The real parts of the nontrivial Riemann zeta function zeros
Igor Turkanov
to my love and wife Mary

ABSTRACT

This theorem is based on holomorphy of studied functions and the fact that near a singularity point the real part of some rational function can take an arbitrary preassigned value.

The colored markers are as follows:
• - assumption or a fact, which is not proven at present;
• - the statement, which requires additional attention;
• - statement, which is proved earlier or clearly understandable.

THEOREM

• The real parts of all the nontrivial Riemann zeta function zeros $\rho$ are equal $Re(\rho) = \frac{1}{2}$.

PROOF:

• In relation to $\zeta(s)$ - Zeta function of Riemann is known [8, p. 5] two equations each of which can serve as its definition:

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \zeta(s) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1}, \quad Re(s) > 1, \quad (1)
$$

where $p_1, p_2, \ldots, p_n, \ldots$ is a series of primes.

• According to the functional equality [8, p. 22], [4, p. 8-11] by part $\Gamma(s)$ is the Gamma function:

$$
\Gamma\left(s \frac{1}{2}\right) \pi^{-\frac{1}{2}} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{-\frac{1}{2}} \zeta(1-s), \quad Re(s) > 0. \quad (2)
$$
• From [4, p. 8-11] $\zeta(\bar{s}) = \overline{\zeta(s)}$, it means that $\forall \rho = \sigma + it$: $\zeta(\rho) = 0$ and $0 \leq \sigma \leq 1$ we have:

$$\zeta(\bar{\rho}) = \zeta(1 - \rho) = \zeta(1 - \bar{\rho}) = 0 \quad (3)$$

• From [9], [7, p. 128], [8, p. 45] we know that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ and consequently on the line $\sigma = 0$ also, in accordance with (3) they don’t exist.

• Let’s denote the set of nontrivial zeros $\zeta(s)$ through $\mathcal{P}$ (multiset with consideration of multiplicitiy):

$$\mathcal{P} \overset{\text{def}}{=} \{ \rho : \zeta(\rho) = 0, \rho = \sigma + it, 0 < \sigma < 1 \}.$$

And:

$$\mathcal{P}_1 \overset{\text{def}}{=} \{ \rho : \zeta(\rho) = 0, \rho = \sigma + it, 0 < \sigma < \frac{1}{2} \},$$

$$\mathcal{P}_2 \overset{\text{def}}{=} \{ \rho : \zeta(\rho) = 0, \rho = \frac{1}{2} + it \},$$

$$\mathcal{P}_3 \overset{\text{def}}{=} \{ \rho : \zeta(\rho) = 0, \rho = \sigma + it, \frac{1}{2} < \sigma < 1 \}.$$

Then:

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \quad \text{and} \quad \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_3 = \emptyset,$$

$$\mathcal{P}_1 = \emptyset \iff \mathcal{P}_3 = \emptyset.$$

• Hadamard’s theorem (Weierstrass preparation theorem) about the decomposition of function through the roots gives us the following result [8, p. 30], [4, p. 31], [10]:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)}{s(s-1)} \prod_{\rho \in \mathcal{P}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \quad Re(s) > 0 \quad (4)$$

$$a = \ln 2\sqrt{\pi} - \frac{\gamma}{2} - 1, \quad \gamma - \text{Euler’s constant and}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2} \ln \pi + a - \frac{1}{s} + \frac{1}{1 - s} - \frac{1}{2} \Gamma'(\frac{s}{2}) \sum_{\rho \in \mathcal{P}} \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right) \quad (5)$$

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According to the fact that \( \frac{\Gamma'(\frac{s}{2})}{\Gamma\left(\frac{s}{2}\right)} \) - Digamma function of [8, p. 31], [4, p. 23] we have:

\[
\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) + \sum_{n=1}^{\infty} \left( \frac{1}{s + 2n} - \frac{1}{2n} \right) + C, \tag{6}
\]

\( C = \text{const.} \)

- From [3, p. 160], [6, p. 272], [2, p. 81]:

\[
\sum_{\rho \in \mathcal{P}} \frac{1}{\rho} = 1 + \frac{\gamma}{2} - \ln 2\sqrt{\pi} = 0.0230957 \ldots \tag{7}
\]

- Indeed, from (3):

\[
\sum_{\rho \in \mathcal{P}} \frac{1}{\rho} = \frac{1}{2} \sum_{\rho \in \mathcal{P}} \left( \frac{1}{1-\rho} + \frac{1}{\rho} \right).
\]

