$$

$$f_{1,m} = 1$$

3. Fórmulas que involucran la constante π

$$2^m \left(\frac{B_m}{2(2m)!} \right)^{1/2} \pi^m = 1 + \sum_{n=1}^{\infty} \frac{(n!)^m}{(n+1)^m \left((n+1)^m \sqrt{h_{n,2m}} + \sqrt{h_{n+1,2m}} \right)} \qquad (3.1)$$

$$\left(\frac{\left(2^{2m}-1\right) B_m}{2(2m)!} \right)^{1/2} \pi^m = 1 + \sum_{n=1}^{\infty} \frac{\left((2n)!/2^n n!\right)^m}{(2n+1)^m \left((2n+1)^m \sqrt{g_{n,2m}} + \sqrt{g_{n+1,2m}} \right)} \qquad (3.2)$$

$$\left(\frac{\left(2^{2m-1}-1\right) B_m}{(2m)!} \right)^{1/2} \pi^m = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (n!)^m}{(n+1)^m \left((n+1)^m \sqrt{f_{n,2m}} + \sqrt{f_{n+1,2m}} \right)} \qquad (3.3)$$

donde B_m son los números de Bernoulli:

$$B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, B_5 = \frac{5}{66}, \ldots.$$

Tabla 1. Algunos valores de $h_{n,m}$

n	$h_{n,2}$	$h_{n,4}$
1	1	1
2	5	17
3	49	1393
4	820	357904
5	21076	224021776
6	773136	290539581696
7	38402064	697854274212096
8	2483133696	2859056348455305216
9	202759531776	18760911610506623282176
10	20407635072000	187626456226399005573120000

Tabla 2. Algunos valores de $g_{n,m}$

n	$g_{n,2}$	$g_{n,4}$
1	1	1
2	10	82
3	259	51331
4	12916	123296356
5	1057221	809068942341
6	128816766	11846375878465206
7	21878089479	338356017569383549191
8	4940831601000	17129606870671774862445000
9	1432009163039625	1430698777932227525446706735625
10	518142759828635250	186451505481090040331197201556276250

Tabla 3. Algunos valores de $f_{n,m}$

n	$f_{n,2}$	$f_{n,4}$
1	1	1
2	3	15
3	31	1231
4	460	313840
5	12076	196481776
6	420336	254433021696
7	21114864	611162423652096
8	1325949696	2502676045996425216
9	109027627776	16422700446075911602176
10	10771080883200	164209664339446343270400000

4. Los Números $U_{n,m}$

Para $n,m \in \mathbb{N}$, los números $U_{n,m}$, se definen por:

$$U_{n,m} = (n!)^m \sum_{k=1}^{n} \frac{\mu(k)}{k^m} = \sum_{k=1}^{n} \mu(k) \left(\frac{n!}{k} \right)^m \qquad (4.1)$$

donde $\mu(k)$ es la función de Mobius:

$$\mu(n) = \begin{cases} 0 \ \ si\ n\ tiene\ uno\ o\ más\ \ factores\ \ primos\ repetidos \\ 1 \ \ si\ n = 1 \\ (-1)^k \ \ si\ n\ es\ el\ \ producto\ de\ k\ diferentes\ \ factores\ \ primos \end{cases} \qquad (4.2)$$

Recurrencia para los números $U_{n,m}$:

$$U_{n+1,m} = (n+1)^m U_{n,m} + \mu(n+1)(n!)^m \qquad (4.3)$$

Fórmula que involucra la constante π:

$$\frac{1}{2^m} \left(\frac{2(2m)!}{B_m} \right)^{1/2} \frac{1}{\pi^m} = 1 + \sum_{n=1}^{\infty} \frac{\mu(n+1)(n!)^m}{(n+1)^m \left((n+1)^m \sqrt{U_{n,2m}} + \sqrt{U_{n+1,2m}} \right)} \qquad (4.4)$$

donde B_m son los números de Bernoulli.

Tabla 4. Algunos valores de $U_{n,m}$

n	$U_{n,2}$	$U_{n,4}$
1	1	1
2	3	15
3	23	1199
4	368	306944
5	8624	191508224
6	324864	248402018304
7	15399936	596144507387904
8	985595904	2441807902260854784
9	79833268224	160207016467334682378 24
10	8115008716800	16022435658864745512960000 0

5. Los Números V_n

Para $n \in \mathbb{N}$, los números V_n se definen por:

$$V_n = \left(\frac{(2n)!}{2^n n!} \right)^3 \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{(2k-1)^3} = \sum_{k=1}^{n} (-1)^{k-1} \left(\frac{(2n)!}{2^n n! (2k-1)} \right)^3 \tag{5.1}$$

Recurrencia para los números V_n :

$$V_{n+1} = (2n+1)^3 V_n + (-1)^n \left(\frac{(2n)!}{2^n n!} \right)^3 \tag{5.2}$$

$$V_1 = 1$$

Fórmula que involucra la constante π :

$$\frac{\pi}{2\sqrt[3]{4}} = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left((2n)!/2^n n! \right)^2}{(2n+1)\left((2n+1)^2 \sqrt[3]{V_n^2} + (2n+1)\sqrt[3]{V_n V_{n+1}} + \sqrt[3]{V_{n+1}^2} \right)} \tag{5.3}$$

Tabla 5. Algunos valores de V_n

n	V_n
1	1
2	26
3	3277
4	1120636
5	818101269
6	1088048880414
7	2391566632649433
8	8069069621683251000
9	39651667753171287803625
10	271929870206854693222673250

6. Referencias

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