Pi Formulas , Part 27

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abstract

In this note we show some formulas related with the constant Pi

Número Pi , Fórmulas

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Resumen-Abstract

En esta nota mostramos algunas fórmulas para la constante Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265\ldots$$

Fórmulas

1. Para $0 < x < \frac{\pi}{2}$, se tiene:

$$\frac{\pi}{2} - x = \sum_{n=1}^{\infty}\frac{1}{n}(\cos x)^n \sin(nx) = \sum_{n=1}^{\infty}\frac{1}{n}(\cos x)^n \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k(\cos x)^{n-2k-1}(\sin x)^{2k+1}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}(\cos x)^{2n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k(\tan x)^{2k+1}$$

Ejemplo 1.1:

$$\pi = 3\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\sqrt{3}}{2}\right)^n \sin\left(\frac{n\pi}{6}\right) = \sqrt{3}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{3}{4}\right)^n \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k 3^{-k}$$

Ejemplo 1.2:

$$\pi = \frac{8}{3}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^n \sin\left(\frac{n\pi}{8}\right) = \frac{8}{3}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{2+\sqrt{2}}{4}\right)^n \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k\left(\sqrt{2}-1\right)^{2k+1}$$

Ejemplo 1.3:

$$\pi = 6\sqrt{3}\sum_{n=1}^{\infty}\frac{1}{n}2^{-n}\sin\left(\frac{n\pi}{3}\right) = 6\sqrt{3}\sum_{n=1}^{\infty}\frac{1}{n}2^{-2n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k 3^k$$

2. Fórmulas del tipo:

$$\pi = A\sum_{n=1}^{\infty}\frac{1}{n}(\sin x)^n\sin(nx) = A\sum_{n=1}^{\infty}\frac{1}{n}(\sin x)^n\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k(\cos x)^{n-2k-1}(\sin x)^{2k+1}$$

$$= A\sum_{n=1}^{\infty}\frac{1}{n}(\sin x)^n(\cos x)^n\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k(\tan x)^{2k+1}$$

Ejemplos:

A	$\sin x$	$\cos x$	$\tan x$
4	$1/\sqrt{2}$	$1/\sqrt{2}$	1
6	$\frac{1}{2}\sqrt{\frac{1+2\sqrt{3}-\sqrt{4\sqrt{3}-3}}{2}}$	$\frac{1}{2}\sqrt{\frac{7-2\sqrt{3}+\sqrt{4\sqrt{3}-3}}{2}}$	$\frac{1}{2}\sqrt{4+3\sqrt{3}-\sqrt{27+16\sqrt{3}}}$
6	$\frac{1}{2}\sqrt{\frac{1+2\sqrt{3}+\sqrt{4\sqrt{3}-3}}{2}}$	$-\frac{1}{2}\sqrt{\frac{7-2\sqrt{3}-\sqrt{4\sqrt{3}-3}}{2}}$	$-\frac{1}{2}\sqrt{4+3\sqrt{3}+\sqrt{27+16\sqrt{3}}}$
8	$\sqrt{1-\frac{4}{5+2\sqrt{2}-\sqrt{1+4\sqrt{2}}}}$	$\frac{2}{\sqrt{5+2\sqrt{2}-\sqrt{1+4\sqrt{2}}}}$	$\frac{1}{2}\sqrt{1+2\sqrt{2}-\sqrt{1+4\sqrt{2}}}$
8	$\sqrt{1-\frac{4}{5+2\sqrt{2}+\sqrt{1+4\sqrt{2}}}}$	$-\frac{2}{\sqrt{5+2\sqrt{2}+\sqrt{1+4\sqrt{2}}}}$	$-\frac{1}{2}\sqrt{1+2\sqrt{2}+\sqrt{1+4\sqrt{2}}}$
12	$\sqrt{1-\frac{4}{4+\sqrt{3}-\sqrt{8\sqrt{3}-13}}}$	$\frac{2}{\sqrt{4+\sqrt{3}-\sqrt{8\sqrt{3}-13}}}$	$\frac{1}{2}\sqrt{\sqrt{3}-\sqrt{8\sqrt{3}-13}}$
12	$\frac{1}{\sqrt{\frac{5}{2}+\sqrt{3}-\frac{1}{2}\sqrt{5+4\sqrt{3}}}}$	$-\frac{1}{2}\sqrt{\frac{4\sqrt{3}-\sqrt{8\sqrt{3}-13}}{2}}$	$-\frac{1}{2}\sqrt{\sqrt{3}+\sqrt{8\sqrt{3}-13}}$

