Pi Formulas , Part 25

Edgar Valdebenito

abstract

In this note we show some formulas related with the constant Pi

Pi fórmulas

Edgar Valdebenito

Febrero 8 , 2010

Resumen

En esta nota se muestran algunas fórmulas que involucran la constante Pi.

Introducción

La constante Pi se define por la serie:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.141592... \qquad (1)$$

En esta nota se muestran algunas representaciones para la constante Pi.

Fórmulas

Fórmula 1.

Para $k \in \mathbb{N}$, se tiene:

$$\pi = 4\sum_{n=1}^{2k-1}(-1)^{n-1}\int_0^1 \frac{x^{2n-2}}{1+x^{4k-2}}\,dx \qquad (2)$$

Fórmula 2.

Para $k \in \mathbb{N}$, se tiene:

$$\pi = 6\sum_{n=1}^{2k-1}(-1)^{n-1}\int_0^{1/\sqrt{3}} \frac{x^{2n-2}}{1+x^{4k-2}}\,dx = \qquad (3)$$

$$= \sqrt{3}\sum_{n=1}^{2k-1}(-1)^{n-1}3^{2k-n}\int_0^1 \frac{x^{n-3/2}}{3^{2k-1}+x^{2k-1}}\,dx$$

Fórmula 3.

Para $k \in \mathbb{N}$, se tiene:

$$\pi = 6\sum_{n=1}^{2k}(-1)^{n-1}\int_0^{3^{-1/2}}\frac{x^{2n-2}}{1-x^{4k}}\,dx =$$
$$= 3^{3/2}\sum_{n=1}^{2k}(-1)^{n-1}3^{2k-n}\int_0^1\frac{x^{n-3/2}}{3^{2k}-x^{2k}}\,dx \tag{4}$$

Fórmula 4.

Para $k\in\mathbb{N}, 0<x<1, y=\dfrac{1-x}{1+x}$, se tiene:

$$\pi = 4\sum_{n=1}^{k}(-1)^{n-1}\left(\int_0^x\frac{u^{2n-2}}{1-(-1)^k u^{2k}}\,du+\int_0^y\frac{u^{2n-2}}{1-(-1)^k u^{2k}}\,du\right) \tag{5}$$

Fórmula 5.

Para $m\in\mathbb{N}-\{1\}$, se tiene:

$$\pi = \frac{8}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{m^n}}{2m^n+1}+4\sum_{n=0}^{\infty}\sum_{k=m^n+1}^{m^{n+1}-1}\frac{(-1)^k}{2k+1} \tag{6}$$

$$(-1)^{m^n} = \begin{cases} 1 & m=par \\ -1 & m=impar\end{cases}\quad ,n\in\mathbb{N} \tag{7}$$

Fórmula 6.

Para $m\in\mathbb{N}-\{1\}$, se tiene:

$$\pi = \sqrt{3}\left(\frac{16}{9}+2\sum_{n=1}^{\infty}\frac{(-1)^{m^n}3^{-m^n}}{2m^n+1}+2\sum_{n=0}^{\infty}\sum_{k=m^n+1}^{m^{n+1}-1}\frac{(-1)^k 3^{-k}}{2k+1}\right) \tag{8}$$

Fórmula 7.

Para $k\in\mathbb{N}$, se tiene:

$$\pi = 4\left(\sum_{n=0}^{2k-1}(-1)^n\int_0^1\frac{x^{2n}}{1+x^{4k}}\,dx+2\sum_{n=2k}^{\infty}(-1)^n\int_0^1\frac{x^{2n}}{1+x^{4k}}\,dx\right) \tag{9}$$

Fórmula 8.

Para $k\in\mathbb{N}\cup\{0\}$, se tiene:

$$\pi = 6\left(\sum_{n=0}^{2k}(-1)^n\int_0^{1/\sqrt{3}}\frac{x^{2n}}{1-x^{4k+2}}\,dx+2\sum_{n=2k+1}^{\infty}(-1)^n\int_0^{1/\sqrt{3}}\frac{x^{2n}}{1-x^{4k+2}}\,dx\right) \tag{10}$$

Fórmula 9.

