Pi Formulas , Part 20

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abstract

In this note we show some formulas related with the constant Pi

Fórmulas Que Involucran a La Constante Pi

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Resumen. En esta nota mostramos algunas fórmulas que involucran la clásica constante Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265\ldots$$

Notación: $\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & , n \geq k \geq 0 \\ 0, en\ otro\ caso \end{cases}$

Fórmulas

(1) $$\pi = \frac{3}{a}\ln\left(\frac{a\sqrt{3}+1}{a\sqrt{3}-1}\right) + 2\sqrt{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k+1}\binom{n+1}{k}\binom{n-k+1}{m}\frac{(-1)^{n+m}2^m(a^2-1)^{n-k-m+1}3^{-k-m}}{2k+2m+1}$$

$$\frac{1}{\sqrt{3}} < a < \sqrt{2}$$

(2) $$\pi = \frac{4}{a}\ln\left(\frac{a+\sqrt{2}-1}{a-\sqrt{2}+1}\right) + 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k+1}\binom{n+1}{k}\binom{n-k+1}{m}\frac{(-1)^{n+m}2^m(a^2-1)^{n-k-m+1}\left(\sqrt{2}-1\right)^{2k+2m+1}}{2k+2m+1}$$

$$\sqrt{2}-1 < a < \sqrt{2}$$

(3) $$\pi = \frac{6}{a}\ln\left(\frac{a+2-\sqrt{3}}{a-2+\sqrt{3}}\right) + 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k+1}\binom{n+1}{k}\binom{n-k+1}{m}\frac{(-1)^{n+m}2^m(a^2-1)^{n-k-m+1}\left(2-\sqrt{3}\right)^{2k+2m+1}}{2k+2m+1}$$

$$2-\sqrt{3}<a<\sqrt{2}$$

(4) $$\pi^2 = 12-\frac{12}{e-1}+12\sum_{n=1}^{\infty}\ln(1+e^{-n})-12\sum_{n=0}^{\infty}\frac{(-1)^n(1-(n+3)e^{-n-2})}{(n+2)^2(e^{n+2}-1)}-12\int_0^1\frac{xe^{-2x}}{1+e^{-x}}dx$$

(5) $$\pi^2 = 6-\frac{6}{e-1}-6\sum_{n=1}^{\infty}\ln(1-e^{-n})+6\sum_{n=0}^{\infty}\frac{(1-(n+3)e^{-n-2})}{(n+2)^2(e^{n+2}-1)}+6\int_0^1\frac{xe^{-2x}}{1-e^{-x}}dx$$

(6) $$\pi^2 = -12\sum_{n=1}^{\infty}\ln(1-e^{-2n})+6\sum_{n=0}^{\infty}\frac{(1-(2n+3)e^{-2n-2})}{(n+1)^2(e^{2n+2}-1)}+24\int_0^1\frac{x}{e^{2x}-1}dx$$

(7) $$\pi = 2\sum_{n=1}^{\infty}\frac{1}{\cosh(n)}+4\sum_{n=0}^{\infty}\frac{(-1)^n(1-(2n+2)e^{-2n-1})}{(2n+1)(e^{2n+1}-1)}+2\int_0^1\frac{x\sinh x}{(\cosh x)^2}dx$$

(8) $$\frac{\pi\,(a+p)}{2\sqrt{p}\,(1+p)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}(-1)^k\left(\frac{a}{a+p}\right)^n\left(\frac{1+p^2}{a}\right)^k\left(\frac{p}{1+p^2}\right)^m\left(\frac{1}{2k+2m+1}+\frac{1}{2k+2m+3}\right)$$

$$a>\frac{1+p^2}{2}\;,p>0$$

(9) $$\frac{\pi\,(1+3p+p^2)}{4\sqrt{p}\,(1+p)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}(-1)^k\left(\frac{1+p+p^2}{1+3p+p^2}\right)^n\left(\frac{2(1+p^2)}{1+p+p^2}\right)^k\left(\frac{p}{1+p^2}\right)^m\left(\frac{1}{2k+2m+1}+\frac{1}{2k+2m+3}\right)$$