3. Fórmulas del tipo:

$$\pi = A\sum_{n=1}^{\infty}\frac{1}{n}(\tan x)^n \sin(nx) = A\sum_{n=1}^{\infty}\frac{1}{n}(\tan x)^n \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k(\cos x)^{n-2k-1}(\sin x)^{2k+1}$$
$$= A\sum_{n=1}^{\infty}\frac{1}{n}(\sin x)^n \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k(\tan x)^{2k+1}$$

Ejemplo 3.1:

$$A = 4$$

$$\sin x = \frac{1}{4} - \frac{1}{4\,s} + \frac{1}{2}\sqrt{\frac{1}{2} - \frac{\left(9+\sqrt{129}\right)^{1/3}}{6^{1/3}} + \frac{2^{2/3}}{\left(3\left(9+\sqrt{129}\right)\right)^{1/3}} + \frac{7}{2}\,s}$$

donde

$$s = \sqrt{\frac{3}{3 - \frac{4\cdot 6^{2/3}}{\left(9+\sqrt{129}\right)^{1/3}} + 2\left(6\left(9+\sqrt{129}\right)\right)^{1/3}}}$$

$$\cos x = -\frac{1}{4} + \frac{1}{4\,c} + \frac{1}{2}\sqrt{\frac{11}{6} - \frac{1}{6}\left(262 - 6\sqrt{129}\right)^{1/3} - \frac{\left(131+3\sqrt{129}\right)^{1/3}}{3\cdot 2^{2/3}} + \frac{3}{2}\,c}$$

donde

$$c = \sqrt{\frac{3}{11 + 2\left(262 - 6\sqrt{129}\right)^{1/3} + 2\left(2\left(131+3\sqrt{129}\right)\right)^{1/3}}}$$

$$\tan x = -\frac{1}{2} - \frac{1}{2}t + \frac{1}{2}\sqrt{2 - \frac{\left(2\left(-9+\sqrt{129}\right)\right)^{1/3}}{3^{2/3}} + \frac{2\cdot 2^{2/3}}{\left(3\left(-9+\sqrt{129}\right)\right)^{1/3}} + \frac{2}{t}}$$

donde

$$t = \sqrt{1 + \frac{\left(2\left(-9+\sqrt{129}\right)\right)^{1/3}}{3^{2/3}} - \frac{2 \cdot 2^{2/3}}{\left(3\left(-9+\sqrt{129}\right)\right)^{1/3}}}$$

Ejemplo 3.2:

$$A = 6 \,, \sin x = \frac{1}{2} \,, \cos x = \frac{\sqrt{3}}{2} \,, \tan x = \frac{1}{\sqrt{3}} \,, x = \frac{\pi}{6}$$

Ejemplo 3.3:

$$A = 8 \,, \sin x = -\frac{1}{\sqrt{2}} \,, \cos x = \frac{1}{\sqrt{2}} \,, \tan x = -1 \,, x = \frac{\pi}{4}$$

Ejemplo 3.4:

$$A = 8$$

$$\sin x = \frac{1}{3} + \frac{s^{1/3}}{3 \cdot 2^{2/3}} - \frac{6\sqrt{2} - 4}{6(2s)^{1/3}}$$

donde

$$s = -23 + 18\sqrt{2} + 3\sqrt{105 - 72\sqrt{2}}$$

$$\cos x = \sqrt{1 - (\sin x)^2} \,, \qquad \tan x = \frac{\sin x}{\sqrt{1 - (\sin x)^2}}$$

Ejemplo 3.5:

$$A = 12$$

$$\sin x = a_1 - \frac{1}{2}\sqrt{a_2 + \frac{s^{1/3}}{6 \cdot 2^{2/3}} - \frac{1}{(2s)^{1/3}}} + \frac{1}{2}\sqrt{2\,a_2 - \frac{s^{1/3}}{6 \cdot 2^{2/3}} + \frac{1}{(2s)^{1/3}} - \frac{a_3}{\sqrt{a_2 + \frac{s^{1/3}}{6 \cdot 2^{2/3}} - \frac{1}{(2s)^{1/3}}}}}$$

donde

$$s = 54 - 27\sqrt{3} + 3\sqrt{3\left(205 - 108\sqrt{3}\right)}$$

$$a_1 = \frac{2-\sqrt{3}}{8} \ , \qquad a_2 = \frac{7-4\sqrt{3}}{16} \ , \qquad a_3 = \frac{17\sqrt{3}-38}{8}$$

$$\cos x = \sqrt{1-(\sin x)^2} \ , \qquad \tan x = \frac{\sin x}{\sqrt{1-(\sin x)^2}}$$

4. Fórmulas del tipo:

$$\pi = A\sum_{n=1}^{\infty}\frac{1}{n}x^n\sin(nx) = A\sum_{n=1}^{\infty}\frac{1}{n}x^n\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}(-1)^k(\cos x)^{n-2k-1}(\sin x)^{2k+1}$$