Para $F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n$, F_n sucesión de Fibonacci , se tiene:

$$\pi = 4\sum_{n=1}^{\infty} F_n \int_0^1 \left(\frac{2}{2+x^2+\left(4+\left(2+x^2\right)^2\right)^{1/2}} \right)^n dx \tag{11}$$

$$\pi = 4\sum_{n=1}^{\infty} F_n \int_1^{\infty} \left(\frac{2}{2+x^2+\left(4+\left(2+x^2\right)^2\right)^{1/2}} \right)^n dx \tag{12}$$

$$\pi = 2\sum_{n=1}^{\infty} F_n \int_0^{\infty} \left(\frac{2}{2+x^2+\left(4+\left(2+x^2\right)^2\right)^{1/2}} \right)^n dx \tag{13}$$

$$\pi = 12\sum_{n=1}^{\infty} F_n \int_{3^{-1/2}}^{1} \left(\frac{2}{2+x^2+\left(4+\left(2+x^2\right)^2\right)^{1/2}} \right)^n dx \tag{14}$$

Fórmula 10.

Para $p_0 = 0, p_1 = 1, p_{n+2} = 2p_{n+1} + p_n$, p_n sucesión de Pell , se tiene:

$$\pi = 4\sum_{n=1}^{\infty} p_n \int_0^1 \left(\frac{2}{3+x^2+\sqrt{4+\left(3+x^2\right)^2}} \right)^n dx \tag{15}$$

$$\pi = 4\sum_{n=1}^{\infty} p_n \int_1^{\infty} \left(\frac{2}{3+x^2+\sqrt{4+\left(3+x^2\right)^2}} \right)^n dx \tag{16}$$

$$\pi = 2\sum_{n=1}^{\infty} p_n \int_0^{\infty} \left(\frac{2}{3+x^2+\sqrt{4+\left(3+x^2\right)^2}} \right)^n dx \tag{17}$$

$$\pi = 12\sum_{n=1}^{\infty} p_n \int_{1/\sqrt{3}}^{1} \left(\frac{2}{3+x^2+\sqrt{4+\left(3+x^2\right)^2}} \right)^n dx \tag{18}$$

Fórmula 11.

Para $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$, F_n sucesión de Fibonacci , se tiene:

$$\pi = 6\sqrt{3}\sum_{n=1}^{\infty} F_n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}\left(\frac{2}{1+3^k+\sqrt{4+\left(1+3^k\right)^2}}\right)^n \tag{19}$$

Fórmula 12.

Para $p_0 = 0, p_1 = 1, p_{n+2} = 2p_{n+1} + p_n$, p_n sucesión de Pell , se tiene:

$$\pi = 6\sqrt{3}\sum_{n=1}^{\infty} p_n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}\left(\frac{2}{2+3^k+\sqrt{4+\left(2+3^k\right)^2}}\right)^n \tag{20}$$

Fórmula 13.

Para $a_0 = 1, a_n = -\sum_{k=1}^{n}\binom{n}{k}(-1)^{[(k+1)/2]} a_{n-k}$, $n \in \mathbb{N}$, se tiene:

$$\pi = 4\lim_{n\to\infty}\frac{(n+1)a_n}{a_{n+1}} \tag{21}$$

Fórmula 14.

Para $\sqrt{2}-1 < x < 1, \alpha = \tan^{-1}\left(\frac{x^2+2x-1}{1+2x-x^2}\right)$, se tiene:

$$\pi^2 = \frac{32}{3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi}{8}\right)\cos\left(\frac{(2n-1)\alpha}{2}\right) \tag{22}$$

Fórmula 15.

Para $H_n = \sum_{m=1}^{n}\frac{1}{m}$, se tiene:

$$\pi = 16\sqrt{3}\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n \sum_{k=1}^{2n-1} kH_k \tag{23}$$

$$-\frac{\pi\left(4-3\sqrt{2}\right)+4\left(7\sqrt{2}-10\right)\ln\left(4-2\sqrt{2}\right)-56\sqrt{2}+80}{448\sqrt{2}-640} = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\sqrt{2}-1\right)^{2n-1}\sum_{k=1}^{2n-1} kH_k \tag{24}$$

$$\frac{\pi\left(10-7\sqrt{2}\right)+\left(16-12\sqrt{2}\right)\ln\left(4-2\sqrt{2}\right)+24\sqrt{2}-32}{448\sqrt{2}-640}= \\ =\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{2}-1\right)^{2n}\sum_{k=1}^{2n}kH_k \tag{25}$$