$$p>0$$

(10) $$\pi = 4\tan^{-1}\left(\frac{2}{x+\sqrt{40-3x^2}}\right)+4\tan^{-1}\left(\frac{2}{x-\sqrt{40-3x^2}}\right)-4\tan^{-1}\left(\frac{1}{x}\right)$$

donde

$$x=\sqrt[3]{11+10\sqrt[3]{11+10\sqrt[3]{11+\cdots}}}$$

(11) $$\pi^2 = 3 - \frac{3}{e-1} - 6\sum_{n=1}^{\infty} \ln(1-e^{-n}) + 3\sum_{n=0}^{\infty}\left(\frac{1-(n+2)e^{-n-1}}{(n+1)^2(e^{n+1}-1)} + \frac{1-(n+3)e^{-n-2}}{(n+2)^2(e^{n+2}-1)}\right) + 3\int_0^1 \frac{x(1+e^{-x})}{e^x-1}dx$$

(12) $$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x^{2n+1}+x^{4n+2})$$

donde

$$x = -\frac{1}{3} + \frac{1}{3}\sqrt[3]{17+3\sqrt{33}} + \frac{1}{3}\sqrt[3]{17-3\sqrt{33}}$$

(13) $$\pi = 6\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x^{2n+1}+x^{4n+2})$$

donde

$$x = -\frac{1}{\sqrt{3}} + \sqrt[3]{1-\frac{\sqrt{3}}{9}+\sqrt{\frac{2}{3}}} + \sqrt[3]{1-\frac{\sqrt{3}}{9}-\sqrt{\frac{2}{3}}}$$

(14) $$\pi = 8\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x^{2n+1}+x^{4n+2})$$

donde

$$x = -\frac{1+\sqrt{2}}{3} + \sqrt[3]{\frac{20+4\sqrt{2}}{27}+\frac{1}{3}\sqrt{\frac{16+6\sqrt{2}}{3}}} + \sqrt[3]{\frac{20+4\sqrt{2}}{27}-\frac{1}{3}\sqrt{\frac{16+6\sqrt{2}}{3}}}$$

(15) $$\pi = 12\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x^{2n+1}+x^{4n+2})$$

donde

$$x = -\frac{2+\sqrt{3}}{3} + \sqrt[3]{\frac{19+3\sqrt{3}}{27}+\frac{1}{3}\sqrt{\frac{14+4\sqrt{3}}{3}}} + \sqrt[3]{\frac{19+3\sqrt{3}}{27}-\frac{1}{3}\sqrt{\frac{14+4\sqrt{3}}{3}}}$$

Las formulas (12),(13),(14),(15), se pueden escribir como:

(16) $$\pi = a \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$$

donde

$$c_{n+6} = -(c_{n+4} + c_{n+2} + c_n)\ , c_0 = 1, c_1 = 2, c_2 = -1, c_3 = 0, c_4 = 1, c_5 = -2, n \in \mathbb{N} \cup \{0\}$$

$a = 4,6,8,12$; x , es el radical correspondiente al valor de a .

(17) $$\pi = 8 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{2n+1}} \sum_{k=0}^{\left[\frac{2n+1}{2}\right]} \binom{2n+1}{2k} (-2)^k$$

(18) $$\pi = 6 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\left(\frac{1+\sqrt{6\sqrt{3}-8}}{3} \right)^{2n+1} + \left(\frac{1-\sqrt{6\sqrt{3}-8}}{3} \right)^{2n+1} \right)$$

$$= 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{2n}} \sum_{k=0}^{\left[\frac{2n+1}{2}\right]} \binom{2n+1}{2k} \left(6\sqrt{3}-8\right)^k$$

(19) $$\pi = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n}} \sum_{k=0}^{\left[\frac{2n+1}{2}\right]} \binom{2n+1}{2k} (-7)^k$$