- From (5):

\[
2 \sum_{\rho \in \mathcal{P}} \frac{1}{\rho} = \lim_{s \to 1} \left( \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{1-s} + \frac{1}{s} - a - \frac{1}{2} \ln \pi + \frac{1}{2} \Gamma'(\frac{s}{2}) \right). \tag{8}
\]

Also it’s known, for example, from [8, p. 49], [2, p. 98] that the number of nontrivial zeros of \( \rho = \sigma + it \) in strip \( 0 < \sigma < 1 \), the imaginary parts of which \( t \) are less than some number \( T > 0 \) is limited, i.e.,

\[
\| \{ \rho : \rho \in \mathcal{P}, \rho = \sigma + it, |t| < T \} \| < \infty.
\]

- Indeed, it can be presented that on the contrary the sum of \( \sum_{\rho \in \mathcal{P}} \frac{1}{\rho} \) would have been unlimited.

- Thus \( \forall T > 0 \exists \delta_x > 0, \delta_y > 0 \) such that

in area \( 0 < t < \delta_y \), \( 0 < \sigma < \delta_x \) there are no zeros \( \rho = \sigma + it \in \mathcal{P} \).
Let’s consider random root \( q \in \mathcal{P} \).

Let’s denote \( k(q) \) the multiplicity of the root \( q \).

Let’s examine the area \( Q(R) \equiv \{ s : \|s - q\| \leq R, R > 0 \} \).

- From the fact of finiteness of set of nontrivial zeros \( \zeta(s) \) in the limited area follows \( \exists R > 0 \), such that \( Q(R) \) does not contain any root from \( \mathcal{P} \) except \( q \) and also does not intersect with the axes of coordinates.

![Diagram](image)

**Fig. 1.**

- From [1], [8, p. 31], [4, p. 23] we know that the Digamma function \( \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} \) in the area \( Q(R) \) has no poles, i.e., \( \forall s \in Q(R) \)

\[
\left\| \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} \right\| < \infty.
\]

Let’s denote:

\[
I_P(s) \overset{\text{def}}{=} -\frac{1}{s} + \frac{1}{1 - s} + \sum_{\rho \in \mathcal{P}} \frac{1}{s - \rho}
\]

and

\[
I_{\mathcal{P}\setminus\{q\}}(s) \overset{\text{def}}{=} -\frac{1}{s} + \frac{1}{1 - s} + \sum_{\rho \in \mathcal{P}\setminus\{q\}} \frac{1}{s - \rho}.
\]

- Hereinafter \( \mathcal{P} \setminus \{q\} \equiv \mathcal{P} \setminus \{(q, k(q))\} \) (the difference in the multiset).
Also we shall consider the summation \(-\sum_{\rho \in P} \frac{1}{s - \rho}\) and \(\sum_{\rho \in \mathcal{P}\backslash\{q\}} \frac{1}{s - \rho}\) further as the sum of pairs \(\left(\frac{1}{s - \rho} + \frac{1}{s - (1 - \rho)}\right)\) and \(\sum_{\rho \in P} \frac{1}{\rho}\) as the sum of pairs \(\left(\frac{1}{\rho} + \frac{1}{1 - \rho}\right)\) as a consequence of division of the sum from (6) \(\sum_{\rho \in P} \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right)\) into \(\sum_{\rho \in P} \frac{1}{s - \rho} + \sum_{\rho \in P} \frac{1}{\rho}\). As specified in [3], [5], [6], [8].

Let’s note that \(I_{\mathcal{P}\backslash\{q\}}(s)\) is holomorphic function \(\forall s \in Q(R)\).

Then from (5) we have:

\[
\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln \pi + a - \frac{1}{2} \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho \in P} \frac{1}{\rho} + I_{\mathcal{P}}(s).
\]

And in view of (4), (7):

\[
\text{Re} \left(\frac{\zeta'(s)}{\zeta(s)}\right) = \frac{1}{2}\ln \pi + \text{Re} \left(-\frac{1}{2} \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + I_{\mathcal{P}}(s)\right).
\] (8)

Let’s note that from the equality of

\[
\sum_{\rho \in P} \frac{1}{1 - s - \rho} = -\sum_{(1 - \rho) \in P} \frac{1}{s - (1 - \rho)} = -\sum_{\rho \in P} \frac{1}{s - \rho}
\] (9)

follows that:

\(I_{\mathcal{P}}(1 - s) = -I_{\mathcal{P}}(s), \quad I_{\mathcal{P}\backslash\{1-q\}}(1 - s) = -I_{\mathcal{P}\backslash\{q\}}(s), \quad \text{Re}(s) > 0.\)

