Navier-Stokes Equations Solutions Completed

Abstract

"5% of the people think; 10% of the people think that they think; and the other 85% would rather die than think."----Thomas Edison

"The simplest solution is usually the best solution"---Albert Einstein

Over nearly a year and half ago, the Navier-Stokes equations in 3-D for incompressible fluid flow were analytically solved by the author. However, some of the solutions contained implicit terms. In this paper, the implicit terms have been expressed explicitly in terms of $x$, $y$, $z$ and $t$. The author proposed and applied a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and each term utilizes gravity to function. The sum of the terms of the ratio is always unity. It was mathematically shown that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known, and also, there would be no magnetohydrodynamics. In addition to the usual method of solving these equations, the N-S equations were also solved by a second method in which the three equations in the system were added to produce a single equation which was then integrated. The solutions by the two methods were identical, except for the constants involved. Ratios were used to split-up the equations; and the resulting sub-equations were readily integrable, and even, the nonlinear sub-equations were readily integrated. The examples in the preliminaries show everyday examples on using ratios to divide a quantity into parts, as well as possible applications of the solution method in mathematics, science, engineering, business, economics, finance, investment and personnel management decisions. The $x$-direction Navier-Stokes equation was linearized, solved, and the solution analyzed. This solution was followed by the solution of the Euler equation of fluid flow. The Euler equation represents the nonlinear part of the Navier-Stokes equation. Following the Euler solution, the Navier-Stokes equation was solved essentially by combining the solutions of the linearized equation and the Euler solution. For the Navier-Stokes equation, the linear part of the relation obtained from the integration of the linear part of the equation satisfied the linear part of the equation; and the relation from the integration of the non-linear part satisfied the non-linear part of the equation. The solutions and relations revealed the role of each term of the Navier-Stokes equations in fluid flow. The gravity term is the indispensable term in fluid flow, and it is involved in the parabolic and forward motion of fluids. The pressure gradient term is also involved in the parabolic motion. The viscosity terms are involved in the parabolic, periodic and decreasingly exponential motion. Periodicity increases with viscosity. The variable acceleration term is also involved in the periodic and decreasingly exponential motion. The fluid flow in the Navier-Stokes solution may be characterized as follows. The $x$-direction solution consists of linear, parabolic, and hyperbolic terms. The first three terms characterize parabolas. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, some of the axes of symmetry of the parabolas would be at right angles to that of laminar flow. The characteristic curve for the integral of the $x$-nonlinear term is such a parabola whose axis of symmetry is at right angles to that of laminar flow. The integral of the $y$-nonlinear term is similar parabolically to that of the $x$-nonlinear term. The integral of the $z$-nonlinear term is a combination of two similar parabolas and a hyperbola. If the above $x$-direction flow is repeated simultaneously in the $y$- and $z$- directions, the flow is chaotic and consequently turbulent.

For a spin-off, the smooth solutions from above are specialized and extended to satisfy the requirements of the CMI Millennium Prize Problems, and thus prove the existence of smooth solutions of the Navier-Stokes equations.
Solutions of the Navier-Stokes Equations

Case 1: Solutions of the Linearized Navier-Stokes Equations ($x$–direction)

Equation

$$-\mu \left( \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2} \right) + \frac{\partial \rho}{\partial t} + 4\rho \left( \frac{\partial \rho}{\partial t} \right) = \rho g_x$$

Solutions

$$V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 z + \frac{f_g x}{4} t + C_9; \quad P(x) = d\rho g_x x$$

Case 2: Solutions of the Euler Equations for Incompressible Fluid Flow ($x$–direction)

Equation

$$\rho \left( \frac{\partial \rho}{\partial t} + V_x \frac{\partial \rho}{\partial x} + V_y \frac{\partial \rho}{\partial y} + V_z \frac{\partial \rho}{\partial z} \right) + \frac{\partial \rho}{\partial x} = \rho g_x$$

Solutions

$$V_x(x,y,z,t) = f g x t \pm \sqrt{2hg_x x + \frac{ng_x y}{V_y} + \frac{q g_x z}{V_z} + \psi_y(V_y) \frac{V_x}{V_y} + \psi_z(V_z) \frac{V_x}{V_z}}; \quad V_y \neq 0, V_z \neq 0; \quad P(x) = d\rho g_x x$$

Case 3: Solutions of the Navier-Stokes Equations (Original) : $x$–direction

Equation

$$-\mu \left( \frac{\partial^2 V_x}{\partial x^2} - \mu \left( \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) + \frac{\partial V_x}{\partial t} + \rho \frac{\partial V_x}{\partial x} + \rho V_x \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} = \rho g_x$$

Solutions

$$V_x = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f g x t \pm \sqrt{2hg_x x + \frac{ng_x y}{V_y} + \frac{q g_x z}{V_z} + \psi_y(V_y) \frac{V_x}{V_y} + \psi_z(V_z) \frac{V_x}{V_z} + C_9}$$

$$P(x) = d\rho g_x x; \quad (a + b + c + d + h + n + q = 1) \quad V_y \neq 0, V_z \neq 0$$

Summary for the fractional terms of the $x$–direction

$$\frac{ng_y}{V_y}$$ and $$\frac{q g_z}{V_z}$$ in terms of $x, y, z$ and $t$ (for Case 3)

$$\frac{ng_y}{V_y} = -\frac{(ng_x)(-\frac{dg_z}{2\mu} (\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_1 x + C_3 y + C_5 z + f g x t \pm \sqrt{2hg_x x})}{\beta_7 g_z}$$

$$\frac{q g_z}{V_z} = -\frac{(q g_x)\{(\beta_7 g_z x)(-\frac{dg_z}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f g x t \pm \sqrt{2hg_x x} - [CE]\}}{(\beta_7 g_z)(q g_z - \beta_6 g_z x)}$$

$$CE = -(ng_x)(-\frac{dg_z}{2\mu} (\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_1 x + C_3 y + C_5 z + f g x t \pm \sqrt{2hg_x x}$$

One observes above that the most important insight of the above solutions is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, $g_x$, were zero, for Case 1, the first three terms, the seventh, and $P(x)$ would all be zero; for Case 2, the first four terms and $P(x)$ would all be zero; and for Case 3, the first three terms, the seventh, the eighth, the ninth, the tenth terms and $P(x)$ would all be zero. These results can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known. It would not therefore be meaningful to write a Navier-Stokes equation for incompressible fluid flow without the gravity term, since there would be no fluid flow.
More Observations  Comparison of the N-S solutions with equations of motion under gravity and liquid pressure of elementary physics

Motion equations of elementary physics:

(B): \( V_f = V_0 + gt \);  (C): \( V_f^2 = V_0^2 + 2gx \);  (D): \( V = \sqrt{2gx} \);  (E): \( x = V_0t + \frac{1}{2} gt^2 \)

The liquid pressure, \( P \) at the bottom of a liquid of depth \( h \) units is given by \( P = \rho gh \)

Observe the following about the Navier-Stokes Solutions (Case 3)

1. The first three terms are parabolic in \( x, y, \) and \( z \); the minus sign shows the usual inverted parabola when a projectile is fired upwards at an acute angle to the horizontal; also note the "\( gt \)" in \( V = gt \) of (B) of the motion equations and the \( fgx \) \( t \) in the Navier-Stokes solution.

2. The pressure, \( P = \rho gh \) of the liquid pressure and the \( P(x) = dpgx \) of the Navier-Stokes solution.

Note that, only the approach in this paper could yield \( P(x) = dpgx \) by integrating \( dp/dx = dpgx \)

3. Observe the "\( \sqrt{2gx} \)" in \( V = \sqrt{2gx} \) of (D) and the \( \sqrt{2hx} \) in the Navier-Stokes solution.

In fact, the N-S solution term \( \sqrt{2hx} \) could have been obtained from \( V_f^2 = V_0^2 + 2gx \) (C), of the equation of motion by letting \( V_0 = 0 \) (for the convective term) ignoring the ratio term "\( h \)" of the N-S radicand. There are eight main terms (ignoring the arbitrary functions) in the N-S solution.

Of these eight terms, six terms, namely, \( -\frac{apgx}{2\mu} x^2 \), \( -\frac{bpgy}{2\mu} y^2 \), \( -\frac{cpgz}{2\mu} z^2 \), \( fgx \) \( t \), \( \sqrt{2hx} \) \( x \) and \( dpgx \) are similar (except for the constants involved) to the terms in the equations of motion and fluid pressure. This similarity means that the approach used in solving the Navier-Stokes equation is sound. One should also note that to obtain these six terms simultaneously on integration, only the equation with the gravity term as the subject of the equation will yield these six terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Navier-Stokes equation, since in this form, one can immediately split-up the equations using ratios, and integrate.

4. With regards to the variables \( x, y, \) and \( z \), the parabolicity of the first three terms and the parabolicity of the eighth, ninth and tenth terms hint at inverse relations.. For examples, \( V_x = x^2 \) and \( V_x = \pm x \) are inverse relations of each other, \( V_x = y^2 \) and \( V_x = \pm \sqrt{y} \) are inverse relations of each other. The implications of knowing these relationships is that if one knows the steps, rules or formulas for designing for laminar flow, one can deduce the steps, rules or formulas for designing for turbulent flow by reversing the steps and using opposite operations in each step of the corresponding laminar flow design. Thus for every method, or formula for laminar flow, there is a corresponding method, formula for turbulent flow design (see also, "Power of Ratios" book by A. A. Frempong, p. 28).

For the velocity profile, the \( x \)-direction solution consists of linear, parabolic, and hyperbolic terms. The first three terms characterize inverted parabolas. Flow distribution for laminar flow is parabolic with the axis of symmetry of the parabola in direction of the fluid flow. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axes of symmetry of some of the parabolas would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the integral of the \( x \)-nonlinear term is such a parabola whose axis of symmetry is at right angles to that of laminar flow. The integral of the \( y \)-nonlinear term is similar parabolically to the integral of the \( x \)-nonlinear term. The characteristic curve for the integral of the \( z \)-nonlinear term is a combination of two similar parabolas and a hyperbola. If the above \( x \)-direction flow is repeated simultaneously in the \( y \)– and \( z \)– directions, the flow is chaotic and consequently turbulent.
The Navier-Stokes equations in three dimensions are three simultaneous equations in Cartesian coordinates for the flow of incompressible fluids. The equations are presented below:

\[
\begin{align*}
\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x &= \rho \left( \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right) \quad (N_x) \\
\mu \left( \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_z}{\partial z^2} \right) - \frac{\partial p}{\partial y} + \rho g_y &= \rho \left( \frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z} \right) \quad (N_y) \\
\mu \left( \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} \right) - \frac{\partial p}{\partial z} + \rho g_z &= \rho \left( \frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} \right) \quad (N_z)
\end{align*}
\]

Equation \((N_x)\) will be the first equation to be solved; and based on its solution, one will be able to write down the solutions for the other two equations, \((N_y)\), and \((N_z)\).

**Dimensional Consistency**

The Navier-Stokes equations are dimensionally consistent as shown below:

\[
\begin{align*}
\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x &= \rho \left( \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right) \\
\text{Using } M L T \\
M(L^{-2}T^{-2} + L^{-2}T^{-2} + L^{-2}T^{-2} - L^{-2}T^{-2} - L^{-2}T^{-2}) &= M(L^{-2}T^{-2} + L^{-2}T^{-2} + L^{-2}T^{-2})
\end{align*}
\]

Using \(kg = m-s\)

\[
\begin{align*}
\text{Using } kg(m^{-2}s^{-2} + m^{-2}s^{-2} + m^{-2}s^{-2} - m^{-2}s^{-2} - m^{-2}s^{-2}) &= kg(m^{-2}s^{-2} + m^{-2}s^{-2} + m^{-2}s^{-2} + m^{-2}s^{-2})
\end{align*}
\]
Option 1
Solution of 3-D Linearized Navier-Stokes Equation in the $x$-direction

The equation will be linearized by redefinition. The nine-term equation will be reduced to six terms.

Given:  \( \mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x = \rho \left( \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right) \)  \[ (A) \]

\[ -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_y}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} = \rho g_x \]  \[ (B) \]

\[ -\mu (\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p}{\partial x} + 4 \rho \left( \frac{\partial V_x}{\partial x} \right) = \rho g_x \]  \[ (C) \]

**Plan:** One will split-up equation (C) into five sub-equations, solve them, and combine the solutions. On splitting-up the equations and proceeding to solve them, the non linear terms could be redefined and made linear. This linearization is possible if the gravitational force term is the subject of the equation as in equation (B). After converting the non-linear terms to linear terms by redefinition, one will have only six terms as in equation (C). One will show logically how equation (C) was obtained from equation (B), using a ratio method.

Three main steps are covered.
In main Step 1, one shows how equation (C) was obtained from equation (B)
In main Step 2, equation (C) will be split-up into five equations.
In main Step 3, each equation will be solved.
In main Step 4, the solutions from the five equations will be combined.
In main Step 5, the combined relation will be checked in equation (C), for identity.

**Preliminaries**
Requirements and procedure for solving a partial differential equation

1. Integrate the partial differential equation.
2. Find the partial derivatives from the integration relation from Step 1
3. Substitute the derivatives from Step 2 in the original partial differential equation and simplify, both sides of the equation.
4. If the left-hand side of the equation is equal to the right-hand side of the equation, then the integration relation from Step 1 is a solution to the partial differential equation.
   (Steps 2-4 can be summarized as checking for identity, or determining if the integration relation satisfies the original partial differential equation.)

Note: If one does not successfully check for identity, one cannot claim a solution.

A ratio method will be used to split-up the partial differential equations into sub-equation which are then integrated.
Example 1: A grandmother left $45,000 in her will to be divided between eight grandchildren, Betsy, Comfort, Elaine, Ingrid, Elizabeth, Maureen, Ramona, Marilyn, in the ratio $\frac{1}{36} : \frac{1}{18} : \frac{1}{12} : \frac{5}{36} : \frac{1}{6} : \frac{7}{36} : \frac{2}{9}$ (Note: $\frac{1}{36} + \frac{1}{18} + \frac{1}{12} + \frac{5}{36} + \frac{1}{6} + \frac{7}{36} + \frac{2}{9} = 1$).

How much does each receive?

Solution:

Betsy's share of $45,000 = \frac{1}{36} \times 45,000 = 1,250$

Comfort's share of $45,000 = \frac{1}{18} \times 45,000 = 2,500$

Elaine's share of $45,000 = \frac{1}{12} \times 45,000 = 3,750$

Ingrid's share of $45,000 = \frac{5}{36} \times 45,000 = 5,000$

Elizabeth's share of $45,000 = \frac{1}{9} \times 45,000 = 5,000$

Maureen's share of $45,000 = \frac{1}{6} \times 45,000 = 6,250$

Ramona's share of $45,000 = \frac{7}{36} \times 45,000 = 7,500$

Marilyn's share of $45,000 = \frac{2}{9} \times 45,000 = 10,000$

Check; Sum of shares = $45,000$

Sum of the fractions = 1

Example 2: Sir Isaac Newton left $\rho g_x$ units in his will to be divided between $-\mu \frac{\partial^2 v_x}{\partial x^2}$, $-\mu \frac{\partial^2 v_x}{\partial y^2}$, $-\mu \frac{\partial^2 v_x}{\partial z^2}$, $-\frac{\partial p}{\partial x}$, $\rho \frac{\partial v_x}{\partial x}$, $\rho V_x \frac{\partial v_x}{\partial y}$, $\rho V_x \frac{\partial v_x}{\partial z}$ in the ratio $a:b:c:d:f:h:m:n$.

where $a + b + c + d + f + h + m + n = 1$. How much does each receive?

Solution

$-\mu \frac{\partial^2 v_x}{\partial x^2}$'s share of $\rho g_x$ units = $a \rho g_x$ units

$-\mu \frac{\partial^2 v_x}{\partial y^2}$'s share of $\rho g_x$ units = $b \rho g_x$ units

$-\mu \frac{\partial^2 v_x}{\partial z^2}$'s share of $\rho g_x$ units = $c \rho g_x$ units

$-\frac{\partial p}{\partial x}$'s share of $\rho g_x$ units = $d \rho g_x$ units

$\rho \frac{\partial v_x}{\partial x}$'s share of $\rho g_x$ units = $f \rho g_x$ units

$\rho V_x \frac{\partial v_x}{\partial y}$'s share of $\rho g_x$ units = $h \rho g_x$ units

$\rho V_x \frac{\partial v_x}{\partial z}$'s share of $\rho g_x$ units = $m \rho g_x$ units

$\rho V_x \frac{\partial v_x}{\partial z}$'s share of $\rho g_x$ units = $n \rho g_x$ units

Sum of shares = $\rho g_x$ units  Note: $a + b + c + d + f + h + m + n = 1$
Example 3:
The returns on investments $A$, $B$, $C$, $D$ are in the ratio $a:b:c:d$. If the total return on these four investments is $P$ dollars, what is the return on each of these investments? $(a + b + c + d = 1)$

Solution
Return on investment $A = aP$ dollars
Return on investment $B = bP$ dollars
Return on investment $C = cP$ dollars
Return on investment $D = dP$ dollars

Check
$aP + bP + cP + dP = P$
$P(a + b + c + d) = P$

Example 4, Method 2:
One will call this method the multiplier method.

Step 1: From $6x^2 + 11x - 10 = 0$ \hspace{1cm} (1)
$6x^2 + 11x = 10$
$6x^2 = 10a$; \hspace{0.5cm} (Here, $a$ is a multiplier)
$3x^2 = 5a$ \hspace{1cm} (2)
$11x = 10b$ \hspace{0.5cm} (Here, $b$ is a multiplier)
$11x = 10(1 - a)$ \hspace{0.5cm} \hspace{0.5cm} $(a + b = 1)$
$11x = 10 - 10a$
$x = \frac{10 - 10a}{11}$
$3\left(\frac{10 - 10a}{11}\right)^2 = 5a$ \hspace{0.5cm} (Substituting for $x$ in (2))
$3\left(\frac{100 - 200a + 100a^2}{121}\right) = 5a$

Step 2: $300a^2 - 1205a + 300 = 0$
$60a^2 - 241a + 60 = 0$
$a = \frac{241 \pm \sqrt{241^2 - 4(60)(60)}}{120}$
$a = \frac{241 \pm \sqrt{43681}}{120}$
$a = \frac{241 \pm 209}{120}$
$a = \frac{241 \pm 209}{120} = \frac{241 + 209}{120} \hspace{1cm} \text{or} \hspace{1cm} \frac{241 - 209}{120}$
$a = \frac{450}{120} \hspace{0.5cm} \text{or} \hspace{0.5cm} \frac{32}{120}$
$a = \frac{15}{4} \hspace{0.5cm} \text{or} \hspace{0.5cm} \frac{4}{15}$

Step 3: Since $a + b = 1$, when $a = \frac{15}{4}$ or $\frac{3}{4}$
$b = 1 - 3\frac{3}{4} = -2\frac{3}{4} \hspace{0.5cm} \text{or} \hspace{0.5cm} -\frac{11}{4}$
when $a = \frac{4}{15}, \hspace{0.5cm} b = 1 - \frac{4}{15} = \frac{11}{15}$

Step 4: When $b = -\frac{11}{4}, \hspace{0.5cm} 11x = 10(-\frac{11}{4})$
$x = -\frac{5}{2}$
When $b = \frac{11}{15}, \hspace{0.5cm} 11x = 10(\frac{11}{15})$
$x = \frac{10}{11}\frac{11}{15}; \hspace{0.5cm} x = \frac{2}{3}$

Again, one obtains the same solution set $\{-\frac{5}{2}, \frac{2}{3}\}$ as by the factoring method.