Ejemplo 4.1:

$$A = 4$$

$$x = 0.709166368008\ldots;\ x(\sin x+\cos x)-1=0$$

$$\sin x = \frac{1-\sqrt{2x^2-1}}{2x} \ , \qquad \cos x = \frac{1+\sqrt{2x^2-1}}{2x}$$

Ejemplo 4.2:

$$A = 6$$

$$x = 0.564496836822\ldots;\ x\left(\sqrt{3}\ \sin x+\cos x\right)-1=0$$

$$\sin x = \frac{\sqrt{3}-\sqrt{4x^2-1}}{4x} \ , \qquad \cos x = \frac{\sqrt{6x^2-1+\sqrt{12x^2-3}}}{2\sqrt{2}\ x}$$

Ejemplo 4.3:

$$A = 8$$

$$x = 0.493851379328\ldots;\ x\left(\sin x+\left(\sqrt{2}-1\right)\cos x\right)-\sqrt{2}+1=0$$

$$\sin x = \frac{\left(2+\sqrt{2}\right)\left(-1+\sqrt{2}-\sqrt{-17+12\sqrt{2}+\left(20-14\sqrt{2}\right)x^2}\right)}{4x}$$

$$\cos x = \sqrt{1-(\sin x)^2}$$

Ejemplo 4.4:

$$A = 12$$

$$x = 0.413845287284\ldots;\ x\left(\sin x + \left(2-\sqrt{3}\right)\cos x\right) - 2 + \sqrt{3} = 0$$

$$\sin x = -\frac{\left(2+\sqrt{3}\right)\left(-2+\sqrt{3}+\sqrt{-97+56\sqrt{3}+\left(104-60\sqrt{3}\right)x^2}\right)}{4x}$$

$$\cos x = \sqrt{1-(\sin x)^2}$$

5. Fórmulas del tipo:

$$\pi = A\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}\sin\left((2n+1)x\right) = A\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}\sum_{k=0}^{n}\binom{2n+1}{2k+1}(-1)^k(\cos x)^{2n-2k}(\sin x)^{2k+1}$$

Ejemplo 5.1:

$$A = 8\ ,x = 0.588591410817\ldots;\ x^2 + 2x\sin x - 1 = 0$$

$$\sin x = \frac{1}{2}\left(\frac{1}{x} - x\right)\ ,\cos x = \sqrt{1-(\sin x)^2}$$

Ejemplo 5.2:

$$A = 12\ ,x = 0.480438044486\ldots;\ x^2 + 2\sqrt{3}\,x\sin x - 1 = 0$$

$$\sin x = \frac{1}{2\sqrt{3}}\left(\frac{1}{x} - x\right)\ ,\cos x = \sqrt{1-(\sin x)^2}$$

Ejemplo 5.3:

$$A = 16\ ,x = 0.419288489875\ldots;\ \left(\sqrt{2}-1\right)x^2 + 2x\sin x - \left(\sqrt{2}-1\right) = 0$$

$$\sin x = \frac{\sqrt{2}-1}{2}\left(\frac{1}{x} - x\right)\ ,\cos x = \sqrt{1-(\sin x)^2}$$

Ejemplo 5.4:

$$A = 24\ ,x = 0.346783615668\ldots;\ \left(2-\sqrt{3}\right)x^2 + 2x\sin x - \left(2-\sqrt{3}\right) = 0$$

$$\sin x = \frac{2-\sqrt{3}}{2}\left(\frac{1}{x}-x\right) \ , \cos x = \sqrt{1-(\sin x)^2}$$

6. Fórmulas del tipo:

$$\pi = A\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}\cos\big((2n+1)x\big) = A\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}\sum_{k=0}^{n}\binom{n}{2k}(-1)^k(\cos x)^{2n-2k+1}(\sin x)^{2k}$$

Ejemplo 6.1:

$$A = 8 \ , x = 0.444416809391\ldots; \ x^2+2x\cos x-1=0$$

$$\cos x = \frac{1}{2}\left(\frac{1}{x}-x\right) \ , \ \sin x = \sqrt{1-(\cos x)^2}$$

Ejemplo 6.2:

$$A = 12 \ , x = 0.277080847884\ldots; \ x^2+2\sqrt{3}\,x\cos x-1=0$$

$$\cos x = \frac{1}{2\sqrt{3}}\left(\frac{1}{x}-x\right) \ , \ \sin x = \sqrt{1-(\cos x)^2}$$