Fórmula 16.

$$\pi=2\sqrt{3}\left(1-\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\binom{n}{k}\frac{2^{n-k}3^{-n-k-1}}{(n+k+1)(2n+2k+3)}\right) \tag{26}$$

$$\pi=4\left(\frac{5}{6}-\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\binom{n}{k}\frac{2^{n-k}\left(2^{-2n-2k-3}+3^{-2n-2k-3}\right)}{(n+k+1)(2n+2k+3)}\right) \tag{27}$$

Fórmula 17.

Para $\frac{1-\sqrt{\sqrt{2}-1}}{1+\sqrt{\sqrt{2}-1}}<x<\sqrt{\sqrt{2}-1},\ y=\frac{1-x}{1+x}$, se tiene:

$$\pi=4\left(x+y-\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\binom{n}{k}\frac{2^{n-k}\left(x^{2n+2k+3}+y^{2n+2k+3}\right)}{(n+k+1)(2n+2k+3)}\right) \tag{28}$$

Fórmula 18.

Para $0\le x\le 1,\ y=\frac{1-x}{1+x}$, se tiene:

$$\frac{\pi\left(e-e^{-1}\right)}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\sum_{k=1}^{n}\frac{(-1)^{k-1}\left(x^{2k-1}+y^{2k-1}\right)}{2k-1}+ \\ +\sum_{n=0}^{\infty}\frac{(-1)^n\left(x^{2n+1}+y^{2n+1}\right)}{2n+1}\sum_{k=0}^{n}\frac{1}{(2k+1)!} \tag{29}$$

$$\frac{\pi\left(e+e^{-1}\right)}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{k=1}^{n}\frac{(-1)^{k-1}\left(x^{2k-1}+y^{2k-1}\right)}{2k-1}+ \\ +\sum_{n=0}^{\infty}\frac{(-1)^n\left(x^{2n+1}+y^{2n+1}\right)}{2n+1}\sum_{k=0}^{n}\frac{1}{(2k)!} \tag{30}$$

Fórmula 19.

$$\frac{\pi\left(e-e^{-1}\right)}{4\cdot 3^{1/2}}=\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\sum_{k=1}^{n}\frac{(-1)^{k-1}3^{-k+1}}{2k-1}+ \\ +\sum_{n=0}^{\infty}\frac{(-1)^{n}3^{-n}}{2n+1}\sum_{k=0}^{n}\frac{1}{(2k+1)!} \tag{31}$$

$$\frac{\pi\left(e+e^{-1}\right)}{4\cdot 3^{1/2}}=\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{k=1}^{n}\frac{(-1)^{k-1}3^{-k+1}}{2k-1}+ \\ +\sum_{n=0}^{\infty}\frac{(-1)^{n}3^{-n}}{2n+1}\sum_{k=0}^{n}\frac{1}{(2k)!} \tag{32}$$

Fórmula 20.

Para $a>5/3$, se tiene:

$$\pi=\frac{4}{a}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}I_k}{a^k} \tag{33}$$

donde

$$I_k=\int_{1/2}^{3}\left(x+\frac{1}{x}\right)^{k}\frac{1}{x}dx \quad ,k\in\mathbb{N}\cup\{0\} \tag{34}$$

$$I_{2k}=\binom{2k}{k}\ln 6+\sum_{\substack{0\le m\le 2k\\ m\ne k}}\binom{2k}{m}\frac{3^{2k-2m}-(1/2)^{2k-2m}}{2k-2m}\,,\ k\in\mathbb{N}\cup\{0\} \tag{35}$$

$$I_{2k+1}=\sum_{m=0}^{2k+1}\binom{2k+1}{m}\frac{3^{2k-2m+1}-(1/2)^{2k-2m+1}}{2k-2m+1}\,,\ k\in\mathbb{N}\cup\{0\} \tag{36}$$

Fórmula 21.

