The Happy Mothers Theorem

F. Portela

Abstract: We revisit a 25 years old approach of the twin primes conjecture, and after a simple adjustment, push it forward by means of ordinary sieves to a possibly important conclusion.

Keywords: twin primes, sieves

The twin prime conjecture is one of the most famous unsolved problems in mathematics. Recent developments\cite{3,4} have incrementally taken the bar down to a bounded gap of only 246. It is still unclear though, whether the same techniques will allow the mathematical community to go all the way down to just 2.

Over the centuries, other (and generally less advanced) methods and techniques have been used to attempt to prove the conjecture. In some cases\cite{2}, authors came close to what we describe below. The inspiration for the attempt presented here comes from Gold & Tucker (1991)\cite{1}. We believe that the roadblock they hit was caused by the usage of their $G(n)$ function. And thus, we will start where they left off, with their Theorem 2, simply coming back to a much more natural formulation of it.

**Theorem 1:** Any twin prime pair greater than $(3; 5)$ is of the form $[6z - 1; 6z + 1]$ where $z \in \mathbb{N}^*$ and satisfies all inequalities in (1), for all $(x, y) \in \mathbb{N}^2$; and conversely, for any such integer $z$, the pair $[6z - 1; 6z + 1]$ is a twin prime pair.

$$
\begin{align*}
6xy + 5x + 5y + 4 & \neq z \\
6xy + 7x + 5y + 6 & \neq z \\
6xy + 7x + 7y + 8 & \neq z
\end{align*}
$$

**Proof:** see Gold & Tucker (1991)\cite{1}. Briefly, we observe that their mention of “$[G(n), G(n + 1)]$, $n$ being odd” is equivalent to $[3(2k + 1) + 2, 3(2k + 2) + 1]$ which we can write $[6(k + 1) - 1, 6(k + 1) + 1]$. We simply pose $z = k + 1$. 

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\footnote{nando8888@gmail.com}
Since there is a one-to-one correspondence between the twin prime pairs and their z “seeds” or “happy mothers” (as we heard them called online), we will aim to count and measure the density over the integers of such numbers z.

A tabulated view will help visualize the effect of the above 3 inequations, seen as sieves:

Table 1: $6xy + 5x + 5y + 4$

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Table 2: $6xy + 7x + 5y + 6$

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Table 3: $6xy + 7x + 7y + 8$

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Sidenote: one can verify that the first missing elements from these 3 tables, \{1,2,3,5,7,10,12,17,18,\ldots\} match exactly the [http://oeis.org/A002822](http://oeis.org/A002822) list.
We first define the infinite sets \( A_k \subset \mathbb{N} \), \( B_k \subset \mathbb{N} \) and \( C_k \subset \mathbb{N} \) of the form

\[
A_k = \{ m \equiv 5k - 1 \pmod{6k - 1} \} \\
\quad \cup \{ m \equiv k \pmod{6k - 1} \} \setminus \{k\}
\]

\[
B_k = \{ m \equiv 5k + 1 \pmod{6k + 1} \} \\
\quad \cup \{ m \equiv k \pmod{6k + 1} \} \setminus \{k\}
\]

\[
C_k = A_k \cup B_k
\]

with \((m, k) \in \mathbb{N} \times \mathbb{N}^*
\]

For instance, with \( k = 1 \) we have

\[
A_1 = \{m \equiv 4 \pmod{5}\} \cup \{m \equiv 1 \pmod{5}\} \setminus \{1\}
\]

\[
B_1 = \{m \equiv 6 \pmod{7}\} \cup \{m \equiv 1 \pmod{7}\} \setminus \{1\}
\]

\[
C_1 = A_1 \cup B_1
\]

And with \( k = 2 \)

\[
A_2 = \{m \equiv 9 \pmod{11}\} \cup \{m \equiv 2 \pmod{11}\} \setminus \{2\}
\]

\[
B_2 = \{m \equiv 11 \pmod{13}\} \cup \{m \equiv 2 \pmod{13}\} \setminus \{2\}
\]

\[
C_2 = A_2 \cup B_2
\]

etc.

How these sets act as sieves is presented below in Table 4. The rightmost column presents the result (in orange, the sifted-out elements, in white, the remaining ones) and each \( A_k \) and \( B_k \) sets are shown by their congruences.
The question is: how dense is $\sum_{k=1}^{\infty} C_k$? And does it leave infinitely many "holes"?
Let’s start with $A_1$. Its arithmetic density is (obviously) exactly:

$$\delta(A_1) = \frac{2}{5}$$

We know that $\delta(C_1)$ will satisfy

$$\delta(C_1) = \delta(A_1 \cup B_1) = \delta(A_1) + \delta(B_1) - \delta(A_1 \cap B_1)$$

Since $A_1$ and $B_1$ are composed of arithmetic progressions with differences (moduli 5 and 7) that are coprimes, we deduce that

$$\delta(A_1 \cap B_1) = \delta(A_1) \cdot \delta(B_1)$$

and we get

$$\delta(C_1) = \frac{2}{5} + \frac{2}{7} - \frac{2}{5} \frac{2}{7} = \frac{14 + 10 - 4}{35} = \frac{20}{35} = \frac{2}{5} + \frac{2}{5} \frac{3}{7}$$