Ejemplo 6.3:

$$A = 16 \ , x = 0.202746234739\ldots; (\sqrt{2}-1)x^2+2x\cos x-(\sqrt{2}-1)=0$$

$$\cos x = \frac{\sqrt{2}-1}{2}\left(\frac{1}{x}-x\right) \ , \ \sin x = \sqrt{1-(\cos x)^2}$$

Ejemplo 6.4:

$$A = 24 \ , x = 0.132781313306\ldots; \ (2-\sqrt{3})x^2+2x\cos x-(2-\sqrt{3})=0$$

$$\cos x = \frac{2-\sqrt{3}}{2}\left(\frac{1}{x}-x\right) \ , \ \sin x = \sqrt{1-(\cos x)^2}$$

7. Fórmulas del tipo:

$$\pi = A\sum_{n=0}^{\infty}\frac{1}{2n+1}(\sin x)^{4n+2}\sin\big((2n+1)x\big)$$
$$= A\sum_{n=0}^{\infty}\frac{1}{2n+1}(\sin x)^{4n+2}\sum_{k=0}^{n}\binom{2n+1}{2k+1}(-1)^k\left(1-(\sin x)^2\right)^{n-k}(\sin x)^{2k+1}$$

Ejemplo 7.1:

$$A = 8$$

$$\sin x = -\frac{1}{2}-\frac{1}{2}\sqrt{1+s}+\frac{1}{2}\sqrt{2-s+\frac{2}{\sqrt{1+s}}}$$

donde

$$s = \frac{\left(2\left(-9+\sqrt{129}\right)\right)^{1/3}}{3^{2/3}} - \frac{2\cdot 2^{2/3}}{\left(3\left(-9+\sqrt{129}\right)\right)^{1/3}}$$

Otra relación para $\sin x$, es:

$$\frac{1}{\sin x} = \sqrt[4]{1+2\sqrt[4]{1+2\sqrt[4]{1+\cdots}}}$$

Ejemplo 7.2:

$$A = 12$$

$$\sin x = -\frac{\sqrt{3}}{2}-\frac{1}{2}\sqrt{3+s}+\frac{1}{2}\sqrt{6-s+6\sqrt{\frac{3}{3+s}}}$$

donde

$$s = \frac{\left(2\left(-27+\sqrt{777}\right)\right)^{1/3}}{3^{2/3}} - \frac{2\cdot 2^{2/3}}{\left(3\left(-27+\sqrt{777}\right)\right)^{1/3}}$$

Otra relación para $\sin x$, es:

$$\frac{1}{\sin x} = \sqrt[4]{1 + 2\sqrt{3}\sqrt[4]{1 + 2\sqrt{3}\sqrt[4]{1 + \cdots}}}$$

Ejemplo 7.3:

$$A = 16$$

$$\sin x = -\frac{1}{2(\sqrt{2}-1)} - \frac{1}{2}\sqrt{\frac{1}{(\sqrt{2}-1)^2} + s} + \frac{1}{2}\sqrt{\frac{2}{(\sqrt{2}-1)^2} - s + \frac{2}{(\sqrt{2}-1)^3\sqrt{\frac{1}{(\sqrt{2}-1)^2} + s}}}$$

donde

$$s = \frac{1}{3\left(\frac{27 + 18\sqrt{2} + \sqrt{1425 + 972\sqrt{2}}}{288}\right)^{1/3}} - 4\left(\frac{27 + 18\sqrt{2} + \sqrt{1425 + 972\sqrt{2}}}{288}\right)^{1/3}$$

Otra relación para $\sin x$, es:

$$\frac{1}{\sin x} = \sqrt[4]{1 + 2(\sqrt{2}+1)\sqrt[4]{1 + 2(\sqrt{2}+1)\sqrt[4]{1 + \cdots}}}$$

Ejemplo 7.4:

$$A = 24$$

$$\sin x = -\frac{1}{2(2-\sqrt{3})} - \frac{1}{2}\sqrt{\frac{1}{(2-\sqrt{3})^2} + s} + \frac{1}{2}\sqrt{\frac{2}{(2-\sqrt{3})^2} - s + \frac{2}{(2-\sqrt{3})^3\sqrt{\frac{1}{(2-\sqrt{3})^2} + s}}}$$

donde

$$s = \frac{1}{3\left(\frac{63 + 36\sqrt{3} + \sqrt{7905 + 4536\sqrt{3}}}{288}\right)^{1/3}} - 4\left(\frac{63 + 36\sqrt{3} + \sqrt{7905 + 4536\sqrt{3}}}{288}\right)^{1/3}$$

Otra relación para $\sin x$, es:

$$\frac{1}{\sin x} = \sqrt[4]{1 + 2(2 + \sqrt{3})\sqrt[4]{1 + 2(2 + \sqrt{3})\sqrt[4]{1 + \cdots}}}$$

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