The objective of presenting examples 1, 2, 3, and 4 was to convince the reader that the principles to be used in splitting the Navier-Stokes equations are valid. In Examples 4, one could have used the quadratic formula directly to solve for $x$, without finding $a$ and $b$ first. The objective was to show that the introduction of $a$ and $b$ did not change the solution set of the original equation.

For the rest of the coverage in this paper, a multiplier is the same as a ratio term. The multiplier method is the same as the ratio method.
Linearization of Non-Linear Terms

Main Step 1
Linearization of the Non-Linear Terms

Step 1: The main principle is to multiply the right side of the equation by the ratio terms. This step is critical to the removal of the non-linearity of the equation.

\[ \rho g_x \] is to be divided by the terms on the left-hand-side of the equation in the ratio

\[ a : b : c : d : f : h : m : n = (a + b + c + d + f + h + m + n) = 1 \]

Linear terms

\[-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial}{\partial x} \left( \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} \right) = \rho g_x \] (1)

Apply the principles involved in the ratio method covered in the preliminaries, to the nonlinear terms (the last three terms.)

Then \( \rho V_z \frac{\partial V_x}{\partial z} = n p g_x \), where \( n \) is the ratio term corresponding to \( \rho V_z \frac{\partial V_x}{\partial z} \).

\[ V_z \frac{\partial V_x}{\partial z} = n g_x \] (2)

\[ V_z \frac{dV_x}{dz} = n g_x \] (One drops the partials symbol, since a single independent variable is involved)

\[ \frac{dz}{dt} \frac{dV_x}{dz} = n g_x \] (3)

Therefore, \[ V_z \frac{\partial V_x}{\partial z} = \frac{dV_x}{dt} = n g_x \] (4)

Step 2: Similarly, let \( \rho V_y \frac{\partial V_x}{\partial y} = m p g_x \) (\( m \) is the ratio term corresponding to \( \rho V_y \frac{\partial V_x}{\partial y} \)).

\[ V_y \frac{dV_x}{dy} = m g_x \] (One drops the partials symbol, since a single independent variable is involved)

\[ \frac{dy}{dt} \frac{dV_x}{dy} = m g_x \] (4)

Therefore, \[ V_y \frac{dV_x}{dy} = \frac{dV_x}{dt} = m g_x \] (7)

Step 3: Let \( \rho V_x \frac{\partial V_x}{\partial x} = h p g_x \) where \( h \) is the ratio term corresponding to \( \rho V_x \frac{\partial V_x}{\partial x} \).

\[ V_x \frac{dV_x}{dx} = h g_x \] (8)

\[ V_x \frac{dV_x}{dx} = h g_x \] (One drops the partials symbol, since a single independent variable is involved)

\[ \frac{dx}{dt} \frac{dV_x}{dx} = h g_x \] (9)

Therefore, \[ V_x \frac{dV_x}{dx} = \frac{dV_x}{dt} = h g_x \] (10)
From equations (4), (7), (10), \[ V_x \frac{\partial V_x}{\partial x} = V_y \frac{\partial V_y}{\partial y} = V_z \frac{\partial V_z}{\partial z} = \frac{dV_x}{dt} \] and
\[ V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_z}{\partial z} = 3 \frac{dV_x}{dt} \] (11)
Thus, the ratio of the linear term \( \frac{\partial V_x}{\partial t} \) to the nonlinear sum \( V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_z}{\partial z} \) in equation (1) is 1 to 3. Unquestionably, there is a ratio between the sum of the nonlinear terms and the linear term \( \frac{\partial V_x}{\partial t} \). This ratio must be verified experimentally.

**Note:** One could have obtained equation (C) from equation (A) by redefining the nonlinear terms by carelessly disregarding the partial derivatives of the nonlinear terms in equation (1). However, the author did not do that, but logically, the terms became linearized. Note also that the above linearization is possible only if \( \rho g_x \) is the subject of the equation, and it will later be learned that a solution to the logically linearized Navier-Stokes equation is obtained only if \( \rho g_x \) is the subject of the equation.

**Step 4:** Substitute the right side of equation (11) for the nonlinear terms on the left-side of nonlinear terms
\[ -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial y} + \rho V_x \frac{\partial V_x}{\partial z} = \rho g_x \] all acceleration terms
(12)
Then one obtains
\[ -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + 3 \rho \frac{\partial V_x}{\partial t} = \rho g_x \] all acceleration terms
(simplifying) (13)
Now, instead of solving equation (1), previous page, one will solve the following equation
\[ -K \frac{\partial^2 V_x}{\partial x^2} - K \frac{\partial^2 V_x}{\partial y^2} - K \frac{\partial^2 V_x}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial x} + 4 \frac{\partial V_x}{\partial t} = g_x \] \( (k = \frac{\mu}{\rho}) \) (14)

**Main Step 2**

**Step 5:** In equation (14) divide \( g_x \) by the terms on the left side in the ratio \( a : b : c : d : f \).
\[ -K \frac{\partial^2 V_x}{\partial x^2} = ag_x; \quad -K \frac{\partial^2 V_x}{\partial y^2} = bg_x; \quad -K \frac{\partial^2 V_x}{\partial z^2} = cg_x; \quad \frac{1}{\rho} \frac{\partial p}{\partial x} = dg_x; \quad 4 \frac{\partial V_x}{\partial t} = fg_x \]
(\( a, b, c, d, f \) are the ratio terms and \( a + b + c + d + f = 1 \)).
As proportions:
\[ \frac{-K \frac{\partial^2 V_x}{\partial x^2}}{a} = g_x; \quad \frac{-K \frac{\partial^2 V_x}{\partial y^2}}{b} = g_x; \quad \frac{-K \frac{\partial^2 V_x}{\partial z^2}}{c} = g_x; \quad \frac{1}{\rho} \frac{\partial p}{\partial x} = g_x; \quad \frac{4 \frac{\partial V_x}{\partial t}}{f} = g_x \]
One can view each of the ratio terms \( a, b, c, d, f \) as a fraction (a real number) of \( g_x \) contributed by each expression on the left-hand side of equation (14) above.
Main Step 3

Step 6: Solve the differential equations in Step 5.

Solutions of the five sub-equations

\[-K \frac{\partial^2 V_x}{\partial x^2} = a_{gx},\]
\[k \frac{\partial^2 V_x}{\partial x^2} = -ag,\]
\[\frac{\partial^2 V_x}{\partial x^2} = \frac{-a}{k} g,\]
\[\frac{\partial V_x}{\partial x} = -ag \frac{x}{k} + C,\]
\[V_{x1} = \frac{ag}{2k} x^2 + C_1x + C_2,\]
\[-K \frac{\partial^2 V_y}{\partial y^2} = b_{gy},\]
\[K \frac{\partial^2 V_y}{\partial y^2} = -bg,\]
\[\frac{\partial^2 V_y}{\partial y^2} = -\frac{b}{k} g,\]
\[\frac{\partial V_y}{\partial y} = -bg \frac{y}{k} + C_3,\]
\[V_{x2} = \frac{-bg}{2k} y^2 + C_3y + C_4,\]
\[-K \frac{\partial^2 V_z}{\partial z^2} = c_{gz},\]
\[K \frac{\partial^2 V_z}{\partial z^2} = -cg,\]
\[\frac{\partial^2 V_z}{\partial z^2} = -\frac{c}{k} g,\]
\[\frac{\partial V_z}{\partial z} = -cg \frac{z}{k} + C_5,\]
\[V_{x3} = \frac{-cg}{2k} z^2 + C_5z + C_6,\]
\[\frac{1}{\rho} \frac{\partial p}{\partial x} = dg_{xx},\]
\[\frac{1}{\rho} \frac{\partial p}{\partial x} = dg,\]
\[\frac{\partial p}{\partial x} = d\rho g,\]
\[p = d\rho g x + C_7,\]
\[\frac{4}{\partial} \frac{\partial V_x}{\partial t} = f_{gx},\]
\[\frac{\partial V_x}{\partial t} = f_{gx} \frac{x}{4},\]
\[V_{x4} = \frac{f_{gx}}{4} t\]

Main Step 4

Step 7: One combines the above solutions

\[V_x = V_{x1} + V_{x2} + V_{x3} + V_{x4}\]
\[= -\frac{ag}{2k} x^2 + C_1x + C_2 - \frac{bg}{2k} y^2 + C_3y + C_4 - \frac{cg}{2k} z^2 + C_5z + C_6 + \frac{f_{gx}}{4} t + C_7\]
\[= -\frac{ag}{2k} x^2 - \frac{bg}{2k} y^2 + C_1x + C_3y + C_5z + \frac{f_{gx}}{4} t + C_9\]
\[= -\frac{ag}{2k} x^2 - \frac{bg}{2k} y^2 - \frac{cg}{2k} z^2 + C_1x + C_3y + C_5z + \frac{f_{gx}}{4} t + C_9\]
\[= -\frac{ag}{2k} x^2 - \frac{bg}{2k} y^2 + C_1x + C_3y + C_5z + \frac{f_{gx}}{4} t + C_9\]
\[= -\frac{ag}{2k} x^2 + \frac{bg}{2k} y^2 + C_1x + C_3y + C_5z + \frac{f_{gx}}{4} t + C_9\]
\[= -\frac{ag}{2k} x^2 + \frac{bg}{2k} y^2 + C_1x + C_3y + C_5z + \frac{f_{gx}}{4} t + C_9\]
\[P(x) = d\rho g x\]

\[V_x = V_{x1} + V_{x2} + V_{x3} + V_{x4}\]
\[V_x(x,y,z,t) = -\frac{\rho g}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{f_{gx}}{4} t + C_9\]
\[P(x) = d\rho g x\]
### Main Step 5

**Checking in equation (C)**

**Step 8:** Find the derivatives, using

\[ V_x = -\frac{\rho g_x}{2\mu} \left(ax^2 + by^2 + cz^2 \right) + C_1 x + C_3 y + C_5 z + \frac{f g_x}{4} t + C_9 \]

\[ P(x) = d\rho g_x x \]

<table>
<thead>
<tr>
<th>1. [ \frac{\partial V_x}{\partial x} = -\frac{\rho g_x}{2\mu} \left(2ax \right) + C_1 ]</th>
<th>2. [ \frac{\partial V_x}{\partial y} = -\frac{\rho g_x}{\mu} \left(by \right) + C_3 ]</th>
<th>3. [ \frac{\partial V_x}{\partial z} = -\frac{\rho g_x}{\mu} \left(cz \right) ]</th>
</tr>
</thead>
</table>
| 4. \[ \frac{\partial p}{\partial x} = d\rho g_x \] | 5. \[ \frac{\partial V_x}{\partial t} = \frac{f g_x}{4} \]

**Step 9:** Substitute the derivatives from Step 8 in \(-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x \)

to check for identity (to determine if the relation obtained satisfies the original equation).

\[ -\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x \]

\[ \mu \left( \frac{a \rho g_x}{\mu} - \frac{b \rho g_x}{\mu} - \frac{c \rho g_x}{\mu} \right) + d \rho g_x + 4\rho \frac{f}{4} g_x = \rho g_x \]

\[ \frac{a \rho g_x + b \rho g_x + c \rho g_x + d \rho g_x + \rho f g_x}{?} = \rho g_x \]

\[ \frac{a g_x + b g_x + c g_x + d g_x + f g_x}{?} = g_x \]

\[ g_x (a + b + c + d + f) = g_x \]

\[ g_x (1) = g_x \quad (a + b + c + d + f = 1) \]

Scrapwork

<table>
<thead>
<tr>
<th>[ \frac{\partial^2 V_x}{\partial x^2} = -\frac{a \rho g_x}{\mu} ]</th>
<th>[ \frac{\partial^2 V_x}{\partial y^2} = -\frac{b \rho g_x}{\mu} ]</th>
<th>[ \frac{\partial^2 V_x}{\partial z^2} = -\frac{c \rho g_x}{\mu} ]</th>
</tr>
</thead>
</table>
| \[ \frac{\partial p}{\partial x} = d \rho g_x \] | \[ \frac{\partial V_x}{\partial t} = \frac{f g_x}{4} \]

An identity is obtained and therefore, the solution of equation (C), p.5, is given by

\[ V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu} \left(ax^2 + by^2 + cz^2 \right) + C_1 x + C_3 y + C_5 z + \frac{f g_x}{4} t + C_9 \]

\[ P(x) = d\rho g_x x \]

The above solution is unique, because all possible equations were integrated but only a single equation, the equation with the gravity term as the subject of the equation produced the solution.
Solution Summary for $V_x$, $V_y$ and $V_z$

For $V_x$ \( a + b + c + d + f = 1 \)
\[
\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) - \frac{\partial p}{\partial x} + \rho g_x = \rho(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z})
\]
\[
-k \frac{\partial^2 V_x}{\partial x^2} - K \frac{\partial^2 V_x}{\partial y^2} - k \frac{\partial^2 V_x}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial x} + 4 \frac{\partial V_x}{\partial x} = g_x
\]
\[
V_x = V_{x1} + V_{x2} + V_{x3} + V_{x4}
\]
\[
V_x(x,y,z,t) = -\frac{ag_x}{2k} x^2 + C_1 x + C_2 - \frac{bg_x}{2k} y^2 + C_3 y + C_4 - \frac{cg_x}{2k} z^2 + C_5 + C_6 + \frac{g_x}{4} t + C_7 + C_8
\]
\[
V_y(x,y,z,t) = -\frac{ag_y}{2k} x^2 + C_1 x - \frac{bg_y}{2k} y^2 + C_3 y - \frac{cg_y}{2k} z^2 + C_5 + \frac{g_y}{4} t + C_9
\]
\[
V_z(x,y,z,t) = -\frac{ag_z}{2k} x^2 - \frac{bg_z}{2k} y^2 + \frac{cg_z}{2k} z^2 + C_1 x + C_3 y + C_5 + \frac{g_z}{4} t + C_9
\]

For $V_y$ \( h + j + m + n + q = 1 \)
\[
\mu(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}) - \frac{\partial p}{\partial y} + \rho g_y = \rho(\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z})
\]
\[
-k \frac{\partial^2 V_y}{\partial x^2} - K \frac{\partial^2 V_y}{\partial y^2} - k \frac{\partial^2 V_y}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial y} + 4 \frac{\partial V_y}{\partial y} = g_y
\]
\[
V_y = -\frac{hg_y}{2k} x^2 + C_1 x - \frac{jg_y}{2k} y^2 + C_3 y - \frac{mg_y}{2k} z^2 + C_5 + \frac{g_y}{4} t
\]
\[
V_y(x,y,z,t) = -\frac{ag_y}{2k} x^2 + C_1 x + C_2 - \frac{bg_y}{2k} y^2 + C_3 y + C_4 - \frac{cg_y}{2k} z^2 + C_5 + \frac{g_y}{4} t + C
\]

For $V_z$ \( r + s + u + v + w = 1 \)
\[
\mu(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}) - \frac{\partial p}{\partial z} + \rho g_z = \rho(\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z})
\]
\[
-k \frac{\partial^2 V_z}{\partial x^2} - k \frac{\partial^2 V_z}{\partial y^2} - \frac{1}{\rho} \frac{\partial p}{\partial z} + 4 \frac{\partial V_z}{\partial z} = g_z
\]
\[
V_z = -\frac{rg_z}{2k} x^2 + C_1 x - \frac{sg_z}{2k} y^2 + C_3 y - \frac{ug_z}{2k} z^2 + C_5 + \frac{w_z}{4} t
\]
\[
V_z(x,y,z,t) = -\frac{ag_z}{2k} x^2 + C_1 x + C_2 - \frac{bg_z}{2k} y^2 + C_3 y + C_4 - \frac{cg_z}{2k} z^2 + C_5 + \frac{g_z}{4} t + C
\]
Discussion About Linearized N-S Solutions

A solution to equation \(-\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x\) (C) is

\[
V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_{x,t}}{4} + C_9, \quad P(x) = d\rho g_x; \quad (a + b + c + d + f = 1)
\]

This relation gives an identity when checked in Equation (C) above.

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, \(g\), were zero, the first three terms, the seventh term, and \(P(x)\) would all be zero. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known. The above result will be the same when one covers the general case, Option 4.

The above parabolic solution is also encouraging. It reminds one of the parabolic curve obtained when a stone is projected upwards at an acute angle to the horizontal.

More Observations    Comparison of the Navier-Stokes solutions with equations of motion under gravity and liquid pressure of elementary physics

Motion equations of elementary physics:

(B): \(V_f = V_0 + gt\);  (C): \(V_f^2 = V_0^2 + 2gx\);  (D): \(V = \sqrt{2gx}\);  (E): \(x = V_0t + \frac{1}{2}gt^2\)

Liquid Pressure,

The liquid pressure, \(P\) at the bottom of a liquid of depth \(h\) units is given by \(P = \rho gh\)

\(x-\) direction linearized Navier–Stokes equation:

\[-\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x\]

\(x-\) direction Navier–Stokes solution :

\[
V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_{x,t}}{4} + C_9; \quad P(x) = d\rho g_x
\]

Observe the following above:

1. Observe that the first three terms of the solution are parabolic in \(x, y,\) and \(z\); the minus sign showing the inverted parabola when a projectile is fired upwards at an acute angle to the horizontal; also note the "gt" in \(V = gt\) of (B) of the motion equations and the \(fg_{x,t}\) in the Navier-Stokes solution.

2. Observe the \(P = \rho gh\) of the liquid pressure and the \(P(x) = d\rho g_x\) of the Navier-Stokes solution. Note that \(d\) is a ratio term.