Besides

\(I_{\mathcal{P}\backslash\{q\}}(s) = I_{\mathcal{P}}(s) - \frac{k(q)}{s - q}\)

and \(I_{\mathcal{P}\backslash\{q\}}(s)\) is limited in the area of \(s \in Q(R)\) as a result of absence of its poles in this area as well as its differentiability in each point of this area.
If in (5) we replace $s$ with $1 - s$ that in view of (7), in a similar way if we take derivative of the principal logarithm (2):

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1 - s)}{\zeta(1 - s)} = -\frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{1}{2} \frac{\Gamma'(1-s/2)}{\Gamma(1-s/2)} + \ln \pi, \ Re(s) > 0. \quad (10)$$

Let’s examine a circle with the center in a point $q$ and radius $r \leq R$, laying in the area of $Q(R)$:

For $s = x + iy, \ q = \sigma_q + it_q$

$$Re \frac{k(q)}{s - q} = Re \frac{k(q)}{x + iy - \sigma_q - it_q} = \frac{k(q)(x - \sigma_q)}{(x - \sigma_q)^2 + (y - t_q)^2} = k(q) \frac{x - \sigma_q}{r^2}.$$  

Let’s prove a series of statements:

**STATEMENT A**

In an arbitrarily small neighborhood of any nontrivial zero there is a point with the following properties:

$$\forall \ q \in \mathcal{P}$$

$$\exists 0 < R_m \leq R : \ \forall 0 < r \leq R_m \ \exists m_r : \|m_r - q\| = r, \ Re(m_r) \leq Re(q),$$

$$Re \frac{\zeta'(m_r)}{\zeta(m_r)} - Re \frac{\zeta'(1 - m_r)}{\zeta(1 - m_r)} + Re \frac{\zeta'(Re(m_r))}{\zeta(Re(m_r))} - Re \frac{\zeta'(Re(1 - m_r))}{\zeta(Re(1 - m_r))} = 0.$$
PROOF:

Let's define function for \( s = x + iy \in Q(R) \):

\[
T(s) \overset{\text{def}}{=} \frac{1}{2} \left( -\frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{1}{2} \frac{\Gamma'(1-s/2)}{\Gamma(1-s/2)} \right) + \\
+ \frac{1}{2} \left( -\frac{1}{2} \frac{\Gamma'(x/2)}{\Gamma(x/2)} - \frac{1}{2} \frac{\Gamma'(1-x/2)}{\Gamma(1-x/2)} \right) + \ln \pi.
\]

For \( s = x + iy \in Q(R) \) consider the following function:

\[
Re \left( \frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\zeta'(x)}{\zeta(x)} - \frac{\zeta'(1-x)}{\zeta(1-x)} - 2 \frac{k(q)}{s-q} \right)
\]

\[\text{From (8) and (9) it is equal to:} \]

\[
Re \left( -\frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{1}{2} \frac{\Gamma'(1-s/2)}{\Gamma(1-s/2)} + 2I_{p\setminus\{q\}}(s) \right) + \\
+ \text{Re} \left( -\frac{1}{2} \frac{\Gamma'(x/2)}{\Gamma(x/2)} + \frac{1}{2} \frac{\Gamma'(1-x/2)}{\Gamma(1-x/2)} + 2I_p(x) \right) = \\
= 2Re \left( T(s) + I_{p\setminus\{q\}}(s) + I_p(x) \right).
\]

Since all the terms in parentheses are limited in the area of \( Q(R) \), then
\[ \exists H_1(R) > 0, \ H_1(R) \in \mathbb{R}, \ \forall s = x + iy \in Q(R) : \]
\[
\left| \text{Re} \left( \frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\zeta'(x)}{\zeta(x)} - \frac{\zeta'(1-x)}{\zeta(1-x)} - 2 \frac{k(q)}{s-q} \right) \right| < H_1(R). \]

- On each of the semicircles: the left -
  \{ s : \| s - q \| = r, \ \sigma_q - r \leq x \leq \sigma_q \} and right -
  \{ s : \| s - q \| = r, \ \sigma_q \leq x \leq \sigma_q + r \} the function \( \text{Re} \frac{k(q)}{s-q} \) is continuous and takes values from \(- \frac{k(q)}{r} \) to \( \frac{k(q)}{r} \), \( r > 0 \).