(20) $$\pi = 3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n}} \sum_{k=0}^{\left[\frac{2n+1}{2}\right]} \binom{2n+1}{2k} \left(8\sqrt{3}-15\right)^k$$

(21) $$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n}} \sum_{k=0}^{\left[\frac{2n+1}{2}\right]} \binom{2n+1}{2k} \left(8\sqrt{2}-7\right)^k$$

(22) $$\pi = \frac{8}{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n}} \sum_{k=0}^{\left[\frac{2n+1}{2}\right]} \binom{2n+1}{2k} (-14)^k$$

$$(23)\qquad \pi = 8\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\left(\frac{1+\sqrt{10\sqrt{2}-14}}{5}\right)^{2n+1}+\left(\frac{1-\sqrt{10\sqrt{2}-14}}{5}\right)^{2n+1}\right)$$

$$= \frac{16}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)5^{2n}}\sum_{k=0}^{\left[\frac{2n+1}{2}\right]}\binom{2n+1}{2k}\left(10\sqrt{2}-14\right)^k$$

$$(24)\qquad \pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\left(\frac{1+\sqrt{10\sqrt{3}-4}}{5}\right)^{2n+1}+\left(\frac{1-\sqrt{10\sqrt{3}-4}}{5}\right)^{2n+1}\right)$$

$$= \frac{24}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)5^{2n}}\sum_{k=0}^{\left[\frac{2n+1}{2}\right]}\binom{2n+1}{2k}\left(10\sqrt{3}-4\right)^k$$

$$(25)\qquad \pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\left(\frac{1+\sqrt{12\sqrt{3}-11}}{6}\right)^{2n+1}+\left(\frac{1-\sqrt{12\sqrt{3}-11}}{6}\right)^{2n+1}\right)$$

$$= 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)6^{2n}}\sum_{k=0}^{\left[\frac{2n+1}{2}\right]}\binom{2n+1}{2k}\left(12\sqrt{3}-11\right)^k$$

$$(26)\qquad \pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\left(\frac{1+\sqrt{14\sqrt{3}-20}}{7}\right)^{2n+1}+\left(\frac{1-\sqrt{14\sqrt{3}-20}}{7}\right)^{2n+1}\right)$$

$$= \frac{24}{7}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)7^{2n}}\sum_{k=0}^{\left[\frac{2n+1}{2}\right]}\binom{2n+1}{2k}\left(14\sqrt{3}-20\right)^k$$

$$(27)\qquad \pi = \frac{4}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)10^{2n}}\sum_{k=0}^{\left[\frac{2n+1}{2}\right]}\binom{2n+1}{2k}(-79)^k$$

$$(28)\qquad \pi = \frac{6}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)10^{2n}}\sum_{k=0}^{\left[\frac{2n+1}{2}\right]}\binom{2n+1}{2k}\left(20\sqrt{3}-99\right)^k$$

$$(29)\qquad \pi = \frac{8}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)10^{2n}}\sum_{k=0}^{\left[\frac{2n+1}{2}\right]}\binom{2n+1}{2k}\left(20\sqrt{2}-79\right)^k$$

(30) $$\pi = \frac{12}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)10^{2n}}\sum_{k=0}^{\left[\frac{2n+1}{2}\right]}\binom{2n+1}{2k}\left(20\sqrt{3}-59\right)^k$$

(31) $$\pi = 2\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k 2^{-k}}{n+k+2}$$

(32) $$\pi = 3\sqrt{2}\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}+1}\right) + 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k 2^n}{(n+k+2)\left(\sqrt{3}+1\right)^{n+k+2}}$$

(33) $$\pi = 8\sqrt{2}\left(\sqrt{2}-1\right)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k\left(2-\sqrt{2}\right)^n\left(1-\frac{1}{\sqrt{2}}\right)^k}{n+k+2}$$

(34) $$\pi = 6\sqrt{2}\tan^{-1}\left(\frac{\sqrt{2}\left(3-\sqrt{3}\right)}{6}\right) + 6\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k 2^{-k}}{n+k+2}\left(\frac{3-\sqrt{3}}{3}\right)^{n+k+2}$$