There are five main terms in the solution of the linearized Navier-Stokes equation. All of these five terms, namely, \(-\frac{ap_x}{2\mu}x^2, -\frac{bp_x}{2\mu}y^2, -\frac{cp_x}{2\mu}z^2, fg_{x,t}\), and \(d\rho g_x\) are similar (except for the constants involved) to the terms in the equations of motion and fluid pressure of elementary physics. This similarity means that the approach used in solving the Navier-Stokes equation is sound. One should also note that to obtain these five terms simultaneously, only the equation with the gravity term as the subject of the equation will yield these six terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the linearized Navier-Stokes equation, since in this form, one can immediately split-up the equation using ratios, and integrate.
The author also tried the following possible approaches: (D), (E) and (F), but none of the possible solutions completely satisfied the corresponding original equations (D), (E) or (F).

\[
\begin{align*}
\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4 \frac{\partial V_x}{\partial t} &= \frac{\partial p}{\partial x} \quad \text{(D)} \quad \text{(One uses the subject } \frac{\partial V_x}{\partial t} \text{)} \\
\frac{K}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4} \frac{\partial V_x}{\partial t} + g_x &= \frac{\partial V_x}{\partial t} \quad \text{(E), (One uses the subject } \frac{\partial V_x}{\partial x} \text{)} \\
- \frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + 4 \frac{\rho}{\mu} + \frac{1}{\mu} \frac{\partial V_x}{\partial t} &= \frac{\partial^2 V_x}{\partial x^2} \quad \text{(F) \quad (One uses subject } \frac{\partial^2 V_x}{\partial x^2} \text{)}
\end{align*}
\]

**Integration Results Summary**

**Case 1:**

\[
-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) + \frac{\partial p}{\partial x} + 4 \rho \frac{\partial V_x}{\partial t} = \rho g_x \quad \text{(C)}
\]

\[
V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{f g_x}{4} t + C
\]

\[
P(x) = d\rho g_x x; \quad (a + b + c + d + f = 1)
\]

**Case 2:**

\[
\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4 \frac{\partial V_x}{\partial t} \right) = \frac{\partial p}{\partial x} \quad \text{(D). (One uses the subject } \frac{\partial p}{\partial x} \text{)}
\]

\[
V_x(x,y,z,t) = \frac{\lambda x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + \lambda_y x + C_3 y + C_5 z - \frac{f \lambda}{4 \rho} t + C
\]

\[
P(x) = \frac{1}{d} \rho g_x x
\]

**Case 3:**

\[
\frac{K}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4} \frac{\partial V_x}{\partial t} + g_x = \frac{\partial V_x}{\partial t} \quad \text{(E). (One uses the subject } \frac{\partial V_x}{\partial t} \text{)}
\]

\[
V_x(x,y,z,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-(\lambda^2 / \beta) t} + (C_3 \cos \lambda_y x + C_4 \sin \lambda_y x) e^{-(\lambda^2 / \alpha) t}
\]

\[
+ (C_5 \cos \lambda_z x + C_6 \sin \lambda_z x) e^{-(\lambda^2 / \varepsilon) t} + \frac{g}{4 f} t + \lambda x + C
\]

\[
P(x) = \lambda x = d\rho g_x x
\]

**Case 4:**

\[
- \frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + 4 \frac{\rho}{\mu} + \frac{1}{\mu} \frac{\partial V_x}{\partial t} = \frac{\partial^2 V_x}{\partial x^2} \quad \text{(F). (One uses the subject } \frac{\partial^2 V_x}{\partial x^2} \text{)}
\]

\[
V_x(x,y,z,t) = \frac{A \cos \lambda_y + B \sin \lambda_y}{C e^{\left( \frac{\lambda^2}{a} \right) x} + D e^{\left( -\frac{\lambda^2}{a} \right) x}}
\]

\[
+ (E \cos \lambda_z + F \sin \lambda_z \left( \frac{\lambda^2}{b} \right) x) + L e^{-\left( \frac{\lambda^2}{b} \right) x} - \frac{\rho g_x x^2}{2 C \mu} + A x + B + (A_1 \cos \lambda x + B_1 \sin \lambda x) e^{-(\lambda^2 / \alpha) t}
\]

\[
+ \frac{\lambda x}{2 \mu} x^2 + C_2 x + C_3
\]

\[
P(x) = d\rho g_x x
\]

**Note:** Relations for equations with subjects \( g_x \) and \( \frac{\partial p}{\partial x} \) are almost identical.

By comparing possible solutions for equations (C) and (D), \( \lambda_x = -\rho g_x \) in relation for (D).

\[
V_x(x,y,z,t) = \frac{\lambda x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + \lambda_y x + C_3 y + C_5 z - \frac{f \lambda}{4 \mu} t + C; \quad P(x) = \frac{1}{d} \rho g_x x
\]
Discussion About Solutions

Comparative analysis of the possible solutions when checked in each corresponding equation

<table>
<thead>
<tr>
<th>Equation</th>
<th>Equation Subject</th>
<th>Number of terms of possible solutions not satisfying original equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1:</td>
<td>(-\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x)</td>
<td>(\rho g_x)</td>
</tr>
<tr>
<td>Case 2:</td>
<td>(\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4\rho \frac{\partial V_x}{\partial t} = \frac{\partial p}{\partial x})</td>
<td>(\frac{\partial p}{\partial x})</td>
</tr>
<tr>
<td>Case 3:</td>
<td>(\frac{K}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4\rho} \frac{\partial p}{\partial x} + \frac{g_x}{4} \frac{\partial V_x}{\partial t})</td>
<td>(\frac{\partial V_x}{\partial t})</td>
</tr>
<tr>
<td>Case 4:</td>
<td>(-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \rho g_x \frac{\partial V_x}{\partial x} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2})</td>
<td>(\frac{\partial^2 V_x}{\partial x^2})</td>
</tr>
<tr>
<td>Case 5:</td>
<td>(-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \rho g_x \frac{\partial V_x}{\partial x} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2})</td>
<td>(\frac{\partial^2 V_x}{\partial x^2})</td>
</tr>
<tr>
<td>Case 6:</td>
<td>(-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \rho g_x \frac{\partial V_x}{\partial x} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2})</td>
<td>(\frac{\partial^2 V_x}{\partial x^2})</td>
</tr>
</tbody>
</table>

Note above that only Case 1 is the solution, and this may imply that the solution to the Navier-Stokes equation is unique. Out of six possible subjects, only one subject produced a solution. The above results show that a relation obtained by the integration of a partial differential equation must be checked in the corresponding equation for identity before the relation becomes a solution. Cases 2, 3, 4, 5 and 6, are not solutions but integration relations. For example, it would be incorrect to say that the equation in Case 3 has a periodic solution; but it would be correct to say that the equation in Case 3 has a periodic relation, since the relation obtained by integration does not satisfy its corresponding equation. It would be correct to say that the equation in Case 1 has a parabolic solution or a parabolic relation.

Below are detailed explanation of results of the identity checking process.

**Outcome 1:** With \(g_x\) included and with \(g_x\) as the subject of the equation. The solution is straightforward and the possible solution checks well in the original equation (C). Also, if \(g_x\) or \(\rho g_x\) is not the subject of the equation, the linearization of the nonlinear terms could not be justified.

**Outcome 2:** With \(g_x\) included but with \(\frac{\partial V_x}{\partial t}\) as the subject of the equation.

There are two problems when checking: 1. For \(\frac{\partial V_x}{\partial t} = -\frac{1}{4\rho} \frac{\partial p}{\partial x} \rightarrow -\frac{\lambda t}{4\rho d}\); 2. \(\frac{g_x}{4} = \frac{\partial V_x}{\partial t} \rightarrow \frac{g_x t}{4 f}\)

With \(d\) and \(f\) in the denominators, the multipliers sum \(a + b + c + d + f = 1\) is false.

**Outcome 3:** With \(g_x\) excluded, and \(\frac{\partial V_x}{\partial t}\) as the subject of the equation, there is one problem:

\[-\frac{1}{4\rho} \frac{\partial p}{\partial x} = \frac{\partial V_x}{\partial t} \rightarrow -\frac{\lambda t}{4\rho d}\]  With \(d\) in the denominator \(a + b + c + d + f = 1\) is false

**Outcome 4:** With \(g_x\) included, and \(\frac{\partial^2 V_x}{\partial x^2}\) as the subject of the equation, there are at least, two problems in the checking with the multipliers \(c\) and \(f\) in the denominators.

Checking for \(a + b + c + d + f = 1\) is impossible.

**Outcomes 5 and 6 are similar to Outcome 4.**
Characteristic curves of the integration results

<table>
<thead>
<tr>
<th>Equations</th>
<th>Equation Subject</th>
<th>Curve characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: $-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right) + \frac{\partial p}{\partial x} + 4 \rho \frac{\partial V_x}{\partial t} = \rho g_x$</td>
<td>$\rho g_x$</td>
<td>Parabolic and Inverted</td>
</tr>
<tr>
<td>Case 2: $\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_y}{\partial y^2} + \mu \frac{\partial^2 V_z}{\partial z^2} + \rho g_x - 4 \rho \frac{\partial V_x}{\partial t} = \frac{\partial p}{\partial x}$</td>
<td>$\frac{\partial p}{\partial x}$</td>
<td>Parabolic</td>
</tr>
<tr>
<td>Case 3: $\frac{1}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{1}{4} \frac{\partial^2 V_y}{\partial y^2} + \frac{1}{4} \frac{\partial^2 V_z}{\partial z^2} - \frac{1}{4} \frac{\partial p}{\partial x} + \frac{s_z - \partial V_x}{\partial t}$</td>
<td>$\frac{\partial V_x}{\partial t}$</td>
<td>Periodic and decreasingly exponential</td>
</tr>
<tr>
<td>Case 4: $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4 \rho \partial V_x}{\mu \partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2}$</td>
<td>$\frac{\partial^2 V_x}{\partial x^2}$</td>
<td>Periodic, parabolic, and decreasingly exponential</td>
</tr>
<tr>
<td>Case 5: $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4 \rho \partial V_x}{\mu \partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial y^2}$</td>
<td>$\frac{\partial^2 V_x}{\partial y^2}$</td>
<td>Periodic, parabolic, and decreasingly exponential</td>
</tr>
<tr>
<td>Case 6: $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4 \rho \partial V_x}{\mu \partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial z^2}$</td>
<td>$\frac{\partial^2 V_x}{\partial z^2}$</td>
<td>Periodic, parabolic, and decreasingly exponential</td>
</tr>
</tbody>
</table>

The following are possible interpretations of the roles of the terms based on the types of curves produced when using the terms as subjects of the equations.

1. $g_x$ and $\frac{\partial p}{\partial x}$ are involved in the parabolic motion; $g_x$ is responsible for the forward motion.

2. $\frac{\partial V_x}{\partial t}$ is involved in the periodic and decreasingly exponential behavior.

3. $\frac{\partial^2 V_x}{\partial x^2}$, $\frac{\partial^2 V_x}{\partial y^2}$ and $\frac{\partial^2 V_x}{\partial z^2}$ are involved in the parabolic, periodic and decreasingly exponential motion. As $\mu$ increases, the periodicity increases.
Definitions and Classification of Equations

\[-K \frac{\partial^2 V_x}{\partial x^2} - K \frac{\partial^2 V_x}{\partial y^2} - K \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4 \frac{\partial V_x}{\partial t} = g_x\quad (k = \frac{\mu}{\rho})\]

One may classify the equations involved in Option 1 according to the following:

**Driver Equation**: A differential equation whose integration relation satisfies its corresponding equation.

**Supporter equation**: A differential equation which contains the same terms as the driver equation but whose integration relation does not satisfy its corresponding equation but provides useful information about the driver equation.

Note that the driver equation and a supporter equation differ only in the subject of the equation.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Equation Subject</th>
<th>Type of equation</th>
<th># of terms of relation not satisfying original equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1:  [-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) + \frac{\partial p}{\partial x} + 4 \rho \frac{\partial V_x}{\partial t} = \rho g_x]</td>
<td>(\rho g_x)</td>
<td>Driver Equation</td>
<td>None</td>
</tr>
<tr>
<td>Case 2:  [\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4 \rho \frac{\partial V_x}{\partial t} = \frac{\partial p}{\partial x}]</td>
<td>(\frac{\partial p}{\partial x})</td>
<td>Supporter equation</td>
<td>One term</td>
</tr>
<tr>
<td>Case 3:  [\frac{K}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial z^2} + \frac{1}{4} \frac{\partial p}{\partial x} + \frac{g_x}{4} = \frac{\partial V_x}{\partial t}]</td>
<td>(\frac{\partial V_x}{\partial t})</td>
<td>Supporter equation</td>
<td>At least 2 terms</td>
</tr>
<tr>
<td>Case 4:  [-\frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_x}{\partial y^2} - \frac{\rho g_x}{\mu} + 4 \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial t} = \frac{\partial^2 V_x}{\partial x^2}]</td>
<td>(\frac{\partial^2 V_x}{\partial x^2})</td>
<td>Supporter equation</td>
<td>At least 2 terms</td>
</tr>
<tr>
<td>Case 5:  [-\frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_x}{\partial y^2} - \frac{\rho g_x}{\mu} + 4 \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial t} = \frac{\partial^2 V_x}{\partial y^2}]</td>
<td>(\frac{\partial^2 V_x}{\partial y^2})</td>
<td>Supporter equation</td>
<td>At least 2 terms</td>
</tr>
<tr>
<td>Case 6:  [-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + 4 \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial t} = \frac{\partial^2 V_x}{\partial z^2}]</td>
<td>(\frac{\partial^2 V_x}{\partial z^2})</td>
<td>Supporter equation</td>
<td>At least 2 terms</td>
</tr>
</tbody>
</table>

The uniqueness of the above solution will guide one to save time and not try to solve some forms of Euler or Navier-Stokes equation which do not produce solutions. That is, one will solve only the equations with the gravity term as the subject. This uniqueness will also guide one to solve the magnetohydrodynamic equations.
Discussion About Solutions

Applications of the splitting technique in science, engineering, business fields

The approach used in solving the equations allows for how the terms interact with each other. The author has not seen this technique anywhere, but the results are revealing and promising.

Fluid flow design considerations:
1. Maximize the role of \( g_x \) forces, followed by; 2. \( \frac{\partial p}{\partial x} \) forces; then 3. \( \frac{\partial V_x}{\partial t} \).

Make \( g_x \) happy by always providing a workable slope.

For long distance flow design such as for water pipelines, water channels, oil pipelines, whenever possible, the design should facilitate and maximize the role of gravity forces, and if design is impossible to facilitate the role of gravity forces, design for \( \frac{\partial p}{\partial x} \) to take over flow.

The performance of \( \frac{\partial^2 V_x}{\partial x^2} \) should be studied further, since its role is the most complicated: periodic, parabolic, and decreasingly exponential.

Tornado Effect Relief

Perhaps, machines can be designed and built to chase and neutralize or minimize tornadoes during touch-downs. The energy in the tornado at touch-down can be harnessed for useful purposes.

Business and economics applications.
1. Figuratively, if \( g_x \) is the president of a company, it will have good working relationships with all the members of the board of directors, according to the solution of the Navier-Stokes equation. If \( g_x \) is present at a meeting \( g_x \) must preside over the meeting for the best outcome.

2. If \( g_x \) is absent from a meeting, let \( \frac{\partial p}{\partial x} \) preside over the meeting, and everything will workout well.

However, if \( g_x \) is present, \( g_x \) must preside over the meeting.

To apply the results of the solutions of the Navier-Stokes equations in other areas or fields, the properties, characteristics and functions of \( g_x, \frac{\partial p}{\partial x}, \frac{\partial V_x}{\partial t} \) must be studied to determine analogous terms in those areas of possible applications. Other areas of applications include investments choice decisions, financial decisions, personnel management and family relationships.

Option 2

Solutions of 4-D Linearized Navier-Stokes Equations

One advantage of the pairing approach is that the above solution can easily be extended to any number of dimensions.

If one adds \( \mu \frac{\partial^2 V_x}{\partial x^2} \) and \( \rho V_s \frac{\partial V_x}{\partial s} \) to the 3-D \( x \)-direction equation, one obtains the 4-D Navier-Stokes equation:

\[-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial^2 V_s}{\partial s^2} \right) + \frac{\partial p}{\partial x} + 4\rho \left( \frac{\partial V_x}{\partial t} \right) + \rho V_s \frac{\partial V_x}{\partial s} = \rho g_x \]

After linearization,

\[-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial^2 V_s}{\partial s^2} \right) + \frac{\partial p}{\partial x} + 5\rho \left( \frac{\partial V_x}{\partial t} \right) = \rho g_x \]

and its solution is

\[V_x(x,y,z,s,t) = -\frac{\rho g_x}{2\mu} \left( ax^2 + by^2 + cz^2 + es^2 \right) + C_1 x + C_3 y + C_5 z + C_7 s + \frac{f g_x}{3} t + C_9 \]

\[P(x) = d\rho g_x \quad (a + b + c + d + e + f = 1)\]

For \( n \)-dimensions one can repeat the above as many times as one wishes.

Back to Options
Solutions of the Euler Equations of Fluid flow

In the Navier-Stokes equation, if \( \mu = 0 \), one obtains the Euler equation. From

\[
\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) - \frac{\partial p}{\partial x} + \rho g_x = \rho\left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z}\right),
\]

one obtains

Euler equation : \( \mu \frac{\partial}{\partial x} \rho V_x = \rho \left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z}\right) + \frac{1}{\rho} \frac{\partial p}{\partial x} = g_x \)

--- driver equation.

Euler equation \( (\mu = 0) \): \( \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = g_x \) --- driver equation

Split the equation using the ratio terms \( f, h, n, q, d, \), and solve. \( f + h + n + q + d = 1 \)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial V_x}{\partial t} = fg_x )</td>
<td>( V_x = fg_x )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial x} = h g_x )</td>
<td>( V_x = h g_x )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial y} = n g_x )</td>
<td>( V_y = n g_x )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial z} = q g_x )</td>
<td>( V_z = q g_x )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial \rho} = d g_x )</td>
<td>( \frac{1}{\rho} \frac{\partial p}{\partial x} = d g_x )</td>
</tr>
</tbody>
</table>

Find the test derivatives to check in the original equation.