Similarly, for $\delta(C_1 \cup A_2)$ we obtain

$$\delta(C_1 \cup A_2) = \frac{20}{35} + \frac{2}{11} - \frac{20}{35} \frac{2}{11} = \frac{220 + 70 - 40}{385} = \frac{250}{385} = \frac{2}{5} + \frac{2}{5} \frac{3}{7} + \frac{2}{5} \frac{3}{7} \frac{5}{11}$$

And for $\delta(C_1 \cup C_2)$ we obtain

$$\delta(C_1 \cup C_2) = \delta(C_1 \cup A_2 \cup B_2) = \frac{2}{5} + \frac{2}{5} \frac{3}{7} + \frac{2}{5} \frac{3}{7} \frac{5}{11} + \frac{2}{5} \frac{3}{7} \frac{5}{11} \frac{9}{13}$$

One can observe\[5\] that

$$\delta(C_1) = \frac{2}{5} \left( 1 + \frac{3}{7} \right) = \frac{2}{5} \frac{10}{7} = \frac{4}{7}$$

and

$$\delta(C_1 \cup C_2) - \delta(C_1) = \frac{2}{5} \frac{3}{7} \frac{5}{11} \left( 1 + \frac{9}{13} \right) = \frac{2}{5} \frac{3}{7} \frac{5}{11} \frac{22}{13} = \frac{4}{7} \frac{3}{13}$$

Similarly, the natural density increment for $C_3$ simplifies to

$$\delta(C_1 \cup C_2 \cup C_3) - \delta(C_1 \cup C_2) = \frac{4}{7} \frac{3}{13} \frac{9}{19}$$
If all the moduli of the $A_k$ and $B_k$ sets would be prime, the following generic formula would be exact:

$$\delta \left( \bigcup_{k=1}^{n} C_k \right) = \frac{4}{7} \sum_{k=0}^{n} \frac{\left(3 \atop 6\right)_k}{9^6}$$

(2)

where $(\alpha)_n = \prod_{k=1}^{n-1} (\alpha + k)$ is the Pochhammer symbol (rising factorial).

Though, a potential issue seems to appear for $A_k$ and $B_k$ sets with composite moduli, like $B_4 \pmod{25}$, $A_6 \pmod{35}$, etc. It is shown in Annex A that such sets always fully “collide” with previous sets. For instance, if $6k - 1$ is composite, then $\exists j < k$ such that either $A_k \subset A_j$ or $A_k \subset B_j$ and therefore

$$\delta \left( \bigcup_{j=1}^{k-1} C_j \cup A_k \right) = \delta \left( \bigcup_{j=1}^{k-1} C_j \right)$$

(3)

the density of the global sieve stays unaffected by these sets.

The pattern identified above (2) and the favorable cases of the composite moduli (3) allow us to define the $\Delta(n)$ function as an estimate and (admittedly mediocre) upper-bound for the arithmetic density of all $C_k$ sieves up to $n$:

$$\delta \left( \bigcup_{k=1}^{n} C_k \right) \leq \Delta(n) = \frac{4}{7} \sum_{k=0}^{n} \frac{\left(3 \atop 6\right)_k}{9^6}$$

(4)

Since $5n + 4$ is the smallest integer possibly affected by the $\bigcup_{k=n}^{\infty} C_k$ sieves, we can define

$$U(n) = (5n + 3) \cdot (1 - \Delta(n))$$

as a mediocre lower-bound estimate for the total number of unsifted elements lower than $5n + 3$. Or alternatively,

$$V(n) = n \cdot (1 - \Delta(floor\left(\frac{n-3}{5}\right)))$$

as a lower-bound estimate for the total number of unsifted elements up to $n$. 

Result: We find

\[
\lim_{n \to \infty} U(n) = \lim_{n \to \infty} (5n + 3) \cdot (1 - \Delta(n)) = \infty
\]

We therefore conclude that there are infinitely many “happy mothers”, and consequently, infinitely many twin prime pairs.

dedicated to Bernard Capelle

Acknowledgments: Many thanks to A. Hui (of Math.SE) for his brilliant insight.

References:

Annex A

The set of congruence classes $\{\bar{1}, \bar{5}\}$ is closed under multiplication. Therefore, all non-prime integers of the form $6k + 1$ can be written as either

$$6k + 1 = (6i - 1)(6j - 1)$$

or

$$6k + 1 = (6i + 1)(6j + 1)$$

And similarly, all non-prime integers of the form $6k - 1$ can be written as

$$6k - 1 = (6i - 1)(6j + 1)$$

Case by case study:

- $6k + 1 = (6i - 1)(6j - 1) \implies k = 6ij - i - j$

  If $X \equiv k \ (mod \ 6k + 1)$, then $\exists \ n$ such that

  $$X = (6k + 1)n + k$$

  $$\iff X = (6i - 1)(6j - 1)n + 6ij - i - j$$

  $$\iff X = (6i - 1)[(6j - 1)n + j] - i$$

  $$\iff X = (6i - 1)[(6j - 1)n + j] - 6i + 5i - 1 + 1$$

  $$\iff X = (6i - 1)[(6j - 1)n + j] - (6i - 1) + 5i - 1$$

  $$\iff X = (6i - 1)[(6j - 1)n + j - 1] + (5i - 1)$$

  $$\implies X = 5i - 1 \ (mod \ 6i - 1)$$

If $Y \equiv 5k + 1 \ (mod \ 6k + 1)$, then $\exists \ n$ such that

... 

- $6k + 1 = (6i + 1)(6j + 1) \implies k = 6ij + i + j$

... 

... (tbd)