$$\pi=\sum_{n=0}^{\infty}\frac{(-1)^{n}(12n+7)}{2^{2n}(4n+1)(4n+3)}+ \\ +\frac{4}{5}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}}{10^{k}}\sum_{m=0}^{k}\binom{k}{m}\frac{2^{2m}\left(2^{2k-4m-1}-1\right)}{2k-4m-1} \tag{37}$$

Fórmula 22.

$$\pi=\sum_{n=0}^{\infty}\frac{(-1)^{n}(12n+7)}{2^{2n}(4n+1)(4n+3)}+\frac{2}{5}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}I_k}{20^{k}} \tag{38}$$

donde

$$I_k = \int_1^2 \left(x^3 + \frac{4}{x} \right)^k \frac{1}{x} dx \quad , k \in \mathbb{N} \cup \{0\} \tag{39}$$

$$I_{4k} = \binom{4k}{3k} 2^{6k} \ln 2 + \sum_{\substack{0 \le m \le 4k \\ m \ne 3k}} \binom{4k}{m} \frac{2^{2m} \left(2^{12k-4m} - 1 \right)}{12k - 4m} , \; k \in \mathbb{N} \cup \{0\} \tag{40}$$

Para $k = 4j+1, 4j+2, 4j+3, \; j \in \mathbb{N} \cup \{0\}$, se tiene

$$I_k = \sum_{m=0}^{k} \binom{k}{m} \frac{2^{2m} \left(2^{3k-4m} - 1 \right)}{3k - 4m} \tag{41}$$

Fórmula 23.

Para $a > 34/15$, se tiene:

$$\pi = \frac{8}{a} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k I_k}{a^k} \tag{42}$$

donde

$$I_k = \int_{3/5}^{\left(7+17\sqrt{2}\right)/23} \left(x + \frac{1}{x} \right)^k \frac{1}{x} dx \quad , k \in \mathbb{N} \cup \{0\} \tag{43}$$

Fórmula 24.

$$\pi = 3\sqrt{3} \left(\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \left((3/4)^{n+k+1} - (1/2)^{n+k+1} \right)}{n+k+1} + \right. \\ \left. + \frac{10}{9} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} \frac{(-1)^{n+k+m} 10^k \left(1 - (3/4)^{k+m+1} \right)}{9^n (k+m+1)} \right) \tag{44}$$

Fórmula 25.

Para $a > 4825/3456$, se tiene:

$$\pi = 3\sqrt{3} \left(\sum_{n=0}^{\infty} (-1)^n \left(\frac{(3/4)^{6n+2}}{6n+2} + \frac{(4/5)^{6n+4}}{6n+4} \right) + \right. \\ \left. + \frac{1}{a} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} \frac{(-1)^k \left((5/4)^{3k-6m-1} - (3/4)^{3k-6m-1} \right)}{a^k (3k - 6m - 1)} \right) \tag{45}$$

Fórmula 26.

$$\pi=\frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{n+k+1} \tag{46}$$

Fórmula 27.

$$\pi=3\sqrt{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\left(v^{n+k+1}-u^{n+k+1}\right)}{n+k+1} \tag{47}$$

Algunos valores para (u,v) son:

$$\{(u,v)\}=\left\{\left(-\frac{1}{2},\frac{1}{5}\right),\left(0,\frac{1}{2}\right),\left(\frac{1}{2},1\right),\left(\frac{1}{3},\frac{4}{5}\right),\left(-\frac{1}{3},\frac{2}{7}\right),\left(\frac{3-13^{1/2}}{2},\frac{13^{1/2}-3}{2}\right)\right\} \tag{48}$$

Fórmula 28.

$$\pi=\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{12^n(2n+1)}+9\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n(5n+3)}{2^{6n+4}(3n+1)(3n+2)} \tag{49}$$

$$\pi=\sqrt{3}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{3n+2}\frac{(-1)^n(21n+11)}{(3n+1)(3n+2)}-\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{12^n(2n+1)} \tag{50}$$

$$\pi=\sqrt{3}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n}\frac{(-1)^n}{2n+1}+\sqrt{3}\sum_{n=0}^{\infty}\frac{90n+79}{2^{6n+4}(6n+1)(6n+5)} \tag{51}$$

$$\pi=4\sqrt{3}\sum_{n=0}^{\infty}\frac{90n+79}{2^{6n+5}(6n+1)(6n+5)}-4\sum_{n=0}^{\infty}\frac{(-1)^n\left(2-\sqrt{3}\right)^{4n+2}}{2n+1} \tag{52}$$