1. \( \frac{\partial V_x}{\partial t} = fg_x \)
2. \( V_x = fg_x \), \( \frac{\partial V_x}{\partial x} = 2 h g_x \), \( \frac{\partial V_x}{\partial y} = n g_x \), \( \frac{\partial V_x}{\partial z} = q g_x \), \( \frac{\partial V_x}{\partial \rho} = d g_x \)
3. \( \frac{\partial V_x}{\partial y} = n g_x \), \( \frac{\partial V_x}{\partial z} = q g_x \), \( \frac{\partial V_x}{\partial \rho} = d g_x \)
4. \( \frac{\partial V_x}{\partial z} = q g_x \), \( \frac{\partial V_x}{\partial \rho} = d g_x \)
5. \( \frac{\partial V_x}{\partial \rho} = d g_x \)

\[
\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = g_x
\]

(Above, \( \psi_y(V_y) \) and \( \psi_z(V_z) \) are arbitrary functions)

\[
fg_x + V_x \frac{h g_x}{V_x} + V_y \frac{n g_x}{V_y} + V_z \frac{q g_x}{V_z} + \frac{1}{\rho} \frac{\partial p g_x}{\partial x} = g_x
\]

\[
f g_x + h g_x + n g_x + q g_x + d g_x = g_x
\]

\[
g_x(f + h + n + q + d) = g_x
\]

\[
g_x(1) = g_x \quad (f + h + n + q + d = 1)
\]

\[
g_x = g_x \quad \text{Yes}
\]
The relation obtained satisfies the Euler equation. Therefore the solution to the Euler equation

\[
\frac{\partial p}{\partial t} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_z}{\partial z} = \rho g_x
\]

is

\[
V_x(x,y,z,t) = f g_s t \pm \sqrt{2h_s} x + \frac{ng_s y}{V_x} + \frac{q g_s z}{V_z} + \frac{\psi_x(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}; \quad P(x) = \rho g_s x
\]

\[V_y \neq 0, \quad V_z \neq 0; \quad (d + f + n + q = 1)\]

Similarly, the equations and solutions for the other two directions are respectively

For \(V_y\),

\[
\frac{\partial p}{\partial t} + \rho \frac{\partial V_y}{\partial t} + \rho V_x \frac{\partial V_y}{\partial x} + \rho V_y \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_y}{\partial z} = \rho g_y
\]

\[
V_y(x,y,z,t) = \lambda g_s t \pm \sqrt{2\lambda g_s} y + \frac{\lambda g_s x}{V_x} + \frac{\lambda g_s z}{V_z} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}; \quad P(y) = \lambda \rho g_s y
\]

\[V_x \neq 0, \quad V_z \neq 0; \quad (\lambda + \lambda_3 + \lambda_6 + \lambda_7 + \lambda_8 = 1)\]

For \(V_z\),

\[
\frac{\partial p}{\partial t} + \rho \frac{\partial V_z}{\partial t} + \rho V_x \frac{\partial V_z}{\partial x} + \rho V_y \frac{\partial V_z}{\partial y} + \rho V_z \frac{\partial V_z}{\partial z} = \rho g_z
\]

\[
V_z(x,y,z,t) = \beta g_s t \pm \sqrt{2\beta g_s} z + \frac{\beta g_s x}{V_x} + \frac{\beta g_s y}{V_y} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z}; \quad P(z) = \beta \rho g_s z
\]

\[V_x \neq 0, \quad V_y \neq 0; \quad (\beta + \beta_5 + \beta_6 + \beta_7 + \beta_8 = 1)\]

One will next solve the above system of solutions for \(V_x, V_y, V_z\) in order to express \(\frac{ng_s y}{V_y}\) and \(\frac{q g_s z}{V_z}\) in terms of \(x, y, z,\) and \(t\).

**Solving for** \(V_x, V_y, V_z\)

\[
V_x = f g_s t \pm \sqrt{2h_s} x + \frac{ng_s y}{V_y} + \frac{q g_s z}{V_z} + \frac{\psi_x(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}; \quad (A)
\]

\[
V_y = \lambda g_s t \pm \sqrt{2\lambda g_s} y + \frac{\lambda g_s x}{V_x} + \frac{\lambda g_s z}{V_z} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}; \quad (B)
\]

\[
V_z = \beta g_s t \pm \sqrt{2\beta g_s} z + \frac{\beta g_s x}{V_x} + \frac{\beta g_s y}{V_y} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z}; \quad (C)
\]

Let \(V_x = x, \quad V_y = y\) and \(V_z = z\). \((x, y\) and \(z\) are being used for simplicity. They will be changed back to \(V_x, V_y,\) and \(V_z\) later, and they do not represent the variables \(x, y\) and \(z\) in the system of solutions.)
### Step 1
From the above system of solutions, let

\[
A = (fgt + \sqrt{2hg_x}x) \quad D = (qg_xz) \quad E = (ng_xy)
\]

\[
B = (\lambda_8 g_t + \sqrt{2\lambda_7 g_y}y) \quad ; F = (\lambda_8 g_x, x) \quad G = (\lambda_8 g_x, z)
\]

\[
C = (\beta_8 g_t + \sqrt{2\beta_7 g_z}z) \quad ; J = (\beta_8 g_x, x) \quad ; L = (\beta_7 g_z, z)
\]

### Step 2:
Then the solutions to the Euler system of equations become (ignoring the arbitrary functions)

\[
x = A + \frac{D}{z} + \frac{E}{y}
\]

\[
y = B + \frac{F}{x} + \frac{G}{z}
\]

\[
z = C + \frac{J}{x} + \frac{L}{y}
\]

### Step 3
Maples software was used to solve system P to obtain

\[
x y z A y z D y + E z
\]

\[
x y z B x z F z + G x
\]

\[
x y z C x y J y + L x
\]

### Step 4
Note:
None of the popular academic programs could solve the system in M.

Maples solved system P (step 4 above) for \(x, y, z\) in terms of \(A, B, C, D, E, F, G, J, \) and \(L.\)

Note also that \(x, y, z\) are not the same as \(x, y, \) and \(z\) in the system of equations.. They were used for convenience and simplicity.

### Step 5:
Apply and substitute in steps 6-8 below

\[
A = (fgt + \sqrt{2hg_x}x) \quad B = (\lambda_8 g_t + \sqrt{2\lambda_7 g_y}y) \quad C = (\beta_8 g_t + \sqrt{2\beta_7 g_z}z) \quad D = (qg_xz) ; E = (ng_xy) ; F = (\lambda_8 g_x, x) ; G = (\lambda_8 g_x, z) ; J = (\beta_8 g_x, x) ; L = (\beta_7 g_z, z)
\]

### Step 6

\[
V_x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)} \quad (\text{back to } V_x)
\]

\[
y = -\frac{L}{C} \quad (\text{changing back to } V_y \text{ as agreed to})
\]

\[
z = -\frac{L(D - J)}{LA - CE} \quad (\text{changing back to } V_z \text{ as agreed to})
\]

### Step 7

\[
V_y = \frac{\beta_7 g_y}{\beta_8 g_t + 2\beta_7 g_z} \quad (\text{changing back to } V_y \text{ as agreed to})
\]

\[
ng_y \frac{V_y}{V_z} = -\frac{[\beta_7 g_y](ng_x y) + \sqrt{2\beta_7 g_z}](ng_x y)}{\beta_7 g_y} \quad (y \neq 0)
\]

\[
ng_y \frac{V_y}{V_z} = -\frac{\beta_7 g_t + \sqrt{2\beta_7 g_z}](ng_x y)}{\beta_7 g_z}
\]
Step 8:  
\[ V_c = - \frac{L(D-J)}{LA-CE} \]

\[ V_c = - \frac{(\beta_7 g_{z,y})[q g_{x,z} - \beta_6 g_{x,y}]}{(\beta_7 g_{x,y})[f_{g,t} \pm \sqrt{2 h_{g,x}}] - (\beta_5 g_{x,y} \pm \sqrt{2 \beta g_{x,z}})(n g_{x,y})} \]

\[ \frac{q g_{x,z}}{V_c} = (q g_{x,z}) \cdot (\beta_7 g_{x,y})[f_{g,t} \pm \sqrt{2 h_{g,x}}] - (\beta_5 g_{x,y} \pm \sqrt{2 \beta g_{x,z}})(n g_{x,y}) \]

\[ = - (q g_{x,z}) \cdot (\beta_7 g_{x,y})[f g_{t} \pm \sqrt{2 h_{g,x}}] - (\beta_5 g_{x,y} \pm \sqrt{2 \beta g_{x,z}})(n g_{x,y}) \]

\[ = - (q g_{x,z}) \cdot (\beta_7 g_{z,y} f_{g,t} \pm \sqrt{2 h_{g,x}} \beta_7 g_{x, y} - \beta_5 g_{x,y} n g_{x,y} \pm \sqrt{2 \beta g_{x,z}} n g_{x,y}) \]

\[ = - (q g_{x,z}) \cdot (\beta_7 g_{z,y})[q g_{x,z} - \beta_6 g_{x,y}] \]

\[ q g_{x,z} = - \frac{g_{x,z} \beta_7 f_{g,t} \pm \sqrt{2 h_{g,x}} x \beta_7 g_{z,y} - g_{x,y} n q t z \pm \sqrt{2 \beta g_{x,z}} g_{x,y} n q t z}{(\beta_7 g_{z,y})[q g_{x,z} - \beta_6 g_{x,y}]} \]

\[ \left( \text{Dividing out the "} y \text{" in the numerator and the denominator} \right) \]

\[ q g_{x,z} = - \frac{g_{x,z} \beta_7 f_{g,t} \pm \sqrt{2 h_{g,x}} x \beta_7 g_{z,y} - g_{x,y} n q t z \pm \sqrt{2 \beta g_{x,z}} g_{x,y} n q t z}{(\beta_7 g_{z,y})[q g_{x,z} - \beta_6 g_{x,y}]} \]

Summary for the fractional terms of the x-direction solution

\[ \frac{n g_{y,y}}{V_y} \text{ and } \frac{q g_{z,x}}{V_z} \text{ in terms of } x, y, z \text{ and } t \]

\[ \frac{n g_{y,y}}{V_y} = \frac{\frac{n \beta_5 g_{x,t} \pm \sqrt{2 \beta g_{z}}(n x)}{g_{z}}}{B} \]

\[ \frac{n g_{y,y}}{V_y} = \frac{-k_1 g_{x,t} \pm \sqrt{2 k_2 g_{z} \cdot g_{x,k_3}}}{g_{z}} \]

\[ k_1 = \frac{n \beta_5}{g_{z}}; \quad k_2 = \beta_8; \quad k_3 = \frac{n}{g_{z}} \]

\[ \frac{q g_{z,x}}{V_z} = \frac{\beta_7 f_{g,x} g_{z,y} q - g_{x,y} n q t z \pm \sqrt{2 h_{g,x} \beta_7 g_{z,y} - g_{x,y} n q t z \pm \sqrt{2 \beta g_{z}} g_{x,y} n q t z}}{\beta_7 g_{z,y} g_{x,y} x - g_{x,y} q t z} \]

\[ \frac{q g_{z,x}}{V_z} = \frac{\beta_7 f_{g,x} g_{z,y} q - g_{x,y} n q t z \pm \sqrt{2 h_{g,x} \beta_7 g_{z,y} - g_{x,y} n q t z \pm \sqrt{2 \beta g_{z}} g_{x,y} n q t z}}{\beta_7 g_{z,y} g_{x,y} x - g_{x,y} q t z} \]

or

\[ \frac{q g_{z,x}}{V_z} = \frac{\beta_7 f_{g,x} g_{z,y} q - g_{x,y} n q t z \pm \sqrt{2 h_{g,x} \beta_7 g_{z,y} - g_{x,y} n q t z}}{\beta_7 g_{z,y} g_{x,y} x - g_{x,y} q t z} \]

\[ k_4 = \beta_7 f_{g,x}; \quad k_5 = \beta_7 n q t; \quad k_6 = h; \quad k_8 = \beta_8; \quad k_9 = n q t \]

\[ k_{10} = \beta_7 b_6; \quad k_{11} = \beta_7 q \]
Analysis of the Euler Solutions

\[ V_x(x,y,z,t) = f_{g_x}t \pm \sqrt{2}g_{s,x}x + \frac{ng_{x,y}}{V_y} + qg_{x,z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}; \quad P(x) = d\rho g_x x \]

\[ V_y \neq 0, \quad V_z \neq 0; \quad (d + f + h + n + q = 1) \]

\[ \frac{ng_{x,y}}{V_y} = -\frac{n\beta_5 g_x t}{\beta_7} \pm \frac{(\sqrt{2}\beta_8 g_z n g_x)}{\beta_7 g_z} \]

\[ \frac{qg_{x,z}}{V_z} = \left\{ \frac{\beta_7 f_{g_z} g_x^2 q - \beta_8 g_x g_n q n z \pm \sqrt{2}g_{s,x} x \beta_7 g_{s,z} g_z \pm \sqrt{2}g_{s,x} x g_x^2 n q z}{\beta_7 \beta_6 g_z ^2 x - \beta_7 g_{s,x} g_Z z} \right\} \]

\[ d + f + h + n + q = 1; \quad \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1; \quad \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 = 1 \]

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, \( g_x \) were zero, the first four terms of the velocity solution and \( P(x) \) would all be zero. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.

More Observations: Comparison of the Euler solutions with equations of motion under gravity and liquid pressure of elementary physics

Motion under gravity equations: (B): \( V = gt; \quad (C): \quad V = \sqrt{2}gx \)

Liquid Pressure. \( P \) at the bottom of a liquid of depth \( h \) units is given by \( P = \rho gh \)

Observe the following similarities above:
1. Observe the "\( gt \)" in \( V = gt \) of (B) of the motion equations and the \( f_{g_x}t \) in the Euler solution.
2. Observe the "\( \sqrt{2}gx \)" in \( V = \sqrt{2}gx \) of (C) and the \( \sqrt{2}g_{s,x}x \) in the Euler solution.
3. Observe the \( P = \rho gh \) of the liquid pressure and the \( P(x) = d\rho g_x x \) of the Euler solution.

There are five main terms (ignoring the arbitrary functions) in the Euler solution. Of these five terms, three terms, namely, \( f_{g_x}t, \sqrt{2}g_{s,x}x, d\rho g_x x \) are the same (except for the constants involved) as the terms in the equations of motion under gravity. This similarity means that the approach used in solving the Euler equation is sound. One should also note that to obtain these three terms simultaneously, only the equation with the gravity term as the subject of the equation will yield these three terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Euler equation, since in this form, one can immediately split-up the equations using ratios, and integrate.

The velocity profile of the \( x \)-direction solution consists of linear, parabolic, and hyperbolic terms. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would be at right angles to that for laminar flow. The characteristic curve for the integral of the \( x \)-nonlinear term is such a parabola whose axis of symmetry is at right angles to that of laminar flow. The integral of the \( y \)-nonlinear term is similar parabolically to that of the \( x \)-nonlinear term. The characteristic curve for the integral of the \( z \)-nonlinear term is a combination of two similar parabolas and a hyperbola. If the above \( x \)-direction flow is repeated simultaneously in the \( y \)- and \( z \)-directions, the flow is chaotic and consequently turbulent.
Solutions of the Euler Equations

Standard form of the x-direction Euler equation for incompressible fluid flow

One will call the Euler equation with the gravity term as the subject of equation in (A), the standard form of the Euler equation for the ratio method of solving these equations, since this form produces a solution on integration. None of the other forms in (B), (C), (D), (E), or (F), produces a solution. That is, the integration results of each of the other five equations do not satisfy the corresponding equation.

\[
\rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} + \frac{\partial p_x}{\partial x} = \rho g_x \quad \text{(A)} < \text{standard form}
\]

\[
-\rho \frac{\partial V_x}{\partial t} - \rho V_x \frac{\partial V_x}{\partial x} - \rho V_y \frac{\partial V_x}{\partial y} - \rho V_z \frac{\partial V_x}{\partial z} + \frac{\partial p_x}{\partial x} = \rho \frac{\partial V_x}{\partial t} \quad \text{(B)}
\]

\[
-\rho V_x \frac{\partial V_x}{\partial y} - \rho V_y \frac{\partial V_x}{\partial y} - \rho V_z \frac{\partial V_x}{\partial z} + \frac{\partial p_x}{\partial x} + \rho g_x = \rho V_y \frac{\partial V_x}{\partial y} \quad \text{(C)}
\]

\[
-\rho V_x \frac{\partial V_x}{\partial z} - \rho V_y \frac{\partial V_x}{\partial z} - \rho V_z \frac{\partial V_x}{\partial z} + \frac{\partial p_x}{\partial x} + \rho g_x = \rho V_z \frac{\partial V_x}{\partial z} \quad \text{(D)}
\]

\[
-\rho V_x \frac{\partial V_x}{\partial t} - \rho V_y \frac{\partial V_x}{\partial y} - \rho V_z \frac{\partial V_x}{\partial z} + \frac{\partial p_x}{\partial x} + \rho g_x = \rho V_z \frac{\partial V_x}{\partial z} \quad \text{(E)}
\]

\[
-\rho V_x \frac{\partial V_x}{\partial t} - \rho V_y \frac{\partial V_x}{\partial y} - \rho V_z \frac{\partial V_x}{\partial z} + \frac{\partial p_x}{\partial x} + \rho g_x = \rho V_z \frac{\partial V_x}{\partial z} \quad \text{(F)}
\]

Uniqueness of the solution of the Euler equation

When each term of the linearized Navier-Stokes equation was made subject of the N-S equation, only the equation with the gravity term as the subject of the equation produced a solution. (vixra:1405.0251 of 2014). Similarly, the solution of the Euler solution is unique.

Extra:

Linearized Euler Equation: If one linearizes the Euler equation as was done in the linearization of the Navier-Stokes equation, one obtains \( 4 \frac{\partial V_x}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} = g_x \); whose solution is

\[
V_x = \frac{f_{gs,t}}{4} + C; \quad P(x) = d\rho g_x x.
\]

**Euler solutions in terms of** \( x, y, z, \) and \( t. \)

\[
V_x(x,y,z,t) = \frac{fg_{s,t}t \pm \sqrt{2hg_x x} \beta g_x \gamma}{\gamma} + \frac{\sqrt{2g_x g_y z} (ng_x)}{\beta g_z} + \frac{(g_x^2 g_y k_4 - g_y g_z k_2) t z \pm \sqrt{2g_x g_y g_z k_5}}{\beta g_y} + \frac{g_x^2 k_{10} t x - g_y g_z k_{11} t z}{\beta g_x} + \frac{\psi_y(V_x)}{V_y} + \frac{\psi_z(V_z)}{V_z}; \quad P(x) = d\rho g_x x
\]

\[
V_y \neq 0, \quad V_z \neq 0; \quad (d + f + h + n + q = 1)
\]

**Note:** By comparison with Navier-Stokes equation and its relation, a relation to Euler equation can be found by deleting the Navier-Stokes relation resulting from the \( \mu \)-terms.
Option 4
Solutions of 3-D Navier-Stokes Equations (Original)

Mehod 1
As in Option 1 for solving these equations, the first step here, is to split-up the equation into eight sub-equations using the ratio method. One will solve only the driver equation, based on the experience gained in solving the linearized equation. There are 8 supporter equations.