Consequently \( \forall 0 < r < \frac{2k(q)}{H_1(R)} \), \( \exists m_{\text{min},r}, \ m_{\text{max},r} : \)
\[ \| m_{\text{min},r} - q \| = r, \ \| m_{\text{max},r} - q \| = r : \]
\[ \text{Re} \frac{2k(q)}{m_{\text{min},r} - q} < -H_1(R), \ \text{Re} \frac{2k(q)}{m_{\text{max},r} - q} > H_1(R) \]
and the sum of two functions:
\[ \text{Re} \left( \frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\zeta'(x)}{\zeta(x)} - \frac{\zeta'(1-x)}{\zeta(1-x)} - 2 \frac{k(q)}{s-q} \right) \]
and
\[ \text{Re} \frac{2k(q)}{s-q} \]

at the points of \( m_{\text{min},r} \) and \( m_{\text{max},r} \) will have values with different signs.

- Properties of continuous functions on take all intermediate values between their extremes, it follows that \( \exists R_m \in \mathbb{R}, \ R_m > 0 : \)
\[ R_m \leq R, \ \frac{2k(q)}{R_m} > H_1(R) \]
and then $\forall 0 < r \leq R_m$
exists on the left semicircle point $m_r \overset{\text{def}}{=} x_{m_r} + iy_{m_r}$ such that:

$$\Re \left( \frac{\zeta'(m_r)}{\zeta(m_r)} - \frac{\zeta'(1 - m_r)}{\zeta(1 - m_r)} + \frac{\zeta'(x_{m_r})}{\zeta(x_{m_r})} - \frac{\zeta'(1 - x_{m_r})}{\zeta(1 - x_{m_r})} \right) = 0.$$ 

- From this equality and (10), it follows that $\forall 0 < r \leq R_m$:

$$\Re \zeta'(m_r) \zeta(m_r) + \Re \zeta'(x_{m_r}) \zeta(x_{m_r}) = \Re \zeta'(1 - m_r) \zeta(1 - m_r) + \Re \zeta'(1 - x_{m_r}) \zeta(1 - x_{m_r}) =$$

$$= \frac{1}{2} \Re \left( -\frac{1}{2} \frac{\Gamma'(\frac{m_r}{2})}{\Gamma(\frac{m_r}{2})} - \frac{1}{2} \frac{\Gamma'(\frac{1 - m_r}{2})}{\Gamma(\frac{1 - m_r}{2})} \right) +$$

$$+ \frac{1}{2} \Re \left( -\frac{1}{2} \frac{\Gamma'(\frac{x_{m_r}}{2})}{\Gamma(\frac{x_{m_r}}{2})} - \frac{1}{2} \frac{\Gamma'(\frac{1 - x_{m_r}}{2})}{\Gamma(\frac{1 - x_{m_r}}{2})} \right) + \ln \pi =$$

$$= \Re T(m_r) = \Re T(1 - m_r) = O(1)_{r \to 0}. \quad (11)$$

□

- From (1) you can write:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = 2 \sum_{n=2,4,\ldots} \frac{1}{n^s} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^s} = 2^{1-s} \sum_{n=1}^{\infty} \frac{1}{n^s},$$

i.e.,

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1 - 2^{1-s}} \eta(s). \quad (12)$$

- The Dirichlet eta function is the function $\eta(s)$ defined by an alternating series:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \forall s : \Re s > 0.$$
This series in accordance with [8, §3, p. 29] converges $\forall s : \Re(s) > 0$.

- And the formula (12) is true for $\forall s : \Re(s) > 0$, $s \neq 1$.

- Lots of numbers type
  
  \[ p_1^{k_1} p_2^{k_2} \cdots p_\pi(X)^{k_\pi(X)}, \quad 0 \leq k_i \leq \log_{p_i} X, \quad 1 \leq i \leq \pi(X), \]

  where $p_1, p_2, \ldots, p_n, \ldots$ is a series of primes and $\pi(X)$ is the prime counting function:

  \[ \pi(X) = \sum_{p_n \leq X} 1, \]

  in accordance with the main theorem of arithmetic on decomposition of natural numbers into the product of the powers of prime numbers contains all natural numbers less than or equal to $p_{\pi([X])+1} - 1$ exactly once.