(35) $$\pi = 6\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{6}-1}\right) - 2\sqrt{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k}{n+k+2}\left(\sqrt{\frac{2}{3}}\right)^n\left(\frac{1}{\sqrt{6}}\right)^k$$

(36) $$\pi = 6\sum_{n=1}^{\infty} c_n\, x^n$$

donde

$$x = -\frac{1}{3} + \sqrt[3]{\frac{7}{54}+\frac{\sqrt{3}}{6}+\sqrt{\frac{1}{9}+\frac{7\sqrt{3}}{162}}} + \sqrt[3]{\frac{7}{54}+\frac{\sqrt{3}}{6}-\sqrt{\frac{1}{9}+\frac{7\sqrt{3}}{162}}}$$

(37) $$\pi = 8\sum_{n=1}^{\infty} c_n\, x^n$$

donde

$$x = -\frac{1}{3} + \sqrt[3]{\frac{\sqrt{2}}{2}-\frac{10}{27}+\sqrt{\frac{35}{54}-\frac{10\sqrt{2}}{27}}} + \sqrt[3]{\frac{\sqrt{2}}{2}-\frac{10}{27}-\sqrt{\frac{35}{54}-\frac{10\sqrt{2}}{27}}}$$

(38) $$\pi = 12\sum_{n=1}^{\infty} c_n\, x^n$$

donde

$$x = -\frac{1}{3} + \sqrt[3]{\frac{61}{54} - \frac{\sqrt{3}}{2} + \sqrt{\frac{55}{27} - \frac{61\sqrt{3}}{54}}} + \sqrt[3]{\frac{61}{54} - \frac{\sqrt{3}}{2} - \sqrt{\frac{55}{27} - \frac{61\sqrt{3}}{54}}}$$

En las fórmulas (36),(37),(38), los números c_n , se definen por:

$$c_n = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \sum_{k=0}^{2m+1} \binom{2m+1}{k} \binom{k}{n-k-2m-1} \frac{(-1)^m}{2m+1} \quad , n \in \mathbb{N}$$

$$\{c_n\} = \left\{1, 1, \frac{2}{3}, -1, -\frac{9}{5}, -\frac{4}{3}, \frac{6}{7}, 4, \frac{43}{9}, \frac{1}{5}, -\frac{100}{11}, -\frac{47}{3}, \dots\right\}$$

(39) $$\frac{\pi\sqrt{3}}{6} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1/3)^k}{n-k+1}\left(\frac{1-e^{-2/(2k+1)}}{2}\right)^{n-k+1} + \ln\left(\prod_{n=0}^{\infty}\left(\cosh\left(\frac{1}{2n+1}\right)\right)^{\left(-\frac{1}{3}\right)^n}\right)$$

(40) $$\frac{\pi\sqrt{3}}{6} = \frac{3\ln 2}{4} - \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(1/3)^k(-1)^n}{n-k+1} e^{-\frac{2(n-k+1)}{2k+1}} + \ln\left(\prod_{n=0}^{\infty}\left(\cosh\left(\frac{1}{2n+1}\right)\right)^{\left(-\frac{1}{3}\right)^n}\right)$$

(41) $$\frac{\pi\sqrt{3}}{6} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)(n-k+1)}\left(\frac{1-e^{-2\cdot 3^{-k}}}{2}\right)^{n-k+1} + \ln\left(\prod_{n=0}^{\infty}\left(\cosh(3^{-n})\right)^{\frac{(-1)^n}{2n+1}}\right)$$

(42) $$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n r^{2n+1}\cos\left((2n+1)\theta\right)}{2n+1} + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=0}^{\left[\frac{2n+1}{2}\right]}(-1)^k\binom{2n+1}{2k} p^{2n-2k+1} q^{2k}$$

donde

$$p = \frac{1-r^2}{1+2r\cos\theta + r^2} \,, q = -\frac{2r\sin\theta}{1+2r\cos\theta + r^2} \,, 0 < r < 1 \,, 0 < \theta < \frac{\pi}{2}$$