Step 1: Apply the ratio method to equation (B) to obtain the following equations:

1. \(-K \frac{\partial^2 V_x}{\partial x^2} = a g_x\); 2. \(-K \frac{\partial^2 V_y}{\partial y^2} = b g_x\); 3. \(-K \frac{\partial^2 V_z}{\partial z^2} = c g_x\); 4. \(\frac{1}{\rho} \frac{\partial p}{\partial x} = d g_x\); 5. \(\frac{\partial V_x}{\partial t} = f g_x\).

6. \(V_x \frac{\partial V_x}{\partial x} = h g_x\); 7. \(V_y \frac{\partial V_y}{\partial y} = q g_x\); 8. \(V_z \frac{\partial V_z}{\partial z} = n g_x\).

where \(a, b, c, d, f, h, n, q\) are the ratio terms and \(a+b+c+d+f+h+n+q = 1\).

Step 2: Solve the differential equations in Step 1.

Note that after splitting the equations, the equations can be solved using techniques of ordinary differential equations.

One can view each of the ratio terms \(a, b, c, d, f, h, n, q\) as a fraction (a real number) of \(g_x\) contributed by each expression on the left-hand side of equation (B) above.

Solutions of the eight sub-equations

| 1. \(-k \frac{\partial^2 V_x}{\partial x^2} = ag_x\) | 2. \(-k \frac{\partial^2 V_y}{\partial y^2} = bg_x\) | 3. \(-k \frac{\partial^2 V_z}{\partial z^2} = cg_x\) | 4. \(\frac{1}{\rho} \frac{\partial p}{\partial x} = d g_x\) |
| \(\frac{k \frac{\partial^2 V_x}{\partial x^2}}{= -ag_x}\) | \(\frac{k \frac{\partial^2 V_y}{\partial y^2}}{= -bg_x}\) | \(\frac{k \frac{\partial^2 V_z}{\partial z^2}}{= -cg_x}\) | \(\frac{1}{\rho} \frac{\partial p}{\partial x}\) |
| \(\frac{\partial^2 V_x}{\partial x^2} = -ag_x\) \(\frac{\partial^2 V_y}{\partial y^2} = -bg_x\) \(\frac{\partial^2 V_z}{\partial z^2} = -cg_x\) | \(\frac{\partial V_x}{\partial x} = \frac{a}{k} g_x\) \(\frac{\partial V_y}{\partial y} = \frac{b}{k} g_x\) \(\frac{\partial V_z}{\partial z} = \frac{c}{k} g_x\) | \(\frac{\partial V_x}{\partial x} = \frac{a}{k} g_x\) \(\frac{\partial V_y}{\partial y} = \frac{b}{k} g_x\) \(\frac{\partial V_z}{\partial z} = \frac{c}{k} g_x\) | \(\frac{\partial p}{\partial x} = \frac{a}{k} \frac{\partial V_x}{\partial x} + C_i\) \(\frac{\partial p}{\partial y} = \frac{b}{k} \frac{\partial V_y}{\partial y} + C_j\) \(\frac{\partial p}{\partial z} = \frac{c}{k} \frac{\partial V_z}{\partial z} + C_k\) |
| \(V_{x1} = \frac{-ag_x}{2k} \frac{x^2}{x^2 + C_1} + C_2\) \(V_{x2} = -\frac{bg_x}{2k} \frac{y^2}{y^2 + C_3} + C_4\) \(V_{x3} = -\frac{cg_x}{2k} \frac{z^2}{z^2 + C_5} + C_6\) | \(V_{y4} = \frac{\psi_y(V_y)}{V_y}\) \(V_{y5} = \frac{\psi_y(V_y)}{V_y}\) \(V_{y6} = \frac{\psi_y(V_y)}{V_y}\) | \(V_{z7} = \frac{\psi_z(V_z)}{V_z}\) \(V_{z8} = \frac{\psi_z(V_z)}{V_z}\) | \(V_{z9} = \frac{\psi_z(V_z)}{V_z}\) |

Note:
\(\psi_y(V_y), \psi_z(V_z)\) are arbitrary functions, (integration constants)
\(V_y \neq 0\)
\(V_z \neq 0\)
Step 3: One combines the above solutions

\[ V_x(x,y,z,t) = V_{x1} + V_{x2} + V_{x3} + V_{x4} + V_{x5} + V_{x6} + V_{x7} \]

\[ = -\frac{ag_x}{2\kappa} x^2 + C_1 x - \frac{bg_y}{2\kappa} y^2 + C_3 y - \frac{cg_z}{2\kappa} z^2 + C_5 z + f_{gz} t \pm \sqrt{2hg_x x + \frac{ng_y y}{V_y} + \frac{qg_z z}{V_z} + \psi_y(V_y) + \psi_z(V_z)} \]

relation for linear terms

\[ -\frac{pg_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f_{gz} t \pm \sqrt{2hg_x x + \frac{ng_y y}{V_y} + \frac{qg_z z}{V_z} + \psi_y(V_y) + \psi_z(V_z)} + C_9 \]

arbitrary functions

\[ P(x) = dpg_s x; \quad (a + b + c + d + f + h + n + q = 1) \quad V_y \neq 0, \quad V_z \neq 0 \]

Step 4: Find the test derivatives

<table>
<thead>
<tr>
<th>Test derivatives for the linear part</th>
<th>Test derivatives for the non-linear part</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial^2 V_x}{\partial x^2} = -\frac{apg_s x}{\mu} )</td>
<td>( \frac{\partial^2 V_x}{\partial y^2} = -\frac{bpg_s x}{\mu} )</td>
</tr>
<tr>
<td>( \frac{\partial^2 V_x}{\partial z^2} = -\frac{cpg_s x}{\mu} )</td>
<td>( \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} (f_{gs} x) )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial t} = \frac{\partial}{\partial t} (f_{gs} x) )</td>
<td>( V_x^2 = 2hg_x x )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y} (f_{gs} x) )</td>
<td>( 2V_x \frac{\partial V_x}{\partial x} = 2hg_x )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial z} = \frac{\partial}{\partial z} (f_{gs} x) )</td>
<td>( \frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y} (f_{gs} x) )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial z} = \frac{\partial}{\partial z} (f_{gs} x) )</td>
<td>( V_x \neq 0 )</td>
</tr>
</tbody>
</table>

Step 5: Substitute the derivatives from Step 4 in equation (A) for the checking.

\[ -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial y} + \rho V_x \frac{\partial V_x}{\partial z} = \rho g_x \quad (A) \]

\[ -\mu (\frac{apg_s x}{\mu} - \frac{bpg_s x}{\mu} - \frac{cpg_s x}{\mu}) + dpg_s x + fpg_s x + \rho (V_x \frac{hg_x}{V_x}) + \rho V_y (\frac{ng_y y}{V_y}) + \rho V_z (\frac{qg_z z}{V_z} = \rho g_x \]

\[ ag_s x + bg_x + cg_x + dg_x + f_{gs} x + hg_x + ng_x + qg_x = g_x \]

\[ g_x (a + b + c + d + f + h + n + q = 1) \]

Step 6: The linear part of the relation satisfies the linear part of the equation; and the non-linear part of the relation satisfies the non-linear part of the equation. (B) below is the solution.

Analogy for the Identity Checking Method: If one goes shopping with American dollars and Japanese yens (without any currency conversion) and after shopping, if one wants to check the cost of the items purchased, one would check the cost of the items purchased with dollars against the receipts for the dollars; and one would also check the cost of the items purchased with yens against the receipts for the yens purchase. However, if one converts one currency to the other, one would only have to check the receipts for only a single currency, dollars or yens. This conversion case is similar to the linearized equations, where there was no partitioning in identity checking. Note that for the Euler equations, there was no partitioning in taking derivatives for identity checking.

Note: After expressing \( \frac{ng_y y}{V_y} \) and \( \frac{qg_z z}{V_z} \) in terms of \( x, y, z, \) and \( t \), there would be no partitioning in identity checking.
Summary of solutions for \( V_x, V_y, V_z \) (\( P(x) = d\rho g_x x; \ P(y) = \lambda_x \rho g_x y; \ P(z) = \beta_x \rho g_x z \))

\[
V_x = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f g_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{ng_z z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C_9
\]

\[P(x) = d\rho g_x x; \quad (a + b + c + d + e + n + q + 1) \quad V_y \neq 0, \ V_z \neq 0\]

\[
V_y = -\frac{\rho g_y}{2\mu} (\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2) + C_{10} x + C_{11} y + C_{12} z + \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y} + \frac{\lambda_6 g_x x}{V_x} + \frac{\lambda_6 g_y y}{V_y} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}
\]

\[
V_z = -\frac{\rho g_z}{2\mu} (\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_{14} x + C_{15} y + C_{16} z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z} + \frac{\beta_6 g_x x}{V_x} + \frac{\beta_6 g_y y}{V_y} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}
\]

The above solutions are unique, because from the experience in Option 1, only the equations with the gravity terms as the subjects of the equations produced the solutions.

One will next solve the above system of solutions for \( V_x, V_y, V_z \) in order to express \( \frac{ng_x y}{V_y} \) and \( \frac{q g_z z}{V_z} \) in terms of \( x, y, z, t \). The author used the help of the Maples software for the simultaneous algebraic solutions for \( V_y, V_z \). The basic expressions are of the forms

\[
-\frac{\rho g_x}{2\mu} ax^2, -\frac{\rho g_x}{2\mu} by^2, -\frac{\rho g_x}{2\mu} cz^2, f g_x t, \sqrt{2hg_x x} , \quad \frac{d\rho g_x x}{x}
\]

These expressions are similar to the terms of the equations of motion under gravity and liquid pressure of elementary physics. Note that the explicit solutions will be the results of the basic operations (addition, subtraction, multiplication, division, power finding and root extraction) on the expressions in Step 1 below.

**Solving for** \( V_x, V_y, V_z \), \( \frac{ng_x y}{V_y} \) and \( \frac{q g_z z}{V_z} \)

Let \( V_x = x, \ V_y = y \) and \( V_z = z \). (\( x, y \) and \( z \) are being used for simplicity. They will be changed back to \( V_x, V_y, V_z \) later, and they do not represent the variables \( x, y \) and \( z \) in the solutions)

**Step 1** From the above system of solutions, let

\[
A = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f g_x t \pm \sqrt{2hg_x x}
\]

\[
B = -\frac{\rho g_y}{2\mu} (\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2) + C_{10} x + C_{11} y + C_{12} z + \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y}
\]

\[
C = -\frac{\rho g_z}{2\mu} (\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_{14} x + C_{15} y + C_{16} z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}
\]

\[
D = q g_x x; \quad E = ng_x y; \quad F = \lambda_6 g_x x
\]

\[
G = \lambda_6 g_y y; \quad J = \beta_6 g_x x; \quad L = \beta_7 g_y y
\]

**Step 2** Then the solutions to the N-S system of equations become (ignoring the arbitrary functions)

\[
x = A + \frac{D}{z} + \frac{E}{y}
\]

\[
y = B + \frac{F}{x} + \frac{G}{z}
\]

\[
z = C + \frac{J}{x} + \frac{L}{y}
\]
Solutions of the 3-D Navier-Stokes Equations (Original)

Step 3
\[
\begin{align*}
xyz &= Ayz + Dy + Ez \\
xyz &= Bxz + Fz + Gx \\
xyz &= Cxy + Jy + Lx
\end{align*}
\]

\[\text{Step 4}\]
\[
\begin{align*}
0 &= Ayz + Dy - Bxz - Fz - Gx \\
0 &= Ayz + Dy - Cxy - Jy - Lx \\
0 &= Bxz + Fz + Gx - Cxy - Jy - Lx
\end{align*}
\]

Maples software was used to solve system P to obtain

Step 5
\[
\begin{align*}
x &= \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)} \\
V_x &= \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)} \text{ (back to }V_x) \\
y &= -\frac{L}{C} \\
V_y &= -\frac{L}{C} \text{ (changing back to }V_y \text{ as agreed to)} \\
z &= -\frac{L(D - J)}{LA - CE} \\
V_z &= -\frac{L(D - J)}{LA - CE} \text{ (changing back to }V_z \text{ as agreed to)}
\end{align*}
\]

Note:
None of the popular academic programs could solve the system in M. Maples solved system P (step 4 above) for \(x, \ y,\) and \(z\) in terms of \(A, B, C, D, E, F, G, J,\) and \(L.\)

Note also that \(x, y\) and \(z\) are not the same as the \(x, y\) and \(z\) in the system of equations. They were used for convenience and simplicity.

Step 5: Apply the following and substitute for \(A, B, C, D, E, F, G, J,\) and \(L\) in steps 6-8 below
\[
\begin{align*}
A &= -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + f_{g,x} + t \pm \sqrt{2h_{g,x}x} \\
B &= -\frac{\rho g_y}{2\mu}(\lambda_1x^2 + \lambda_2y^2 + \lambda_3z^2) + C_{10}x + C_{11}y + C_{12}z + \lambda_5y + t \pm \sqrt{2\lambda_7g,y} \\
C &= -\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_{14}x + C_{15}y + C_{16}z + \beta_5z + t \pm \sqrt{2\beta_8g,z} \\
D &= q_{g,z} x ; \ E = q_{g,y} y ; \ J = \lambda_{g,z} x ; \ L = \beta_{g,z} y
\end{align*}
\]

Step 6
\[
\begin{align*}
V_y &= -\frac{L}{C} = \frac{(\beta_7 g_z,y)}{-\frac{\rho g_x}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_z + t \pm \sqrt{2\beta_8g_z}z} \\
\frac{ng_x,y}{V_y} &= ng_x,y + \frac{(\beta_7 g_z,y)}{-\frac{\rho g_y}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_z + t \pm \sqrt{2\beta_8g_z}z} \\
\frac{ng_x,y}{V_y} &= \frac{-\left(\beta_7 g_z,y\right)}{-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_z + t \pm \sqrt{2\beta_8g_z}z} \\
\frac{ng_x,y}{V_y} &= \frac{-(\beta_7 g_z,y)}{-\frac{\rho g_x}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_z + t \pm \sqrt{2\beta_8g_z}z} \\
\frac{ng_x,y}{V_y} &= \frac{ng_x,y}{\beta_7 g_z} \text{ (cancel "y")}
\end{align*}
\]
Step 7: $V_z = \frac{-L(D-J)}{LA-CE} = \frac{JL-DL}{LA-CE}$

\[
\frac{q_g z}{V_z} = (q_g z) \cdot \frac{(\beta g_z y)[-(\frac{pg_y}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + fg_x t \pm \sqrt{2hg_x x})]}{(\beta g_z y)[\beta g_x z - q_g z]}
\]

\[
\frac{q_g z}{V_z} = (q_g z) \cdot \frac{(\beta g_z y)[-(\frac{pg_y}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + fg_x t \pm \sqrt{2hg_x x})]}{(\beta g_z y)[\beta g_x z - q_g z]}
\]

**Summary for the fractional terms of the x-direction solution**

\[
\begin{align*}
\frac{ng_y}{V_y} \text{ and } \frac{q_g z}{V_z} \text{ in terms of } x, y, z \text{ and } t
\end{align*}
\]

\[
\frac{ng_y}{V_y} = -(ng_x)(\frac{pg_z}{2\mu}(\beta g_x^2 + \beta z^2 + \beta z^2) + C_1 x + C_3 y + C_5 z + \beta g_x t \pm \sqrt{2\beta g_z z})
\]

\[
\frac{q_g z}{V_z} = (q_g z) \cdot \frac{(\beta g_z y)[-(\frac{pg_y}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + fg_x t \pm \sqrt{2hg_x x})]}{(\beta g_z y)[\beta g_x z - q_g z]}
\]

\[
(CE = -(ng_x)(\frac{pg_z}{2\mu}(\beta g_x^2 + \beta z^2 + \beta z^2) + C_1 x + C_3 y + C_5 z + \beta g_x t \pm \sqrt{2\beta g_z z}) \quad d + f + a + \lambda = 1; \quad \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1; \quad \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 = 1
\]

**Expanded N-S Solutions** (in explicit solutions)

\[
V_z(x, y, z, t) = A + B + F
\]

\[
A = -(\frac{pg_y}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + fg_x t \pm \sqrt{2hg_x x} + C_9)
\]

\[
B = -(ng_x)(\frac{pg_z}{2\mu}(\beta g_x^2 + \beta z^2 + \beta z^2) + C_1 x + C_3 y + C_5 z + \beta g_x t \pm \sqrt{2\beta g_z z}) + \psi_y(V_y)
\]

\[
F = (q_g z) \cdot \frac{(\beta g_z y)[-(\frac{pg_y}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + fg_x t \pm \sqrt{2hg_x x})]}{(\beta g_z y)[\beta g_x z - q_g z]}
\]

\[
P(x) = dpg_x, \quad (a + b + c + d + h + n + q = 1) \quad \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 = 1
\]
x-direction solution

\[ v_x = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f_{gx, t} \pm \sqrt{2hg_x x} + \frac{ng_{y,y}}{V_y} + \frac{ng_{z,z}}{V_z} \psi_y(V_y) + \psi_z(V_z) + C_9 \]

\[ P(x) = d\rho g_x, x; \quad (a + b + c + d + h + n + q = 1) \quad v_y \neq 0, \quad v_z \neq 0 \]

\[ \frac{ng_{y,y}}{V_y} = -\frac{(ng_x)(-\rho g_x)}{2\mu} (\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_1 x + C_3 y + C_5 z + \beta_5 g_x t \pm \sqrt{2\beta_6 g_z z}) \]

\[ \frac{ng_{z,z}}{V_z} = \frac{-(ng_{x,z})((\beta_7 g_y, y) - \rho g_x/2\mu (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f_{gx, t} \pm \sqrt{2hg_x x} - [CE])}{(\beta_7 g_y, y)(q g_{x,z} - \beta_6 g_z z)} \]

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, \( g_x \), were zero, the first three terms, the seventh, the eighth, the ninth, the tenth terms of the velocity solution and \( P(x) \) would all be zero. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.

More Observations

Comparison of the Navier-Stokes solutions with equations of motion under gravity and liquid pressure of elementary physics

Motion equations of elementary physics:

(B): \( V_f = V_0 + gt \)

(C): \( V_f^2 = V_0^2 + 2gx \)

(D): \( V = \sqrt{2gx} \)

(E): \( x = V_0 t + \frac{1}{2} gt^2 \)

Liquid Pressure

The liquid pressure, \( P \) at the bottom of a liquid of depth \( h \) units is given by \( P = \rho gh \)

Observe the following above:

1. Observe that the first three terms are parabolic in \( x, y, \) and \( z \); the minus sign showing the usual inverted parabola when a projectile is fired upwards at an acute angle to the horizontal. Also note the "gt" in \( V = gt \) of (B) of the motion equations and the \( f_{gx, t} \) in the Navier-Stokes solution.