- For arbitrary positive real numbers $X$, define a function $\forall s : \Re(s) > 0$:

  \[ \eta_X(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \sum_{n=p_1^{k_1} p_2^{k_2} \cdots p_\pi(X)^{k_\pi(X)}, k_i \in \mathbb{N}_0} (-1)^{n-1} \frac{(−1)^{n-1}}{n^s}. \]

  For $\forall s : \Re(s) > 0$ is executed:

  \[ \eta_X(s) = \sum_{n=1, n=p_1^{k_1} p_2^{k_2} \cdots p_\pi(X)^{k_\pi(X)}, k_i \in \mathbb{N}_0} \frac{1}{n^s} - \sum_{n=1, n=p_1^{k_1} p_2^{k_2} \cdots p_\pi(X)^{k_\pi(X)}, k_i \in \mathbb{N}_0, k_1 \in \mathbb{N}_1} \frac{2}{n^s}. \]

  - I.e., the first sum of the cost components of type

  \[ \frac{1}{p_1^{k_1 s} p_2^{k_2 s} \cdots p_\pi(X)^{k_\pi(X)s}}, \quad k_i \in \mathbb{N}_0, \]

  and in the second - double composed with an even index $n$:

  \[ \frac{1}{p_1^{k_1 s} p_2^{k_2 s} \cdots p_\pi(X)^{k_\pi(X)s}}, \quad k_2, \ldots, k_\pi(X) \in \mathbb{N}_0, k_1 \in \mathbb{N}_1. \]
That can be written as:

\[
\eta_X(s) = \left(1 - \frac{2}{2^s}\right) \sum_{n=1, n=p_1^{k_1} p_2^{k_2} \cdots p_{\pi(X)}^{k_{\pi(X)}}}^\infty \frac{1}{n^s} = \\
= \left(1 - \frac{2}{2^s}\right) \prod_{p_n \leq X} \left(1 - \frac{1}{p_n^s}\right)^{-1}. \tag{13}
\]

- For an arbitrary positive real number \(X\) define function \(\forall s: \text{Re}(s) > 0, s \neq 1:\)

  \[
  \zeta_X(s) \overset{\text{def}}{=} \frac{1}{1 - 2^{1-s}} \eta_X(s).
  \]

- I.e., \(\forall s: \text{Re}(s) > 0, s \neq 1\) and arbitrary fixed \(X > 0:\)

  \[
  \zeta_X(s) = \prod_{p_n \leq X} \left(1 - \frac{1}{p_n^s}\right)^{-1}. \tag{14}
  \]

- **STATEMENT B**

  For any value of the argument: \(s: \text{Re}(s) > 0\) function \(\eta_X(s)\) has a limit when \(X \to \infty\) and it is:

  \[
  \lim_{X \to \infty} \eta_X(s) = \eta(s), \quad \forall s: \text{Re}(s) > 0.
  \]

**PROOF:**

- For any \(s: \text{Re}(s) > 1\) this statement follows from the definition of an infinite product, taking into account (1), (12), (13).

Let’s consider \(\forall s: \text{Re}(s) > 0\) a difference \(\eta(s)\) and \(\eta_X(s)\), denoting its:

\[
\phi_X(s) \overset{\text{def}}{=} \eta(s) - \eta_X(s).
\]
The function $\phi_X(s)$ is defined and analytic $\forall s : \text{Re}(s) > 0$.

- Consequently $\forall s_0 : \text{Re}(s_0) > 0$ function $\phi_X(s)$ is displayed in Taylor’s number:
  \[ \phi_X(s) = \sum_{k=0}^{\infty} \frac{\phi_X(s_0)^{(k)}}{k!} (s - s_0)^k. \]

Limit $\forall s : \text{Re}(s) > 1$:
  \[ \lim_{X \to \infty} \phi_X(s) = 0. \]

I.e., $\forall k \geq 0$ :
  \[ \lim_{X \to \infty} \frac{\phi_X(s_0)^{(k)}}{k!} = 0. \]

Consequently $\forall s : \text{Re}(s) > 0$:
  \[ \lim_{X \to \infty} \phi_X(s) = 0. \]

□

- This in turn means that $\forall s : \text{Re}(s) > 0$, $s \neq 1$:
  \[ \lim_{X \to \infty} \zeta_X(s) = \zeta(s). \]  \hspace{1cm} (15)

And in particular, because $\forall 0 < r \leq R_m : \zeta(m_r) \neq 0$, $\zeta(\text{Re}(m_r)) \neq 0$, $\zeta(1 - m_r) \neq 0$, $\zeta(\text{Re}(1 - m_r)) \neq 0$:

\[ \lim_{X \to \infty} \ln \|\zeta_X(m_r) \zeta_X(\text{Re}(m_r))\| = \ln \|\zeta(m_r) \zeta(\text{Re}(m_r))\|, \]

\[ \lim_{X \to \infty} \ln \|\zeta_X(1 - m_r) \zeta_X(\text{Re}(1 - m_r))\| = \ln \|\zeta(1 - m_r) \zeta(\text{Re}(1 - m_r))\|. \]
• STATEMENT C