Fórmula 29.

$$\pi=2\sqrt{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\left(\left(1+\sqrt{6}-\sqrt{3}\right)^{n+k+1}-\left(1-\sqrt{6}+\sqrt{3}\right)^{n+k+1}\right)}{2^{n+k+1}(n+k+1)} \tag{53}$$

$$\pi=3\sqrt{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\left(\left(\sqrt{3}-1\right)^{n+k+1}-\left(2-\sqrt{3}\right)^{n+k+1}\right)}{n+k+1} \tag{54}$$

$$\pi=\sqrt{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\left(\left(\sqrt{3}+1\right)^{n+k+1}-\left(1-\sqrt{3}\right)^{n+k+1}\right)}{2^{n+k+1}(n+k+1)} \tag{55}$$

$$\pi=12\sqrt{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\left(1-\left(\left(1+\sqrt{6}-\sqrt{3}\right)/2\right)^{n+k+1}\right)}{n+k+1} \tag{56}$$

Fórmula 30.

$$\pi = 3\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}\left(\frac{3-3^{1/2}}{3}\right)^{4n+1}\left(\frac{2}{4n+1}+\frac{1}{2n+1}\left(\frac{3-3^{1/2}}{3}\right)+\frac{1}{4n+3}\left(\frac{3-3^{1/2}}{3}\right)^2\right) \qquad (57)$$

$$\pi = 3\sum_{n=0}^{\infty}\frac{(-1)^n\left(3^{1/2}-1\right)^{4n+1}}{2^{2n+1}}\left(\frac{2}{4n+1}+\frac{3^{1/2}-1}{2n+1}+\frac{\left(3^{1/2}-1\right)^2}{4n+3}\right) \qquad (58)$$

$$\pi = 2\sum_{n=0}^{\infty}\frac{(-1)^n\left(3^{1/2}-1\right)^{4n+1}}{2^{2n}}\left(\frac{2}{4n+1}+\frac{\left(3^{1/2}-1\right)^2}{4n+3}\right) \qquad (59)$$

Fórmula 31.

Para $a>0$, se tiene:

$$\pi = \frac{2\sqrt{2}}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{1+a}\right)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{a^k}\left(\frac{1}{4k+1}+\frac{1}{4k+3}\right) \qquad (60)$$

Fórmula 32.

$$\pi = 4\sum_{n=0}^{\infty}I_n(0,1) = 6\sum_{n=0}^{\infty}I_n\left(0,\frac{1}{\sqrt{3}}\right) = 3\sum_{n=0}^{\infty}I_n\left(0,\sqrt{3}\right) \qquad (61)$$

$$\pi = 4\sum_{n=0}^{\infty}I_n\left(a,\frac{1+a}{1-a}\right) \quad ,1-\sqrt{2}<a<\sqrt{2}-1 \qquad (62)$$

donde

$$I_n(a,b) = \int_a^b \frac{x^n(1-x)^n}{(1+x)^{n+1}}dx \quad ,n\in\mathbb{N}\cup\{0\} \qquad (63)$$

$$I_0(a,b) = \ln\left(\frac{1+b}{1+a}\right) \qquad (64)$$

Para $n\in\mathbb{N}$, se tiene:

$$I_n(a,b)=\left(\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}3^{n-2k}2^k\right)\ln\left(\frac{1+b}{1+a}\right)+$$

$$+\sum_{k=0}^{[n/2]}\sum_{\substack{0\le m\le 2k\\ m\neq k}}\binom{n}{2k}\binom{2k}{m}\frac{3^{n-2k}2^m\left((1+b)^{2k-2m}-(1+a)^{2k-2m}\right)}{2k-2m} \tag{65}$$

$$-\sum_{k=0}^{[(n-1)/2]}\sum_{m=0}^{2k+1}\binom{n}{2k+1}\binom{2k+1}{m}\frac{3^{n-2k-1}2^m\left((1+b)^{2k-2m+1}-(1+a)^{2k-2m+1}\right)}{2k-2m+1}$$