2. Observe the \( P = \rho gh \) of the liquid pressure and the \( P(x) = d\rho g_x x \) of the Navier-Stokes solution.

Note that \( d \) is a ratio term.

3. Observe the "\( \sqrt{2gx} \)" in \( V = \sqrt{2gx} \) of (D) and the \( \sqrt{2hg_x x} \) in the Navier-Stokes solution.

There are eight main terms (ignoring the arbitrary functions) in the Navier-Stokes solution. Of these eight terms, six terms, namely, \(-\frac{ap_{g_x}}{2\mu} x^2, -\frac{bp_{g_x}}{2\mu} y^2, -\frac{cp_{g_x}}{2\mu} z^2, f_{gx, t}, \sqrt{2hg_x x} \) and \( d\rho g_x x \) are similar (except for the constants involved) to the terms in the equations of motion. This similarity means that the approach used in solving the Navier-Stokes equation is sound. One should also note that to obtain these six terms simultaneously on integration, only the equation with the gravity term as the subject of the equation will yield these six terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Navier-Stokes equation, since in this form, one can immediately split-up the equation using ratios, and integrate.
Velocity Profile, Polynomial and Radical Parabolas, Laminar and Turbulent flow

For communication purposes, each of the terms containing the even powers $x^2$, $y^2$ and $z^2$ will be called a polynomial parabola, and each of the terms containing the square roots $\pm \sqrt{x}$, $\pm \sqrt{y}$ and $\pm \sqrt{z}$ will be called a radical parabola. For each polynomial parabola, the axis of symmetry is in the direction of fluid flow; but for each radical parabola, the axis of symmetry is at right angles to the direction of fluid flow.

The fluid flow in the Navier-Stokes solution may be characterized as follows. The $x$–direction solution consists of linear, parabolic, and hyperbolic terms. The first three terms characterize polynomial parabolas. The characteristic curve for the integral of the $x$–nonlinear term is a radical parabola. The integral of the $y$–nonlinear term is similar parabolically to that of the $x$–nonlinear term. The integral of the $z$–nonlinear term is a combination of two radical parabolas and a hyperbola. If the above $x$–direction flow is repeated simultaneously in the $y$– and $z$– directions, the flow is chaotic and consequently turbulent.

In the N-S solution, during fluid flow, both the polynomial and radical parabolas are present at any speed. The polynomial parabolas are prominent and dominate flow while the radical parabolas are dormant at low speeds. At a low speed, a radical parabola (or polynomial parabolas susceptible to radicalization) is not active, since the radicand of the parabola is small and consequently, the root is small. When the speed becomes large, the "$x$" in $\sqrt{2hg_{xx}}$ becomes large and therefore the radical parabola becomes active. One can also observe how gravity interacts with the "$x$" of the radicand. By "$g$" and "$x$" being factors of the radicand (instead of "$g$" being outside the radical), "$g$" is closely aligned with $x$. Note that the radical parabola will be moving at right angles to the direction of fluid flow, the direction of which is also that of the axis of symmetry of the dominating polynomial parabola. Consequently, the flow profile becomes relatively more uniform or flattened due to the radical parabola moving at right angles to the direction of fluid flow. When viscosity increases, speed decreases, and the radicand (the factor $x$ in $\sqrt{2hg_{xx}}$ decreases) of the radical parabola decreases. Consequently, the disruptive behavior of the radical parabola diminishes. When the fluid flows over an obstacle, the radical parabolas temporarily become significant resulting in turbulence. For a low value of $x$ (i.e., from low fluid velocity), the viscous term dominates and the inertial term is not significant. At high fluid velocity, the factor "$x$" of the radicand is large. Also when density increases, velocity increases and the radicand increases, adding to the effect of the radical parabola.

Analogy:
Imagine a crowded marathon race involving one thousand runners at various positions on the race route, all running in the same direction. Imagine also that at certain points on the route, during the race, some of the runners at various positions suddenly begin to run to the left or to the right in directions at right angles to the direction of the race route; and imagine the resulting collisions and chaos. The polynomial parabolas are those runners following the route of the race, and the radical parabolas are those runners making ninety-degree turns from various positions, literally, the radical parabolas disrupt the laminar flow.

Uniqueness of the solution of the Navier-Stokes equation
When each term of the linearized Navier-Stokes equation was made subject of the N-S equation, only the equation with the gravity term as the subject of the equation produced a solution. Similarly, the solution of the Navier-Stokes equation solution is unique.
Option 5

Solutions of 4-D Navier-Stokes Equations

In the above method, the solution can easily be extended to any number of dimensions.

Adding $\mu \frac{\partial^2 V_x}{\partial s^2}$ and $\rho V_x \frac{\partial V_x}{\partial s}$ to the 3-D $x$-direction equation yields the 4-D N-S equation

$$-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_x}{\partial s^2} \right) + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} + \rho V_s \frac{\partial V_x}{\partial s} = \rho g_x$$

whose solution is given by

$$V_x(x,y,z,s,t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2 + es^2) + C_1 x + C_3 y + C_5 z + C_6 s + f g_x t \pm \sqrt{2 h g_x x + \frac{ng_x y}{V_y} + \frac{ng_x z}{V_z} + \frac{rg_x s}{V_s} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + \frac{\psi_s(V_s)}{V_s}} + C_9$$

$$P(x) = d \rho g_x x \quad (a + b + c + d + e + f + h + n + q + r = 1) \quad V_x \neq 0, \quad V_y \neq 0, \quad V_s \neq 0.$$  

For $n$-dimensions one can repeat the above as many times as one wishes.

Option 5b

Two-term Linearized Navier-Stokes Equation (one nonlinear term)

By linearization as in Option 1, if one replaces $\rho V_y \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_z}{\partial z}$ by $2 \rho \frac{\partial V_z}{\partial t}$ in

$$-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} + \rho V_s \frac{\partial V_x}{\partial s} = \rho g_x$$

one obtains

$$-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) + \frac{\partial p}{\partial x} + 3 \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} = \rho g_x,$$

whose solution is

$$V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{f g_x t}{3} \pm \sqrt{2 h g_x x + C_6}$$

Back to Options
**Conclusion (for Option 4)**

One will begin from the general case and end with the special cases.

**Solutions of the Navier--Stokes equations (general case)**

$x$-direction **Navier-Stokes Equation** (also driver equation)

\[-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} = \rho g_x \]

$x$-direction

V\(x(x,y,z,t)\) =

<table>
<thead>
<tr>
<th>solution for linear terms</th>
<th>solution for non-linear terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + fg_s t \pm \sqrt{2hg_s x + \frac{ng_y y}{V_y} + \frac{qg_z z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C_o})</td>
<td>arbitrary functions</td>
</tr>
</tbody>
</table>

P(x) = d\(\rho g_s x\); \((a + b + c + d + h + n + q = 1)\)

V\(_x \neq 0\), V\(_z \neq 0\)

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, \(g_s\), were zero, the first three terms, the 7th term, the 8th term, the 9th term, the 10th term and \(P(x)\) would all be zero. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known. The author proposed and applied a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and also each term utilizes gravity to function. This law was applied in splitting-up the Navier-Stokes equations. The resulting sub-equations were readily integrable, and even the nonlinear sub-equations were readily integrated.

The \(x\)-direction Navier-Stokes equation was split-up into sub-equations using ratios. The sub-equations were solved and combined. The relation obtained from the integration of the linear part of the equation satisfied the linear part of the equation and the relation obtained from integrating the nonlinear part of the equation satisfied the nonlinear part of the equation. By solving algebraically and simultaneously for \(V_x\), \(V_y\) and \(V_z\), the \((ng_x y)/V_y\) and \((qg_z z)/V_z\) terms were expressed explicitly in terms of \(x, y, z\) and \(t\). The above \(x\)-direction solution is the solution everyone has been waiting for, for nearly 150 years. It was obtained in two simple steps, namely, splitting the equation using ratios and integrating.
Special Cases of the Navier-Stokes Equations

1. Linearized Navier-Stokes equations

One may note that there are six linear terms and three nonlinear terms in the Navier-Stokes equation. The linearized case was covered before the general case, and the experience gained in the linearized case guided one to solve the general case efficiently. In particular, the gravity term must be the subject of the equation for a solution. When the gravity term was the subject of the equation, the equation was called the driver equation. A splitting technique was applied to the linearized Navier-Stokes equations (Option 1). Twenty sub-equations were solved. (Four sets of equations with different equation subjects). The integration relations of one of the sets satisfied the linearized Navier-Stokes equation; and this set was from the equation with \( g_x \) as the subject of the equation.

In addition to finding a solution, the results of the integration revealed the roles of the terms of the Navier-Stokes equations in fluid flow. In particular, the gravity forces and \( \frac{\partial}{\partial \dot{\varepsilon}} \) are involved mainly in the parabolic as well as the forward motion of fluids; \( \mu \frac{\partial^2 V_x}{\partial x^2} \) and \( \frac{\partial^2 V_x}{\partial y^2} \) are involved in the periodic motion of fluids, and one may infer that as \( \mu \) increases, the periodicity increases. One should determine experimentally, if the ratio of the linear term \( \mu \frac{\partial^2 V_x}{\partial x^2} \) to the nonlinear sum \( V_x(\partial V_x/\partial x) + V_y(\partial V_x/\partial y) + V_z(\partial V_x/\partial z) \) is 1 to 3.

Solution to the linearized Navier--Stokes equation

\[
V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{f_{g_x}}{4}t + C_9 \quad ; \quad P(x) = d\rho g_x \]

Linearized Equation

\[
-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial x} = \rho g_x
\]

2. Solutions of the Euler equation

Since one has solved the Navier-Stokes equation, one has also solved the Euler equation.

Euler equation (\( \mu = 0 \)): \( \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} + 1 \frac{\partial p}{\partial x} = g_x \)

\[
V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x + \frac{ng_{xy} y}{V_y} + \frac{ng_{xz} z}{V_z}} \pm \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C
\]

\( P(x) = d\rho g_x \) \quad (\( f + h + n + q + d = 1 \)) \( V_y \neq 0, V_z \neq 0 \)

\[
\begin{align*}
\frac{ng_{xy}}{V_y} &= -\frac{nB_6 g_x}{\beta_7} \left\{ \sqrt{2g_x q - \beta_5 g_x g_q nq} \right\} \\
\frac{ng_{xz}}{V_z} &= \frac{\beta_7 q_g g_x^2 - \beta_7 q g_x g_q nq}{\beta_7 g_x^2 x - \beta_7 q g_x g_q nq} \left\{ \sqrt{2g_x q - \beta_5 g_x g_q nq} \right\}
\end{align*}
\]
Comparison of Linearized N-S Solutions, Euler Solutions, and N-S Solutions

\[ V_x(x,y,z,t) = - \frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4} t + C_6; \quad P(x) = d\rho g_x \]

\[ V_x(x,y,z,t) = f g_x t + \sqrt{2h g_x x} \quad + n g_x y \quad + q g_x z \quad + \psi_y(V_x) \quad + \psi_z(V_z) + C_7 \]

\[ \frac{ng_y}{V_y} = - \frac{\beta_7 g_x}{\beta_7 g_z} + \left( \sqrt{2h g_x x} \beta_7 g_x g_z t \right) \]

\[ \frac{qg_z}{V_z} = \frac{(\beta_7 g_x g_z t - \beta_7 q g_z g_z) \pm \sqrt{2h g_x x}}{\beta_7 g_z} \]

\[ V_x(x,y,z,t) = - \frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4} t + C_6; \quad P(x) = d\rho g_x \]

\[ V_x(x,y,z,t) = f g_x t + \sqrt{2h g_x x} \quad + n g_x y \quad + q g_x z \quad + \psi_y(V_x) \quad + \psi_z(V_z) + C_7 \]

\[ \frac{ng_y}{V_y} = - \frac{\beta_7 g_x}{\beta_7 g_z} + \left( \sqrt{2h g_x x} \beta_7 g_x g_z t \right) \]

\[ \frac{qg_z}{V_z} = \frac{(\beta_7 g_x g_z t - \beta_7 q g_z g_z) \pm \sqrt{2h g_x x}}{\beta_7 g_z} \]
### Option 6

**Solutions of 3-D Navier-Stokes Equations (Method 2)**

Here, the three equations below, will be added together; and a single equation will be integrated

\[
\begin{align*}
-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right) + \frac{\partial p}{\partial x} + \rho \left( \frac{\partial V_x}{\partial x} + V_x \frac{\partial V_x}{\partial y} + V_x \frac{\partial V_x}{\partial z} \right) &= \rho g_x \quad (1) \\
-\mu \left( \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial z^2} \right) + \frac{\partial p}{\partial y} + \rho \left( \frac{\partial V_y}{\partial y} + V_y \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial z} \right) &= \rho g_y \quad (2) \\
-\mu \left( \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial x^2} \right) + \frac{\partial p}{\partial z} + \rho \left( \frac{\partial V_z}{\partial z} + V_z \frac{\partial V_z}{\partial x} + V_z \frac{\partial V_z}{\partial y} \right) &= \rho g_z \quad (3)
\end{align*}
\]

**Step 1:** Apply the axiom, if \(a = b\) and \(c = d\), then \(a + c = b + d\); and therefore, add the left sides and add the right sides of the above equations. That is, \((1) + (2) + (3) = \rho g_x + \rho g_y + \rho g_z

\[
-\mu \left( \frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_y}{\partial y^2} - \frac{\partial^2 V_z}{\partial z^2} \right) - \mu \left( \frac{\partial^2 V_y}{\partial x^2} - \frac{\partial^2 V_z}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} \right) - \mu \left( \frac{\partial^2 V_z}{\partial x^2} - \frac{\partial^2 V_y}{\partial z^2} - \frac{\partial^2 V_x}{\partial x^2} \right)
\]

\[
+ \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} + \rho \left( \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_z}{\partial z} + \rho V_x \frac{\partial V_y}{\partial y} + \rho V_y \frac{\partial V_x}{\partial x}
\]

\[
+ \rho V_y \frac{\partial V_z}{\partial z} + \rho V_z \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_x}{\partial x} = (\rho g_x + \rho g_y + \rho g_z)
\]

(Three lines per equation)

Let \(\rho g_x + \rho g_y + \rho g_z = \rho G\), where \(G = \left| g_x + g_y + g_z \right|\) to obtain

\[
-\mu \left( \frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_y}{\partial y^2} - \frac{\partial^2 V_z}{\partial z^2} \right) - \mu \left( \frac{\partial^2 V_y}{\partial x^2} - \frac{\partial^2 V_z}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} \right) - \mu \left( \frac{\partial^2 V_z}{\partial x^2} - \frac{\partial^2 V_y}{\partial z^2} - \frac{\partial^2 V_x}{\partial x^2} \right)
\]

\[
+ \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} + \rho \left( \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_z}{\partial z} + \rho V_x \frac{\partial V_y}{\partial y} + \rho V_y \frac{\partial V_x}{\partial x}
\]

\[
+ \rho V_y \frac{\partial V_z}{\partial z} + \rho V_z \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_x}{\partial x} + \rho V_x \frac{\partial V_y}{\partial y} + \rho V_y \frac{\partial V_x}{\partial x} + \rho V_z \frac{\partial V_x}{\partial z} + \rho V_z \frac{\partial V_y}{\partial z} + \rho V_y \frac{\partial V_z}{\partial x} = \rho G
\]

**Step 2:** Solve the above 25-term equation using the ratio method. (24 ratio terms)

The ratio terms to be used are respectively the following: (Sum of the ratio terms = 1)

\(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \xi, \omega, \chi, \psi, \phi, \varphi, \theta, \vartheta, \upsilon, \rho, \sigma, \tau, \nu, \xi, \phi, \chi, \psi, \omega, \phi, \Gamma, \Lambda, \Theta, \Xi, \Phi, \Psi, \Omega\)

\[
-\mu \frac{\partial^2 V_x}{\partial x^2} = \alpha \rho G; \quad -\mu \frac{\partial^2 V_y}{\partial y^2} = b \rho G; \quad -\mu \frac{\partial^2 V_z}{\partial z^2} = c \rho G; \quad -\mu \frac{\partial^2 V_x}{\partial x^2} = d \rho G;
\]

\[
-\mu \frac{\partial^2 V_y}{\partial y^2} = e \rho G; \quad -\mu \frac{\partial^2 V_z}{\partial z^2} = f \rho G; \quad -\mu \frac{\partial^2 V_x}{\partial x^2} = m \rho G; \quad -\mu \frac{\partial^2 V_z}{\partial z^2} = n \rho G;
\]

\[
-\mu \frac{\partial^2 V_y}{\partial y^2} = r \rho G; \quad \frac{\partial p}{\partial x} = \beta_1 \rho G; \quad \frac{\partial p}{\partial y} = \beta_2 \rho G; \quad \frac{\partial p}{\partial z} = \beta_3 \rho G;
\]

\[
\rho V_x \frac{\partial V_x}{\partial t} = \beta_4 \rho G; \quad \rho V_y \frac{\partial V_y}{\partial t} = \beta_5 \rho G; \quad \rho V_z \frac{\partial V_z}{\partial t} = \beta_6 \rho G; \quad \rho V_x \frac{\partial V_y}{\partial t} = \beta_7 \rho G;
\]

\[
\rho V_y \frac{\partial V_x}{\partial t} = \lambda_1 \rho G; \quad \rho V_z \frac{\partial V_y}{\partial t} = \lambda_2 \rho G; \quad \rho V_x \frac{\partial V_z}{\partial t} = \lambda_3 \rho G; \quad \rho V_y \frac{\partial V_z}{\partial t} = \lambda_4 \rho G;
\]