Algunos valores para θ, r, p, q, son:

$$\theta = \frac{\pi}{4} \,, r = \frac{1}{4} \,, p = \frac{15}{17+4\sqrt{2}} \,, q = \frac{32-68\sqrt{2}}{257}$$

$$\theta=\frac{\pi}{4}\,,r=\frac{1}{3}\,,p=\frac{8}{10+3\sqrt{2}}\,,q=\frac{9-15\sqrt{2}}{41}$$

$$\theta=\frac{\pi}{4}\,,r=\frac{1}{2}\,,p=\frac{3}{5+2\sqrt{2}}\,,q=\frac{8-10\sqrt{2}}{17}$$

$$\theta=\frac{\pi}{4}\,,r=\frac{3}{4}\,,p=\frac{7}{25+12\sqrt{2}}\,,q=-\frac{12\sqrt{2}}{25+12\sqrt{2}}$$

$$\theta=\frac{\pi}{3}\,,r=\frac{1}{4}\,,p=\frac{5}{7}\,,q=-\frac{4\sqrt{3}}{21}$$

$$\theta=\frac{\pi}{3}\,,r=\frac{1}{3}\,,p=\frac{8}{13}\,,q=-\frac{3\sqrt{3}}{13}$$

$$\theta=\frac{\pi}{3}\,,r=\frac{1}{2}\,,p=\frac{3}{7}\,,q=-\frac{2\sqrt{3}}{7}$$

$$\theta=\frac{\pi}{3}\,,r=\frac{3}{4}\,,p=\frac{7}{37}\,,q=-\frac{12\sqrt{3}}{37}$$

$$\theta=\frac{\pi}{6}\,,r=\frac{1}{4}\,,p=\frac{15}{17+4\sqrt{2}}\,,q=\frac{16\sqrt{3}-68}{241}$$

$$\theta=\frac{\pi}{6}\,,r=\frac{1}{3}\,,p=\frac{8}{10+3\sqrt{3}}\,,q=\frac{9\sqrt{3}-30}{73}$$

$$\theta=\frac{\pi}{6}\,,r=\frac{1}{2}\,,p=\frac{3}{5+2\sqrt{3}}\,,q=\frac{4\sqrt{3}-10}{13}$$

$$\theta=\frac{\pi}{6}\,,r=\frac{3}{4}\,,p=\frac{7}{25+12\sqrt{3}}\,,q=-\frac{12}{25+12\sqrt{3}}$$

$$\theta=\frac{\pi}{8}\,,r=\frac{1}{4}\,,p=\frac{15}{17+4\sqrt{2+\sqrt{2}}}\,,q=\frac{16\sqrt{2}-68\sqrt{2-\sqrt{2}}}{257-16\sqrt{2}}$$

$$\theta=\frac{\pi}{8}\,,r=\frac{1}{3}\,,p=\frac{8}{10+3\sqrt{2+\sqrt{2}}}\,,q=\frac{9\sqrt{2}-30\sqrt{2-\sqrt{2}}}{82-9\sqrt{2}}$$

$$\theta=\frac{\pi}{8}\,,r=\frac{1}{2}\,,p=\frac{3}{5+2\sqrt{2+\sqrt{2}}}\,,q=-\frac{\left(\sqrt{2+\sqrt{2}}-\frac{5}{2}\right)\sqrt{2-\sqrt{2}}}{\sqrt{2}-\frac{17}{4}}$$

$$\theta = \frac{\pi}{8}, r = \frac{3}{4}, p = \frac{7}{25 + 12\sqrt{2 + \sqrt{2}}}, q = -\frac{12\sqrt{2 - \sqrt{2}}}{25 + 12\sqrt{2 + \sqrt{2}}}$$

$$\theta = \frac{\pi}{12}, r = \frac{1}{4}, p = \frac{15}{17 + 2\sqrt{2} + 2\sqrt{6}}, q = \frac{2\left(\sqrt{2} - \sqrt{6}\right)}{17 + 2\sqrt{2} + 2\sqrt{6}}$$