Caso particular $I_n(0,1)$:

$$\pi=4\sum_{n=0}^{\infty}I_n(0,1)=$$

$$=4\left(\ln 2+(3\ln 2-2)+(13\ln 2-9)+\left(63\ln 2-\frac{131}{3}\right)+\right. \tag{66}$$

$$\left.+\left(321\ln 2-\frac{445}{2}\right)+\left(1683\ln 2-\frac{34997}{30}\right)+....\right)$$

Fórmula 33.

$$\pi=\frac{22}{7}-\frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n+9}{2n+4}} \tag{67}$$

Fórmula 34.

$$\pi=\frac{22}{7}-\frac{1}{10}\sum_{n=0}^{\infty}\frac{c_n}{2^n\binom{n+9}{n+4}} \tag{68}$$

donde

$$c_{n+2}=2(c_{n+1}-c_n)\quad ,c_0=1,c_1=2,n\in\mathbb{N}\cup\{0\} \tag{69}$$

Fórmula 35.

$$\pi=\frac{22}{7}-\frac{1}{10}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\,2^{-k}}{\binom{n+k+9}{n+k+4}} \tag{70}$$

Fórmula 36.

Para $a>0$, se tiene:

$$\pi = \frac{22}{7} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+a)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} \left(\frac{1}{2k+5} - \frac{4}{2k+6} + \frac{6}{2k+7} - \frac{4}{2k+8} + \frac{1}{2k+9} \right) \tag{71}$$

$$\pi = \frac{22}{7} - \frac{1}{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+a)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-a)^{n-k}}{\binom{2k+9}{2k+4}} \tag{72}$$

Fórmula 37.

Para $c_{n+4} = -(c_{n+2} + 2c_{n+1} + c_n), c_0 = 1, c_1 = 0, c_2 = -1, c_3 = -2$, se tiene:

$$\pi = 4 \sum_{n=0}^{\infty} c_n \left(\frac{\sqrt{5}-1}{2} \right)^{n+1} \left(\frac{1}{n+1} + \frac{\sqrt{5}-1}{n+2} \right) \tag{73}$$

$$\pi = 6 \sum_{n=0}^{\infty} c_n \left(\frac{2}{\sqrt{3} + \sqrt{3+4\sqrt{3}}} \right)^{n+1} \left(\frac{1}{n+1} + \frac{4 \big/ \left(\sqrt{3} + \sqrt{3+4\sqrt{3}} \right)}{n+2} \right) \tag{74}$$

$$\pi = 8 \sum_{n=0}^{\infty} c_n \left(\frac{\sqrt{4\sqrt{2}-3}-1}{2} \right)^{n+1} \left(\frac{1}{n+1} + \frac{\sqrt{4\sqrt{2}-3}-1}{n+2} \right) \tag{75}$$

Fórmula 38.

Para $n \in \mathbb{N} \cup \{0\}, 0 < a < 1$, se tiene:

$$\frac{(2n)!}{2^{2n+1}(n!)^2} \pi = \sum_{k=0}^{\infty} \frac{(2k)! a^{2k+2n+1}}{2^{2k}(k!)^2 (2k+2n+1)} + \sqrt{2(1-a)} \sum_{k=0}^{\infty} \frac{(2k)!}{2^{3k}(k!)^2} \sum_{m=0}^{2n} \binom{2n}{m} \frac{(-1)^m (1-a)^{m+k}}{2m+2k+1} \tag{76}$$

Con $n = 0, a = \frac{1}{2}$, se tiene:

$$\pi = 3 \sum_{k=0}^{\infty} \frac{(2k)!}{2^{4k}(k!)^2 (2k+1)} \tag{77}$$

Con $n = 0, a = \frac{2}{3}$, se tiene:

$$\pi = 4\sum_{k=0}^{\infty}\frac{(2k)!}{3^{2k+1}(k!)^2(2k+1)} + 2\left(\frac{2}{3}\right)^{1/2}\sum_{k=0}^{\infty}\frac{(2k)!}{2^{3k}3^k(k!)^2(2k+1)} \tag{78}$$

Fórmula 39.