\[
\rho V_z \frac{\partial V_x}{\partial t} = \lambda_5 \rho G; \quad \rho V_x \frac{\partial V_y}{\partial t} = \lambda_6 \rho G; \quad \rho V_y \frac{\partial V_y}{\partial t} = \lambda_7 \rho G; \quad \rho V_z \frac{\partial V_z}{\partial t} = \lambda_8 \rho G
\]
### Solutions of Navier-Stokes Equations (Method 2)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial^2 V_x}{\partial x^2} = -\frac{a}{\mu} \rho G )</td>
<td>( -\mu \frac{\partial^2 V_y}{\partial y^2} = b \rho G )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial x} = -\frac{a}{\mu} \rho G x + C_1 )</td>
<td>( \frac{\partial^2 V_y}{\partial y^2} = c \rho G )</td>
</tr>
<tr>
<td>( V_x = -\frac{a}{\mu} \rho G \frac{x^2}{2} + C_1 x + C_2 )</td>
<td>( \frac{\partial^2 V_y}{\partial y^2} = -\frac{b}{\mu} \rho G y + C_3 )</td>
</tr>
<tr>
<td>( V_x = -\frac{b}{\mu} \rho G \frac{y^2}{2} + C_3 y + C_4 )</td>
<td>( \frac{\partial V_y}{\partial y} = -\frac{c}{\mu} \rho G y + C_9 )</td>
</tr>
<tr>
<td>( -\mu \frac{\partial^2 V_z}{\partial x^2} = d \rho G )</td>
<td>( -\mu \frac{\partial^2 V_y}{\partial y^2} = f \rho G )</td>
</tr>
<tr>
<td>( \frac{\partial^2 V_y}{\partial x^2} = d \rho G )</td>
<td>( \frac{\partial^2 V_z}{\partial y^2} = -\frac{f}{\mu} \rho G y + C_9 )</td>
</tr>
<tr>
<td>( \frac{\partial V_y}{\partial x} = -\frac{d}{\mu} \rho G x + C_7 )</td>
<td>( \frac{\partial V_z}{\partial y} = -\frac{f}{\mu} \rho G y + C_9 )</td>
</tr>
<tr>
<td>( V_y = -\frac{d}{\mu} \rho G \frac{x^2}{2} + C_7 x + C_8 )</td>
<td>( \frac{\partial V_z}{\partial z} = -\frac{g}{\mu} \rho G z + C_{15} )</td>
</tr>
<tr>
<td>( \frac{\partial^2 V_x}{\partial x^2} = m \rho G )</td>
<td>( \frac{\partial^2 V_z}{\partial x^2} = -\frac{m}{\mu} \rho G )</td>
</tr>
<tr>
<td>( \frac{\partial V_x}{\partial x} = -\frac{m}{\mu} \rho G x + C_{13} )</td>
<td>( \frac{\partial V_z}{\partial y} = -\frac{n}{\mu} \rho G y + C_{15} )</td>
</tr>
<tr>
<td>( V_x = -\frac{m}{\mu} \rho G \frac{x^2}{2} + C_{13} x + C_{14} )</td>
<td>( \frac{\partial V_z}{\partial z} = -\frac{r}{\mu} \rho G z + C_{17} )</td>
</tr>
</tbody>
</table>

\begin{align*}
\frac{dp}{dx} &= \beta_1 \rho G \\
\frac{dp}{dx} &= \beta_1 \rho G \\
P(x) &= \beta_1 \rho G x + C_{19} \\
\frac{dp}{dy} &= \beta_2 \rho G \\
\frac{dp}{dy} &= \beta_2 \rho G \\
P(y) &= \beta_2 \rho G y + C_{20} \\
\frac{dp}{dz} &= \beta_3 \rho G \\
\frac{dp}{dz} &= \beta_3 \rho G \\
P(z) &= \beta_3 \rho G z + C_{21}
\end{align*}
<table>
<thead>
<tr>
<th>Step</th>
<th>Equation</th>
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</thead>
<tbody>
<tr>
<td>13</td>
<td>[ \rho \frac{\partial V_x}{\partial t} = \beta_4 \rho G ]</td>
</tr>
<tr>
<td></td>
<td>[ \frac{dV_x}{dt} = \beta_4 G ]</td>
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<tr>
<td></td>
<td>[ V_x = \beta_4 G t + C_{22} ]</td>
</tr>
<tr>
<td>14</td>
<td>[ \rho \frac{\partial V_y}{\partial t} = \beta_5 \rho G ]</td>
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<tr>
<td></td>
<td>[ \frac{dV_y}{dt} = \beta_5 G ]</td>
</tr>
<tr>
<td></td>
<td>[ V_y = \beta_5 G t + C_{23} ]</td>
</tr>
<tr>
<td>15</td>
<td>[ \rho \frac{\partial V_z}{\partial t} = \beta_6 \rho G ]</td>
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<tr>
<td></td>
<td>[ \frac{dV_z}{dt} = \beta_6 G ]</td>
</tr>
<tr>
<td></td>
<td>[ V_z = \beta_6 G t + C_{24} ]</td>
</tr>
<tr>
<td>16</td>
<td>[ \rho V_x \frac{\partial V_x}{\partial x} = \lambda_4 \rho G ]</td>
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<td>[ \frac{dV_x}{dx} = \lambda_4 G ]</td>
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<tr>
<td></td>
<td>[ V_x^2 = \lambda_4 G x ]</td>
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<tr>
<td></td>
<td>[ V_x = \pm \sqrt{\lambda_4 G x} + C_{25} ]</td>
</tr>
<tr>
<td>17</td>
<td>[ \rho V_y \frac{\partial V_y}{\partial y} = \lambda_2 \rho G ]</td>
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<tr>
<td></td>
<td>[ \frac{dV_y}{dy} = \lambda_2 G ]</td>
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<tr>
<td></td>
<td>[ V_y \frac{dV_x}{dy} = \lambda_2 G d y ]</td>
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<tr>
<td></td>
<td>[ V_y \frac{dV_y}{dy} = \lambda_2 G y + \psi_y(V_y) ]</td>
</tr>
<tr>
<td></td>
<td>[ V_x = \frac{\lambda_2 G y}{V_y} + \psi_y(V_y) ]</td>
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<tr>
<td>18</td>
<td>[ \rho V_z \frac{\partial V_z}{\partial z} = \lambda_3 \rho G ]</td>
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<tr>
<td></td>
<td>[ \frac{dV_z}{dz} = \lambda_3 G ]</td>
</tr>
<tr>
<td></td>
<td>[ V_z \frac{dV_x}{dz} = \lambda_3 G d z ]</td>
</tr>
<tr>
<td></td>
<td>[ V_z \frac{dV_y}{dz} = \lambda_3 G z + \psi_z(V_z) ]</td>
</tr>
<tr>
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<td>[ V_z = \frac{\lambda_3 G z}{V_z} + \psi_z(V_z) ]</td>
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<tr>
<td>19</td>
<td>[ \rho V_x \frac{\partial V_y}{\partial x} = \lambda_4 \rho G ]</td>
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<td></td>
<td>[ \frac{dV_y}{dx} = \lambda_4 G ]</td>
</tr>
<tr>
<td></td>
<td>[ V_x V_y = \lambda_4 G x + \psi_x(V_x) ]</td>
</tr>
<tr>
<td></td>
<td>[ V_y = \frac{\lambda_4 G x}{V_x} + \psi_x(V_x) ]</td>
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<tr>
<td>20</td>
<td>[ \rho V_z \frac{\partial V_y}{\partial z} = \lambda_5 \rho G ]</td>
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<td></td>
<td>[ \frac{dV_y}{dz} = \lambda_5 G ]</td>
</tr>
<tr>
<td></td>
<td>[ V_z \frac{dV_y}{dz} = \lambda_5 G d y ]</td>
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<tr>
<td></td>
<td>[ V_z \frac{dV_y}{dz} = \lambda_5 G d y ]</td>
</tr>
<tr>
<td></td>
<td>[ V_y = \frac{\lambda_5 G y}{V_z} + \psi_y(V_z) ]</td>
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<tr>
<td>21</td>
<td>[ \rho V_z \frac{\partial V_z}{\partial x} = \lambda_6 \rho G ]</td>
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<tr>
<td></td>
<td>[ \frac{dV_z}{dx} = \lambda_6 G ]</td>
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<td>[ V_z \frac{dV_x}{dx} = \lambda_6 G d x ]</td>
</tr>
<tr>
<td></td>
<td>[ V_z \frac{dV_y}{dx} = \lambda_6 G z + \psi_z(V_z) ]</td>
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<tr>
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<td>[ V_z = \frac{\lambda_6 G z}{V_z} + \psi_z(V_z) ]</td>
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<tr>
<td>22</td>
<td>[ \rho V_x \frac{\partial V_z}{\partial x} = \lambda_7 \rho G ]</td>
</tr>
<tr>
<td></td>
<td>[ \frac{dV_z}{dx} = \lambda_7 G ]</td>
</tr>
<tr>
<td></td>
<td>[ V_x \frac{dV_x}{dx} = \lambda_7 G d x ]</td>
</tr>
<tr>
<td></td>
<td>[ V_x \frac{dV_z}{dx} = \lambda_7 G y + \psi_x(V_x) ]</td>
</tr>
<tr>
<td></td>
<td>[ V_x = \frac{\lambda_7 G y}{V_x} + \psi_x(V_x) ]</td>
</tr>
<tr>
<td>23</td>
<td>[ \rho V_y \frac{\partial V_z}{\partial y} = \lambda_8 \rho G ]</td>
</tr>
<tr>
<td></td>
<td>[ \frac{dV_z}{dy} = \lambda_8 G ]</td>
</tr>
<tr>
<td></td>
<td>[ V_y \frac{dV_x}{dy} = \lambda_8 G d y ]</td>
</tr>
<tr>
<td></td>
<td>[ V_y \frac{dV_z}{dy} = \lambda_8 G z + \psi_y(V_y) ]</td>
</tr>
<tr>
<td></td>
<td>[ V_y = \frac{\lambda_8 G y}{V_y} + \psi_y(V_y) ]</td>
</tr>
<tr>
<td>24</td>
<td>[ \rho V_z \frac{\partial V_z}{\partial z} = \lambda_9 \rho G ]</td>
</tr>
<tr>
<td></td>
<td>[ \frac{dV_z}{dz} = \lambda_9 G ]</td>
</tr>
<tr>
<td></td>
<td>[ V_z \frac{dV_x}{dz} = \lambda_9 G d z ]</td>
</tr>
<tr>
<td></td>
<td>[ V_z \frac{dV_y}{dz} = \lambda_9 G z ]</td>
</tr>
<tr>
<td></td>
<td>[ V_z = \frac{\lambda_9 G z}{V_z} + \psi_z(V_z) ]</td>
</tr>
</tbody>
</table>
### Step 3:
One collects the integrals of the sub-equations, above, for $V_x$, $V_y$, $V_z$, $P(x)$, $P(y)$, $P(z)$

<table>
<thead>
<tr>
<th>For $V_x$, $P(x)$</th>
<th>For $V_y$, $P(y)$</th>
<th>For $V_z$, $P(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sum of integrals from sub-equations #1, #2, #3, #13, #16, #17, #18, #10</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_x = -\frac{a}{\mu} \rho G \frac{x^2}{2} + C_1 x + C_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_x = -\frac{b}{\mu} \rho G \frac{y^2}{2} + C_3 y + C_4$</td>
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</tr>
<tr>
<td>$V_x = -\frac{c}{\mu} \rho G \frac{z^2}{2} + C_5 z + C_6$</td>
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<tr>
<td>$V_x = \beta_4 G t + C_{22}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_x = \pm \sqrt{2 \lambda_4 G} x + C_{25}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_x = \frac{\lambda_5 G y}{V_y} + \frac{\psi_y(V_y)}{V_y}$</td>
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<td></td>
</tr>
<tr>
<td>$V_x = \frac{\lambda_6 G z}{V_z} + \frac{\psi_z(V_z)}{V_z}$</td>
<td></td>
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</tr>
<tr>
<td>$P(x) = \beta_1 \rho G x + C_{19}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Sum of integrals from sub-equations #4, #5, #6, #14, #19, #20, #21, #11</strong></td>
<td></td>
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</tr>
<tr>
<td>$V_y = -\frac{d}{\mu} \rho G \frac{x^2}{2} + C_7 x + C_8$</td>
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</tr>
<tr>
<td>$V_y = -\frac{f}{\mu} \rho G \frac{y^2}{2} + C_9 y + C_{10}$</td>
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<tr>
<td>$V_y = -\frac{h}{\mu} \rho G \frac{z^2}{2} + C_{11} z + C_{12}$</td>
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<tr>
<td>$V_y = \beta_5 G t + C_{21}$</td>
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</tr>
<tr>
<td>$V_y = \pm \sqrt{2 \lambda_5 G} y + C_{26}$</td>
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<tr>
<td>$V_y = \frac{\lambda_6 G z}{V_z} + \frac{\psi_z(V_z)}{V_z}$</td>
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<tr>
<td>$P(y) = \beta_2 \rho G y + C_{20}$</td>
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</tr>
<tr>
<td><strong>Sum of integrals from sub-equations #7, #8, #9, #15, #22, #23, #24, #12,</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_z = -\frac{m}{\mu} \rho G \frac{x^2}{2} + C_{13} x + C_{14}$</td>
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</tr>
<tr>
<td>$V_z = -\frac{n}{\mu} \rho G \frac{y^2}{2} + C_{15} y + C_{16}$</td>
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<tr>
<td>$V_z = -\frac{r}{\mu} \rho G \frac{z^2}{2} + C_{17} z + C_{18}$</td>
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</tr>
<tr>
<td>$V_z = \beta_6 G t + C_{24}$</td>
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</tr>
<tr>
<td>$V_z = \pm \sqrt{2 \lambda_6 G} z + C_{27}$</td>
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<tr>
<td>$P(z) = \beta_3 \rho G z + C_{21}$</td>
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</tbody>
</table>

From above,

For $V_x(x,y,z,t)$

$$V_x(x,y,z,t) = -\frac{a}{\mu} \rho G \frac{x^2}{2} + C_1 x - \frac{b}{\mu} \rho G \frac{y^2}{2} + C_3 y - \frac{c}{\mu} \rho G \frac{z^2}{2} + C_5 z + \beta_4 G t + \sqrt{2 \lambda_4 G} x + \frac{\lambda_5 G y}{V_y} + \frac{\psi_y(V_y)}{V_y} + \frac{\lambda_6 G z}{V_z} + \frac{\psi_z(V_z)}{V_z}$$

$$P(x) = \beta_1 \rho G x + C_{19}$$

For $V_y(x,y,z,t)$

$$V_y(x,y,z,t) = -\frac{d}{\mu} \rho G \frac{x^2}{2} + C_7 x - \frac{f}{\mu} \rho G \frac{y^2}{2} + C_9 y - \frac{h}{\mu} \rho G \frac{z^2}{2} + C_{11} z + \beta_5 G t + \sqrt{2 \lambda_5 G} y + \frac{\lambda_6 G z}{V_z} + \frac{\psi_z(V_z)}{V_z}$$

$$P(y) = \beta_2 \rho G y + C_{20}$$

For $V_z(x,y,z,t)$

$$V_z(x,y,z,t) = -\frac{m}{\mu} \rho G \frac{x^2}{2} + C_{13} x - \frac{n}{\mu} \rho G \frac{y^2}{2} + C_{15} y - \frac{r}{\mu} \rho G \frac{z^2}{2} + C_{17} z + \beta_6 G t + \sqrt{2 \lambda_6 G} z + \frac{\lambda_8 G y}{V_y} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}$$

$$P(z) = \beta_3 \rho G z + C_{21}$$
Step 4: Simplify the sums of the integrals from above. (Method 2 solutions of N-S equations)

\[ V_x(x,y,z,t) = -\frac{\rho G}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \beta_4 Gt \pm \sqrt{2\lambda_4 G} x + \frac{\lambda_3 G y}{V_y} + \frac{\lambda_3 G z}{V_z} \]

\[ P(x) = \beta_1 \rho G x + C_{19} \quad (V_y \neq 0, \, V_z \neq 0) \]

\[ = \psi_y(V_y) \frac{V_y}{V_y} + \psi_z(V_z) \frac{V_z}{V_z} \quad \text{arbitrary functions} \]

\[ V_y(x,y,z,t) = -\frac{\rho G}{2\mu} (dx^2 + fy^2 + hz^2) + C_7x + C_9y + C_{11}z + C_{10} \beta_5 Gt \pm \sqrt{2\lambda_5 G} y + \frac{\lambda_4 G x}{V_x} + \frac{\lambda_6 G z}{V_z} \]

\[ P(y) = \beta_2 \rho Gy + C_{20} \quad (V_x \neq 0, \, V_z \neq 0) \]

\[ = \psi_x(V_x) \frac{V_x}{V_x} + \psi_z(V_z) \frac{V_z}{V_z} \quad \text{arbitrary functions} \]

\[ V_z(x,y,z,t) = -\frac{\rho G}{2\mu} (mx^2 + ny^2 + rz^2) + C_{13}x + C_{15}y + C_{17}z + \beta_6 Gt \pm \sqrt{2\lambda_6 G} z + \frac{\lambda_7 G x}{V_x} + \frac{\lambda_8 G y}{V_y} \]

\[ P(z) = \beta_3 \rho Gz + C_{21} \quad (V_y \neq 0, \, V_x \neq 0) \]

\[ = \psi_x(V_x) \frac{V_x}{V_x} + \psi_y(V_y) \frac{V_y}{V_y} \quad \text{arbitrary functions} \]

The above are solutions for \( V_x, V_y, V_z, P(x), P(y), P(z) \) of the Navier-Stokes Equations.
Comparison of Method 1 (Option 4) and Method 2 (Option 6) of Solutions of Navier-Stokes Equations

Method 1: $x$-direction solution of Navier-Stokes equation

$$V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + f_{g_x} + n_{g_x}V_y + n_{g_x}V_z + \sqrt{\frac{2h_{g_x}g_x + q_{g_x}}{V_y + V_z}}$$

$$P(x) = d\rho g_x; \quad (a + b + c + d + h + n + q = 1) \quad (V_y \neq 0, \ V_z \neq 0) \quad + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C_9$$

Method 2: $x$-direction solution of Navier-Stokes equation

$$V_x(x,y,z,t) = -\frac{\rho G}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \beta_4 G t + \sqrt{\frac{2\lambda_1 G x + \lambda_2 G y}{V_y} + \frac{\lambda_3 G z}{V_z}}$$

$$P(x) = \beta_1 \rho G x + C_{19} \quad (V_y \neq 0, \ V_z \neq 0) \quad + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}$$

It is pleasantly surprising that the above solutions (A) and (B) are almost identical (except for the constants), even though they were obtained by different approaches as in Option 4 and Option 6. Such an agreement confirms the validity of the solution method for the system of magnetohydrodynamic equations (see viXra:1405.0251). For the system of magnetohydrodynamic equations, there is only a single "driver" equation. For the system of N-S equations, there are three driver equations, since each equation contains the gravity term. Therefore, one was able to solve each of the three simultaneous equations separately (as in Method 1); but in addition, one obtained an identical solution (except for the constants) in solving the simultaneous N-S system by adding the three equations in the system and integrating a single driver equation. In Method 1, the gravity term was $\rho g$. In Method 2, the gravity term was $\rho G$, where $G$ is the magnitude of the vector sum of the gravity terms. Note that in Method 1, the sum of the ratio terms (8 ratio terms for each equation) equals unity, but in Method 2, the sum of the ratio terms (24 ratio terms) for the single driver equation solved equals unity. Note that in Method 2, only a single "driver" equation was solved, but in Method 1, three "driver" equations were solved. In Method 2, one could say that the system of N-S equations was "more simultaneously" solved than in Method 1.