$$\theta = \frac{\pi}{12}, r = \frac{1}{3}, p = \frac{16}{20 + 3\sqrt{2} + 3\sqrt{6}}, q = \frac{3\left(\sqrt{2} - \sqrt{6}\right)}{20 + 3\sqrt{2} + 3\sqrt{6}}$$

$$\theta = \frac{\pi}{12}, r = \frac{1}{2}, p = \frac{3}{5 + \sqrt{2} + \sqrt{6}}, q = \frac{\sqrt{2} - \sqrt{6}}{5 + \sqrt{2} + \sqrt{6}}$$

$$\theta = \frac{\pi}{12}, r = \frac{3}{4}, p = \frac{7}{25 + 6\sqrt{2} + 6\sqrt{6}}, q = \frac{6\left(\sqrt{2} - \sqrt{6}\right)}{25 + 6\sqrt{2} + 6\sqrt{6}}$$

$$\theta = \frac{\pi}{5}, r = \frac{1}{4}, p = \frac{15}{19 + 2\sqrt{5}}, q = \frac{\sqrt{2}\sqrt{5 - \sqrt{5}}\left(4\sqrt{5} - 38\right)}{341}$$

$$\theta = \frac{\pi}{5}, r = \frac{1}{3}, p = \frac{16}{23 + 3\sqrt{5}}, q = \frac{\sqrt{2}\sqrt{5 - \sqrt{5}}\left(9\sqrt{5} - 69\right)}{484}$$

$$\theta = \frac{\pi}{5}, r = \frac{1}{2}, p = \frac{18 - 3\sqrt{5}}{31}, q = \frac{\sqrt{2}\sqrt{5 - \sqrt{5}}\left(\sqrt{5} - 6\right)}{31}$$

$$\theta = \frac{\pi}{5}, r = \frac{3}{4}, p = \frac{7}{31 + 6\sqrt{5}}, q = -\frac{6\sqrt{2}\sqrt{5 - \sqrt{5}}}{31 + 6\sqrt{5}}$$

(43) $$\pi = 8\sum_{n=1}^{\infty} \frac{(-1)^{n-1} u_{2n-1}}{2n - 1} \left(\frac{1 - \sqrt{8\sqrt{2} - 11}}{2\left(\sqrt{2} - 1\right)}\right)^{2n-1}$$

donde

$$u_{n+2} = u_{n+1} + u_n, u_1 = 1, u_2 = 3, n \in \mathbb{N}$$

(44) $$\pi = 12\sum_{n=1}^{\infty} \frac{(-1)^{n-1} u_{2n-1}}{2n - 1} \left(\frac{1 - \sqrt{16\sqrt{3} - 27}}{2\left(2 - \sqrt{3}\right)}\right)^{2n-1}$$

donde

$$u_{n+2} = u_{n+1} + u_n, u_1 = 1, u_2 = 3, n \in \mathbb{N}$$

(45) $$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{3}-1}{2}\right)^{n} u_n\, v_n$$

donde

$$u_{n+2} = u_{n+1} + u_n\ , u_1 = 1, u_2 = 3, n \in \mathbb{N}$$

$$v_{n+2} = -2(v_{n+1} + v_n)\ , v_1 = 1, v_2 = -2, n \in \mathbb{N}$$

Observación: Todas las fórmulas se han tomado de la referencia (5).

Referencias

[1] Abramowitz, M., and Stegun, I.A.: Handbook of Mathematical Functions. Nueva York: Dover , 1965.

[2] Boros, G., Moll, V.: Irresistible Integrals.Cambridge University Press,2004.

[3] Gradshteyn, I.S., and Ryzhik, I.M.: Table of Integrals,Series and Products. 5th ed.,ed. Alan Jeffrey. Academic Press, 1994.

[4] Spiegel,M.R.: Mathematical Handbook, McGraw-Hill Book Company , New York , 1968.

[5] Valdebenito, E.:, Pi Handbook , manuscript , unpublished , 1989 , (20000 formulas).