Para $a > 0, b > 0, c > 0, 0 < a < b < a + 2c, 0 < u < 1$, se tiene:

$$\begin{aligned}\frac{a^{u-1}}{\sin(\pi u)}\pi &= b^{-1+u}\sum_{n=0}^{\infty}\frac{(-a/b)^n}{n+1-u} + \\ &+ \frac{1}{a+c}\sum_{n=0}^{\infty}\left(\frac{1}{a+c}\right)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k c^{n-k}b^{k+u}}{k+u}\end{aligned} \tag{79}$$

Con $a = 1, b = 2, c = 1, u = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$, se tiene:

$$\pi = \sqrt{2}\left(\sum_{n=0}^{\infty}\frac{(-1/2)^n}{2n+1} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2^{n-k}(2k+1)}\right) \tag{80}$$

$$\pi = \frac{3}{4}\sqrt{3}\sqrt[3]{2}\left(\sum_{n=0}^{\infty}\frac{(-1/2)^n}{3n+2} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2^{n-k}(3k+1)}\right) \tag{81}$$

$$\pi = \sqrt{2}\sqrt[4]{2}\left(\sum_{n=0}^{\infty}\frac{(-1/2)^n}{4n+3} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2^{n-k}(4k+1)}\right) \tag{82}$$

$$\pi = \frac{3}{2}\sqrt[6]{2}\left(\sum_{n=0}^{\infty}\frac{(-1/2)^n}{6n+5} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2^{n-k}(6k+1)}\right) \tag{83}$$

Fórmula 40.

Para $0 < a < 1$, se tiene:

$$\begin{aligned}\frac{(\Gamma(1/3))^3}{2\sqrt{3}\sqrt[3]{2}}\frac{1}{\pi} &= \sum_{n=0}^{\infty}\frac{(2n)!a^{3n+1}}{2^{2n}(n!)^2(3n+1)} + \\ &+ \frac{2\sqrt{1-a}}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\sum_{k=0}^{n}\binom{n}{k}\frac{(1-a)^{n+k}}{(-3)^k(2n+2k+1)}\end{aligned} \tag{84}$$

Fórmula 41.

Para $0 < a < 1$, se tiene:

$$\frac{\sqrt{3}\left(\Gamma(2/3)\right)^3}{\sqrt[3]{4}}\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{(2n)!a^{3n+2}}{2^{2n}(n!)^2(3n+2)}+$$

$$+\frac{2\sqrt{1-a}}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\sum_{k=0}^{n}\binom{n}{k}\frac{(1-a)^{n+k}}{(-3)^k}\left(\frac{1}{2n+2k+1}-\frac{1-a}{2n+2k+3}\right) \tag{85}$$

Fórmula 42.

$$\pi\ln 2=8\sum_{n=0}^{\infty}2^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{1-2^{-n-k-1}}{(n+k+1)^2}-\frac{2^{-n-k-1}\ln 2}{n+k+1}\right) \tag{86}$$

$$\pi\ln 2=4\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2^{n-k}}\left(\frac{1-2^{2n-k+1}}{(2n-k+1)^2}+\frac{2^{2n-k+1}\ln 2}{2n-k+1}\right) \tag{87}$$

Fórmula 43.

Para $\frac{1}{2}<a\le 1, c_{n+2}=2\left(c_{n+1}-c_n\right), c_0=1, c_1=2$, se tiene:

$$\pi\ln 2=8\sum_{n=0}^{\infty}c_n\left(\frac{2^{-a(n+1)}-2^{-(n+1)}}{(n+1)^2}+\frac{\left(a\,2^{-a(n+1)}-2^{-(n+1)}\right)\ln 2}{n+1}\right)+$$

$$+8\int_0^{a\ln 2}\frac{x}{e^x+2e^{-x}-2}dx \tag{88}$$

Observación. Todas las fórmulas se han tomado de la referencia (4).

Referencias

1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York:Dover , 1965.
2. I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals,Series, and Products (A.Jeffrey), Academic Press,New York, London, and Toronto, 1980.
3. M.R. Spiegel, Mathematical Handbook, McGraw-Hill Book Company, New York,1968.
4. E. Valdebenito, Pi Handbook,manuscript,unpublished,1989, (20000 formulas).