The limit of a private derivative on axis of ordinates of function

\[ f_X(x, y) \overset{\text{def}}{=} \ln \| \zeta_X(x + iy) \zeta_X(x) \| \]

exists and is equal to a private derivative on a variable \( x \) to function

\[ f(x, y) \overset{\text{def}}{=} \lim_{X \to \infty} f_X(x, y) = \ln \| \zeta(x + iy) \zeta(x) \| \]

in points \((x_{mr}, y_{mr})\) and \((1 - x_{mr}, -y_{mr})\):

\[
\lim_{X \to \infty} \frac{\partial}{\partial x} f_X(x, y_{mr}) \bigg|_{x=x_{mr}} = \frac{\partial}{\partial x} f(x, y_{mr}) \bigg|_{x=x_{mr}},
\]

\[
\lim_{X \to \infty} \frac{\partial}{\partial x} f_X(x, -y_{mr}) \bigg|_{x=1-x_{mr}} = \frac{\partial}{\partial x} f(x, -y_{mr}) \bigg|_{x=1-x_{mr}}.
\]

PROOF:

• Since the function \( \zeta(x + iy) \) is analytic, there are neighborhoods \( U(x_{mr}) \) and \( U(1 - x_{mr}) \) of points \( x_{mr} \) and \( 1 - x_{mr} \) for which is carried out:

\[ \forall x \in U(x_{mr}), \ x \in U(1 - x_{mr}), \ y = y_{mr}, \ y = -y_{mr} : \]

\[ \| \zeta(x + iy) \zeta(x) \| \neq 0. \]

And taking into account (15):

\[ \forall x \in U(x_{mr}), \ x \in U(1 - x_{mr}), \ y = y_{mr}, \ y = -y_{mr}, \]

\[ \exists X_0 > 0 : \ \forall X > X_0 : \]

\[ \| \zeta_X(x + iy) \zeta_X(x) \| \neq 0. \]

Consequently all functions \( f_X(x, y_{mr}), f_X(x, -y_{mr}) \) at \( X > X_0 \) and \( f(x, y_{mr}), f(x, -y_{mr}) \) are correctly certain in neighborhoods \( U(x_{mr}) \) and
$U(1 - x_{m_r})$ accordingly.

From the fact that the derivative:

$$
\frac{\partial}{\partial x} f(x_{m_r}, y_{m_r}) = \frac{\partial}{\partial x} \ln \| \zeta (x_{m_r} + iy_{m_r}) \zeta (x_{m_r}) \| = \\
= Re \frac{\zeta'(m_r)}{\zeta (m_r)} + Re \frac{\zeta'(x_{m_r})}{\zeta (x_{m_r})},
$$

(16)

in accordance with (11) limited for $\forall 0 < r \leq R_m$ should the existence of a neighborhood $U^*(x_{m_r}) \subseteq U(x_{m_r})$ such that for $\forall x \in U^*(x_{m_r})$ will be limited to the derivative:

$$
\left| \frac{\partial}{\partial x} f(x, y_{m_r}) \right| < \infty.
$$

- Based on the mean value theorem:

$$
\forall \Delta x > 0 : x_{m_r} + \Delta x \in U^*(x_{m_r}), \\
\exists 0 < \theta_1 < 1, \ 0 < \theta_2 < 1 :
$$

$$
\frac{f_X(x_{m_r} + \Delta x, y_{m_r}) - f_X(x_{m_r}, y_{m_r})}{\Delta x} = \frac{\partial}{\partial x} f_X(x_{m_r} + \theta_1 \Delta x, y_{m_r})
$$

and

$$
\frac{f(x_{m_r} + \Delta x, y_{m_r}) - f(x_{m_r}, y_{m_r})}{\Delta x} = \frac{\partial}{\partial x} f(x_{m_r} + \theta_2 \Delta x, y_{m_r}).
$$

