Proof of existence of integral solutions \((a_1, a_2, \ldots, a_n)\) of the equation 
\[ a_1p_1^m + a_2p_2^m + \ldots + a_np_n^m = 0 \] 
for any integer “m” greater than or equal to one, 
for sequence of prime \(p_1, p_2, \ldots, p_n\)

Prashanth R. Rao

Abstract: We prove using Bezout’s identity that 
\[ a_1p_1^m + a_2p_2^m + \ldots + a_np_n^m = 0 \] 
has integral solutions for \(a_1, a_2, \ldots, a_n\), where \(p_1, p_2, \ldots, p_n\) is a sequence of distinct prime 
and \(m\) is any integer larger than or equal to 1.

Proof:

If \(p_1, p_2, p_3, \ldots, p_n\) be “n” distinct primes and “m” is an integer greater or equal to one, then there exists integers \(a_1, a_2, a_3, \ldots, a_n\) (not all zero) such that,

\[ a_1p_1^m + a_2p_2^m + \ldots + a_np_n^m = 0 \]

Since \(p_1, p_2, p_3, \ldots, p_n\) are n distinct primes, therefore the terms \(p_1^m, p_2^m, p_3^m, \ldots, p_n^m\) are pair wise co-prime and 
\(\gcd(p_1^m, p_2^m, p_3^m, \ldots, p_n^m) = 1\)

This also implies \(\gcd(p_1^m, p_2^m, p_3^m, \ldots, p_{n-1}^m) = 1\)

Therefore using Bezout’s identity there must exist \((n-1)\) integers 
\(b_1, b_2, b_3, \ldots, b_{n-1}\) (not all zero) such that

\[ b_1p_1^m + b_2p_2^m + \ldots + (b_{n-1})(p_{n-1})^m = 1 \]

Multiplying both sides with \((-a_np_n^m)\) where we choose \(a_n\) is a non-zero integer,

\[ (-a_np_n^m) b_1p_1^m + (-a_np_n^m) b_2p_2^m + \ldots + (-a_np_n^m) (b_{n-1})(p_{n-1})^m = (-a_np_n^m) \]

Replacing \((-a_np_n^m)\) \(b_1\) by \(a_1\),
\[ (-a_np_n^m) b_2\) by \(a_2, \]
\[ ................. \]
\[ (-a_np_n^m) (b_{n-1})\) by \(a_{n-1}\)

We have

\[ a_1p_1^m + a_2p_2^m + \ldots + a_{n-1}p_{n-1}^m = (-a_np_n^m) \]

or

\[ a_1p_1^m + a_2p_2^m + \ldots + a_{n-1}p_{n-1}^m + a_np_n^m = 0 \]

where \(a_1, a_2, a_3, \ldots, a_n\) are integers (not all zero).