To summarize, solving the Navier-Stokes equations by the first method helped one to solve the magnetohydrodynamic equations (not presented in this paper). See viXra:1405.0251) and solving the magnetohydrodynamic equations encouraged one to solve the Navier-Stokes equations by the second method.

("Navier-Stokes equations" scratched the back of magnetohydrodynamic equations; and in return, magnetohydrodynamic equations "scratched the back" of Navier-Stokes equations)

About integrating only a single equation

If one asked for help in solving the N-S equations, and one was told to add the three equations together and then solve them, one would think that one was being given a nonsensical advice; but now, after studying the above Option 6 method, one would appreciate such a suggestion.
Option 7

Solutions of 3-D Linearized Navier-Stokes Equations: Method 2

Here, the three equations below, will be added together; and a single equation will be integrated.

\[
\begin{align*}
-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} &= \rho g_x \quad (1) \\
-\mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial y} + 4\rho \frac{\partial V_y}{\partial t} &= \rho g_y \\
-\mu \frac{\partial^2 V_z}{\partial z^2} - \mu \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial p}{\partial z} + 4\rho \frac{\partial V_z}{\partial t} &= \rho g_z
\end{align*}
\]

Step 1: Apply the axiom, if \( a = b \) and \( c = d \), then \( a + c = b + d \); and therefore, add the left sides and add the right sides of the above equations. That is, \((1) + (2) + (3) = \rho g_x + \rho g_y + \rho g_z\)

\[
-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial y} + 4\rho \frac{\partial V_y}{\partial t} - \mu \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial p}{\partial z} + 4\rho \frac{\partial V_z}{\partial t} = \rho g_x + \rho g_y + \rho g_z \quad \text{(Two lines per equation)}
\]

Let \( \rho g_x + \rho g_y + \rho g_z = \rho G \), where \( G = [g_x, g_y, g_z] \) to obtain

\[
-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial y} + 4\rho \frac{\partial V_y}{\partial t} - \mu \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial p}{\partial z} + 4\rho \frac{\partial V_z}{\partial t} = \rho G \quad \text{(Two lines per equation)}
\]

Step 2: Solve the above 15-term equation using the ratio method. (14 ratio terms)

The ratio terms to be used are respectively the following: (Sum of the ratio terms = 1)

\( a, b, c, d, f, h, j, m, n, q, r, s, u, v, w \). (Sum of the ratio terms = 1)

\[
-\mu \frac{\partial^2 V_x}{\partial x^2} = a\rho G; \quad -\mu \frac{\partial^2 V_y}{\partial y^2} = b\rho G; \quad -\mu \frac{\partial^2 V_z}{\partial z^2} = c\rho G; \quad \frac{\partial p}{\partial x} = d\rho G
\]

\[
4\rho \frac{\partial V_x}{\partial t} = f\rho G; \quad -\mu \frac{\partial^2 V_y}{\partial y^2} = h\rho G; \quad -\mu \frac{\partial^2 V_z}{\partial z^2} = j\rho G; \quad -\mu \frac{\partial^2 V_x}{\partial x^2} = m\rho G;
\]

\[
\frac{\partial p}{\partial y} = n\rho G; \quad 4\rho \frac{\partial V_y}{\partial t} = q\rho G; \quad -\mu \frac{\partial^2 V_z}{\partial z^2} = r\rho G; \quad -\mu \frac{\partial^2 V_y}{\partial y^2} = s\rho G;
\]

\[
-\mu \frac{\partial^2 V_z}{\partial z^2} = u\rho G; \quad \frac{\partial p}{\partial z} = v\rho G; \quad 4\rho \frac{\partial V_z}{\partial t} = w\rho G
\]

\[
\begin{array}{llll}
1. & -\mu \frac{\partial^2 V_x}{\partial x^2} = a\rho G & 2. & -\mu \frac{\partial^2 V_y}{\partial y^2} = b\rho G & 3. & -\mu \frac{\partial^2 V_z}{\partial z^2} = c\rho G \\
\frac{\partial V_x}{\partial x} &= -\frac{a}{\mu} \rho G X + C_1 & \frac{\partial V_y}{\partial y} &= -\frac{b}{\mu} \rho G Y + C_2 & \frac{\partial V_z}{\partial z} &= -\frac{c}{\mu} \rho G Z + C_3 \\
V_x &= -\frac{\rho G a}{2\mu} X^2 + C_1 X + C_2 & V_y &= -\frac{\rho G b}{2\mu} Y^2 + C_2 Y + C_3 & V_z &= -\frac{\rho G c}{2\mu} Z^2 + C_3 Z + C_4
\end{array}
\]
Solutions of 3-D Linearized Navier-Stokes Equations: Method 2

\[ \frac{\partial p}{\partial x} = d\rho G \]
\[ P(x) = d\rho G x + C_7 \]

\[ 4 \rho \frac{\partial V_x}{\partial t} = f\rho G \]
\[ \frac{\partial V_x}{\partial t} = \frac{fG}{4} \]
\[ V_x = \frac{fG}{4} t + C_8 \]

\[ \frac{\partial^2 V_y}{\partial y^2} = j\rho G \]
\[ \frac{\partial^2 V_y}{\partial x^2} = \frac{-j}{\mu} \rho G \]
\[ \frac{\partial V_y}{\partial y} = \frac{-j}{\mu} \rho G y + C_{11} \]
\[ V_y = \frac{-\rho G j}{2\mu} y^2 + C_{11y} + C_{12} \]

\[ 4 \rho \frac{\partial V_y}{\partial t} = q\rho G \]
\[ \frac{\partial V_y}{\partial t} = \frac{qG}{4} \]
\[ V_y = \frac{qG}{4} t + C_{16} \]

\[ \frac{\partial^2 V_z}{\partial z^2} = u\rho G \]
\[ \frac{\partial^2 V_z}{\partial x^2} = \frac{-u}{\mu} \rho G \]
\[ \frac{\partial V_z}{\partial z} = \frac{-u}{\mu} \rho G z + C_{21} \]
\[ V_z = \frac{-\rho G u}{2\mu} z^2 + C_{21z} + C_{22} \]

\[ -\mu \frac{\partial^2 V_y}{\partial x^2} = h\rho G \]
\[ \frac{\partial^2 V_y}{\partial x^2} = \frac{-h}{\mu} \rho G \]
\[ \frac{\partial V_y}{\partial x} = \frac{-h}{\mu} \rho G x + C_9 \]
\[ V_y = \frac{-\rho G h}{2\mu} x^2 + C_9x + C_{10} \]

\[ \frac{\partial p}{\partial y} = n\rho G \]
\[ P(y) = n\rho G y + C_{15} \]

\[ -\mu \frac{\partial^2 V_z}{\partial y^2} = s\rho G \]
\[ \frac{\partial^2 V_z}{\partial y^2} = \frac{-s}{\mu} \rho G \]
\[ \frac{\partial V_z}{\partial y} = \frac{-s}{\mu} \rho G y + C_{19} \]
\[ V_z = \frac{-\rho G s}{2\mu} y^2 + C_{19y} + C_{20} \]

\[ \frac{\partial p}{\partial z} = v\rho G \]
\[ P(z) = v\rho G z + C_{23} \]

\[ 4 \rho \frac{\partial V_z}{\partial t} = w\rho G \]
\[ \frac{\partial V_z}{\partial t} = \frac{wG}{4} \]
\[ V_z = \frac{wG}{4} t + C_{24} \]
Step 3: One collect the solutions from Step 2 for \((V_x, V_y, V_z, P(x), P(y), P(z))\)

For \(V_x\): Sum of integrals from sub-equations #1, #2, #3, #5, and for \(P(x)\), from #4

\[
V_x(x,y,z,t) = -\frac{\rho G}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fG}{4}t + C_8; \quad P(x) = \rho Gx + C_7
\]

For \(V_y\): Sum of integrals from sub-equations #6, #7, #8, #10, and for \(P(y)\), from #9.

\[
V_y(x,y,z,t) = -\frac{\rho G}{2\mu} (hx^2 + jy^2 + mz^2) + C_9x + C_{11}y + C_{13}z + \frac{qG}{4}t + C_{16}; \quad P(y) = npGy + C_{15}
\]

For \(V_z\): Sum of integrals from sub-equations #11, #12, #13, and for \(P(z)\), from #14

\[
V_z(x,y,z,t) = -\frac{\rho G}{2\mu} (rx^2 + sy^2 + uz^2) + C_{17}x + C_{19}y + C_{21}z + \frac{wG}{4}t + C_{24}; \quad P(z) = v\rho Gz + C_{23}
\]

Comparison of the above methods for the solutions of Linearized Navier-Stokes Equations

Note below that the solutions by the two different methods are the same except for the constants involved. Now, one has two different methods for solving the system of Navier-Stokes equations. Such an agreement and consistency confirm the validity of the method used in solving the magnetohydrodynamic equations.

Solutions by Method 1

\[
V_x(x,y,z,t) = -\frac{\rho G}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fG}{4}t + C_9; \quad P(x) = \rho Gx
\]

\[
V_y(x,y,z,t) = -\frac{\rho G}{2\mu} (hx^2 + jy^2 + mz^2) + C_9x + C_{11}y + C_{13}z + \frac{qG}{4}t + C_8; \quad P(y) = npGy + C_{15}
\]

\[
V_z(x,y,z,t) = -\frac{\rho G}{2\mu} (rx^2 + sy^2 + uz^2) + C_{17}x + C_{19}y + C_{21}z + \frac{wG}{4}t + C_{24}; \quad P(z) = v\rho Gz + C_{23}
\]

Solutions by Method 2

\[
V_x(x,y,z,t) = -\frac{\rho G}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fG}{4}t + C_8; \quad P(x) = \rho Gx + C_7
\]

\[
V_y(x,y,z,t) = -\frac{\rho G}{2\mu} (hx^2 + jy^2 + mz^2) + C_9x + C_{11}y + C_{13}z + \frac{qG}{4}t + C_{16}; \quad P(y) = npGy + C_{15}
\]

\[
V_z(x,y,z,t) = -\frac{\rho G}{2\mu} (rx^2 + sy^2 + uz^2) + C_{17}x + C_{19}y + C_{21}z + \frac{wG}{4}t + C_{24}; \quad P(z) = v\rho Gz + C_{23}
\]
Overall Conclusion

The Navier-Stokes (N-S) equations in 3-D and 4-D have been solved analytically for the first time by two different methods. In Method 1, the three equations were separately integrated. In Method 2, the three equations were first added together and a single equation was integrated. The solutions from these two methods were the same, except for the constants involved. The N-S solution is unique. The experience gained in solving the linearized equation helped the author to propose a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and each term utilizes gravity to function. The sum of the terms of the ratio is always unity. The application of this law helped speed-up the solutions of the non-linearized N-S equations, since there was no more experimentation as to the subject of the equation. It was also shown that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.

The solutions and relations revealed the role of each term of the Navier-Stokes equations in fluid flow. Most importantly, the gravity term is the indispensable term in fluid flow, and it is involved in the parabolic as well as the forward motion. The pressure gradient term is also involved in the parabolic motion. The viscosity terms are involved in parabolic, periodic and decreasingly exponential motion. As the viscosity increases, periodicity increases. The variable acceleration term is also involved in the periodic and decreasingly exponential motion. The convective acceleration terms produce square root function behavior and behavior of fractional terms containing square root functions with variables in the denominator. In terms of the velocity profile, the first three terms characterize parabolas. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the $x$–nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The $y$–nonlinear term is similar parabolically to the $x$–nonlinear term. The characteristic curve for the $z$–nonlinear term is a combination of two similar parabolas and a hyperbola. If the above $x$–direction flow is repeated simultaneously in the $y$– and $z$– directions, the flow is chaotic and consequently turbulent.

The following statements can be made:

(a) The N-S equations have unique solutions;  
(b) The N-S equations have parabolic solutions;  
3. The N-S equations have square root function solutions.  
4. The N-S equations do not have periodic solutions but have periodic relations.  
5. The N-S equations do not have decreasingly exponential solutions but have decreasingly exponential relations.

In applications, the ratio terms $a$, $b$, $c$, $d$, $f$, $h$, $n$, $q$ and others may perhaps be determined using information such as initial and boundary conditions or may have to be determined experimentally. The author came to the experimental determination conclusion after referring to preliminaries...The question is how did the grandmother determine the terms of the ratio for her grandchildren? Note that so far as the general solutions of the N-S equations are concerned, one needs not find the specific values of the ratio terms.

Finally, for any fluid flow design, one should always maximize the role of gravity for cost-effectiveness, durability, and dependability. Perhaps, Newton's law for fluid flow should read "Sum of everything else equals $\rho g$"; and this would imply the proposed new law that the other terms divide the gravity term in a definite ratio, and each term utilizes gravity to function.

P.S.

Maples software was used to help express the implicit terms in terms of $x$, $y$, $z$, and $t$, by solving System P (p.27, 28). None of the academic programs could solve the system of solutions M. The author would like to find a software that can solve the original system, System M, for comparison purposes.
Option 8
Spin-off: CMI Millennium Prize Problem Requirements

Proof 1

For the linearized Navier-Stokes equations

Proof of the existence of solutions of the Navier-Stokes equations

Since from page 11, it has been shown that the smooth equations given by

$$V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_2y + C_3z + \frac{fg_x}{4}t + C_9; \quad P(x) = d\rho g_x$$

are solutions of the linearized equation,

$$-\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + \frac{\partial P}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x,$$

it has been shown that smooth solutions to the above differential equation exist, and the proof is complete.

From above, if \( y = 0, \ z = 0, \)

$$V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x + \frac{fg_x}{4}t + C_9; \quad P(x) = d\rho g_x + C_{10}$$

Therefore, \( V_x(x,0) = V_x^0(x) = -\frac{\rho g_x}{2\mu}ax^2 + C_{10}x + C_9 \)

Finding \( P(x,t) \)

1. \( V_x(x,t) = -\frac{\rho g_x}{2\mu}(ax^2) + C_1x + \frac{fg_x}{4}t + C_9; \quad P(x) = d\rho g_x \)

2. \( \frac{\partial P}{\partial x} = d\rho g; \)

Required: To find \( P(x,t) \) (that is, find a formula for \( P \) in terms of \( x \) and \( t \))

$$\frac{dp}{dt} = \frac{dp}{dx} \frac{dx}{dt}$$
$$\frac{dp}{dt} = \frac{dp}{dx} V_x \quad \left( \frac{dx}{dt} = V_x \right)$$
$$\frac{dp}{dt} = d\rho g_x \left( -\frac{\rho g_x}{2\mu}(ax^2) + C_1x + \frac{fg_x}{4}t + C_9 \right) \quad \left( \frac{dp}{dx} = d\rho g_x \right)$$

$$P(x,t) = \int d\rho g_x \left( -\frac{\rho g_x}{2\mu}(ax^2) + C_1x + \frac{fg_x}{4}t + C_9 \right) dt$$
$$P(x,t) = d\rho g_x \left( -\frac{a\rho g_x}{2\mu}x^2t + C_1xt + \frac{fg_x}{8}t^2 + C_9t \right) + C_{10}$$

For the corresponding coverage for the original Navier-Stokes equation, see the next page
Proof 2
For the Non-linearized Navier-Stokes equations (Original Equations)

Proof of the existence of solutions of the Navier-Stokes equations

From page 30, if \( y = 0, z = 0 \) in

<table>
<thead>
<tr>
<th>Solution to Linear part</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_x(x,y,z,t) = \frac{-\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_2 y + C_3 z + \frac{f g_x t}{\sqrt{2h g_x x}} )</td>
</tr>
<tr>
<td>( n g_y y \ V_y V_g z \ V_z + \psi_z (V_x) + \psi_x (V_z) )</td>
</tr>
</tbody>
</table>

Continued:

\( P(x) = d \rho g_x x \)

one obtains

\( V_x(x,t) = \frac{-\rho g_x}{2\mu} ax^2 + C_1 x + f g_x t \pm \sqrt{2h g_x x} + C_9 ; \quad P(x) = d \rho g_x x ; \)

Since previously, from p.21, it has been shown that the smooth equations given by

\( V_x(x,t) = \frac{-\rho g_x}{2\mu} ax^2 + C_1 x + f g_x t \pm \sqrt{2h g_x x} + C_9 ; \quad P(x) = d \rho g_x x ; \)

are solutions of

\( \mu \frac{\partial^2 V_x}{\partial x^2} + \frac{dp}{dx} + \rho V_x \frac{\partial V_x}{\partial x} = \rho g_x \) (deleting the \( y \) and \( z \) terms of (A)), p.25, one has shown that smooth solutions to the above differential equation exist, and the proof is complete.

Finding \( P(x,t) \):  

1. \( V_x(x,t) = \frac{-\rho g_x}{2\mu} ax^2 + C_1 x + f g_x t \pm \sqrt{2h g_x x} + C_9 ; \quad P(x) = d \rho g_x x ; \quad 2. \frac{dp}{dx} = d \rho g \)

\( \frac{dp}{dt} = \frac{dp}{dx} \frac{dx}{dt} \)

\( \frac{dp}{dt} = \frac{dp}{dx} V_x \quad (\frac{dx}{dt} = V_x) \)

\( \frac{dp}{dt} = d \rho g_x \left( \frac{-\rho g_x}{2\mu} (ax^2) + C_1 x \pm \sqrt{2h g_x x} + f g_x t + C_9 \right) \quad (\frac{dp}{dx} = d \rho g_x) \)

\( P(x,t) = \int d \rho g_x \left( \frac{-\rho g_x}{2\mu} (ax^2) + C_1 x \pm \sqrt{2h g_x x} + f g_x t + C_9 \right) dt \)

\( P(x,t) = d \rho g_x \left( \frac{-a \rho g_x}{2\mu} x^2 t + C_1 x t \pm t \sqrt{2h g_x x} + \frac{f g_x t^2}{2} + C_9 t \right) + C_{10} \)

References:
For paper edition of the above paper, see Chapter 11 & Appendix 7 of the book entitled "Power of Ratios" by A. A. Frempong, published by Yellowtextbooks.com. Without using ratios or proportion, the author would never be able to split-up the Navier-Stokes equations into sub-equations which were readily integrable. The impediment to solving the Navier-Stokes equations for over 150 years (whether linearized or non-linearized) has been due to finding a way to split-up the equations. Since ratios were the key to splitting the Navier-Stokes equations, and solving them, the solutions have also been published in the "Power of Ratios" book which covers definition of ratio and applications of ratio in mathematics, science, engineering, economics and business fields.

Adonten