- From the definition of the limit it follows that:

$$
\forall \varepsilon > 0, \exists X_1 > X_0 > 0 : \forall X > X_1 :
$$

$$
|f(x_{m_r}, y_{m_r}) - f_X(x_{m_r}, y_{m_r})| < \frac{\varepsilon}{2} \Delta x,
$$

$$
|f(x_{m_r} + \Delta x, y_{m_r}) - f_X(x_{m_r} + \Delta x, y_{m_r})| < \frac{\varepsilon}{2} \Delta x.
$$
I.e., \( \exists X_1 \geq X_0 : \forall X > X_1 \) the derivative of function \( f_X(x, y_{m_r}) \) also will be limited:

\[
\left| \frac{\partial}{\partial x} f(x, y_{m_r}) \right| < \infty, \quad \forall x \in U^*(x_{m_r})
\]

and

\[
\left| \frac{\partial}{\partial x} f(x_{m_r} + \theta_2 \Delta x, y_{m_r}) - \frac{\partial}{\partial x} f(x_{m_r} + \theta_1 \Delta x, y_{m_r}) \right| < \varepsilon.
\]

Because \( \Delta x > 0 \) can be chosen arbitrarily small, when \( \Delta x \rightarrow 0 \) have:

\[
\left| \frac{\partial}{\partial x} f(x_{m_r}, y_{m_r}) - \frac{\partial}{\partial x} f(x_{m_r}, y_{m_r}) \right| \leq \varepsilon,
\]

this proves the statement for the point \((x_{m_r}, y_{m_r})\).

In a similar way it is possible to lead the same reasonings and for the point \((1 - x_{m_r}, -y_{m_r})\).

\( \square \)

- **STATEMENT D**

Since some instant, the sum of private derivatives on axis of ordinates of function \( f_X(x, y) \) in points \((x_{m_r}, y_{m_r})\) and \((1 - x_{m_r}, -y_{m_r})\) slightly different from 0, i.e.:

\[
\forall \varepsilon > 0, \exists X_\varepsilon > 0 : \forall X > X_\varepsilon : \left| \frac{\partial}{\partial x} f(X_{m_r}, y_{m_r}) + \frac{\partial}{\partial x} f(1 - x_{m_r}, -y_{m_r}) \right| < \varepsilon.
\]

**PROOF:**
From the previous statement it follows that ∀ \( \varepsilon > 0 \), ∃ \( X_\varepsilon > 0 \):
\[
\forall X > X_\varepsilon : 
\left| \frac{\partial}{\partial x} f(x_{m_r}, y_{m_r}) - \frac{\partial}{\partial x} f_X(x_{m_r}, y_{m_r}) \right| < \frac{\varepsilon}{2}
\]
and
\[
\left| \frac{\partial}{\partial x} f(1 - x_{m_r}, -y_{m_r}) - \frac{\partial}{\partial x} f_X(1 - x_{m_r}, -y_{m_r}) \right| < \frac{\varepsilon}{2}.
\]

And taking into account (16) and the same equality:
\[
\frac{\partial}{\partial x} f(1 - x_{m_r}, -y_{m_r}) = \frac{\partial}{\partial x} \ln \| \zeta (1 - x_{m_r} - iy_{m_r}) \zeta (1 - x_{m_r}) \| =
\]
\[
= -\text{Re} \, \frac{\zeta' (1 - m_r)}{\zeta (1 - m_r)} - \text{Re} \, \frac{\zeta' (1 - x_{m_r})}{\zeta (1 - x_{m_r})}.
\]

it follows that:
\[
\left| \text{Re} \, \frac{\zeta' (m_r)}{\zeta (m_r)} + \text{Re} \, \frac{\zeta' (x_{m_r})}{\zeta (x_{m_r})} - \frac{\partial}{\partial x} f_X(x_{m_r}, y_{m_r}) \right| < \frac{\varepsilon}{2}
\]
and
\[
\left| \text{Re} \, \frac{\zeta' (1 - m_r)}{\zeta (1 - m_r)} + \text{Re} \, \frac{\zeta' (1 - x_{m_r})}{\zeta (1 - x_{m_r})} + \frac{\partial}{\partial x} f_X(1 - x_{m_r}, -y_{m_r}) \right| < \frac{\varepsilon}{2}.
\]

And from (11):
\[
\left| \frac{\partial}{\partial x} f_X(x_{m_r}, y_{m_r}) + \frac{\partial}{\partial x} f_X(1 - x_{m_r}, -y_{m_r}) \right| < \varepsilon.
\]

\[\square\]

- Note that:
\[
\frac{\partial}{\partial x} f_X(x_{m_r}, y_{m_r}) = \text{Re} \, \frac{\zeta_X' (m_r)}{\zeta_X (m_r)} + \text{Re} \, \frac{\zeta_X' (x_{m_r})}{\zeta_X (x_{m_r})},
\]

16
\[
\frac{\partial}{\partial x} f_X(1 - x_m, -y_m) = -\text{Re} \frac{\zeta_X'(1 - m_r)}{\zeta_X(1 - m_r)} - \text{Re} \frac{\zeta_X'(1 - x_m)}{\zeta_X(1 - x_m)}.
\]

- And also from (14) for \( s = m_r, \ s = 1 - m_r, \ s = x_m, \ s = 1 - x_m:\

\[
\text{Re} \frac{\zeta_X'(s)}{\zeta_X(s)} = \text{Re} \sum_{p_n \leq X} \frac{\ln p_n}{p_n^{s}} = \text{Re} \sum_{p_n \leq X} \sum_{k=1}^{\infty} \frac{\ln p_n}{p_n^{ks}}. \tag{17}
\]

- **STATEMENT E**

In an arbitrarily small neighborhood of any nontrivial zero, there is a point with a real part equal to \( \frac{1}{2} \).

\[
\forall q \in \mathcal{P},
\exists 0 < R_m \leq R : \forall 0 < r \leq R_m \ \exists m_r : \|m_r - q\| = r, \ \text{Re}(m_r) \leq \text{Re}(q),
\]

\[
m_r = \frac{1}{2}.
\]

**PROOF:**

From the previous statement, taking into account (17), we have:

\[
\forall \varepsilon > 0, \ \exists X_\varepsilon > 0 : \forall X > X_\varepsilon :
\]

\[
\left| \text{Re} \sum_{p_n \leq X} \sum_{k=1}^{\infty} \left( \frac{\ln p_n}{p_n^{km_r}} + \frac{\ln p_n}{p_n^{km_m^r}} - \frac{\ln p_n}{p_n^{k(1-m_r)}} - \frac{\ln p_n}{p_n^{k(1-x_m)}} \right) \right| < \varepsilon.
\]

Or:

\[
\sum_{p_n \leq X} \sum_{k=1}^{\infty} \ln p_n \ (1 + \cos(ky_m, \ln p_n)) \left| \frac{1}{p_n^{km_m^r}} - \frac{1}{p_n^{k(1-x_m)}} \right| < \varepsilon.
\]
Let’s consider, that \( X_\varepsilon > 3 \), then at the same time two sums cannot be equal to 0:

\[
1 + \cos(y_m \ln 2), \quad 1 + \cos(y_m \ln 3),
\]

- because otherwise there would be two integers \( m_1, m_2 \in \mathbb{Z} \):

\[
y_m \ln 2 = \pi + 2\pi m_1, \quad y_m \ln 3 = \pi + 2\pi m_2.
\]

And given the fact that \( y_m \neq 0 \):

\[
\frac{\ln 3}{\ln 2} = \frac{1 + 2m_2}{1 + 2m_1}.
\]

Since \( \frac{\ln 3}{\ln 2} > 0 \) should exist non-negative \( m_1 \) and \( m_2 \):

\[
3^{1+2m_1} = 2^{1+2m_2}.
\]

- That is impossible, since the left part of equality always odd, and right - even.

For definiteness, we assume that:

\[
1 + \cos(y_m \ln 2) > 0,
\]

- then, assuming:

\[
\frac{1}{2^{x_m}} - \frac{1}{2^{(1-x_m)}} \neq 0,
\]

as \( \varepsilon \) take:

\[
\varepsilon = \frac{1}{2} \ln 2 \left(1 + \cos(y_m \ln 2)\right) \left| \frac{1}{2^{x_m}} - \frac{1}{2^{(1-x_m)}} \right| > 0.
\]

- Let’s come to the contradiction:

\[
\sum_{p_n \leq X} \sum_{k=1}^{\infty} \ln p_n (1 + \cos(ky_m \ln p_n)) \left| \frac{1}{p_{n_k \ln x_m}} - \frac{1}{p_{n(1-x_m)}} \right| > \varepsilon, \quad \forall X > X_\varepsilon.
\]
I.e.,

\[
\frac{1}{2^{x_{mr}}} = \frac{1}{2^{(1-x_{mr})}},
\]

that is equivalent to:

\[
x_{mr} = \frac{1}{2}.
\]

Thus, we took a random nontrivial root \( q = \sigma_q + it_q \in \mathcal{P} \) and concluded that:

\[
\sigma_q = \lim_{r \to 0} x_{mr} = \frac{1}{2},
\]

i.e., \( \mathcal{P}_1 = \mathcal{P}_3 = \emptyset \) and

\[
\mathcal{P} = \mathcal{P}_2,
\]

that proves the basic statement and the assumption, which had been made by Bernhard Riemann about of the real parts of the nontrivial zeros of Zeta function.
Список литературы


