Hybrid Vector Similarity Measures and Their Applications to Multi-attribute Decision Making under Neutrosophic Environment

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Abstract

In this paper, we propose new vector similarity measures of single valued and interval neutrosophic sets by hybridizing the concepts of Dice and cosine similarity measures. We present their applications in multi attribute decision making under neutrosophic environment. We use these similarity measures to find out the best alternative by determining the similarity measure values between the ideal alternative and each alternative. The results of the proposed similarity measures have been validated by comparing with other existing similarity measures reported in the literature for multi attribute decision making. The main thrust of the proposed similarity measures will be in the field of practical decision making, medical diagnosis, pattern recognition, data mining, clustering analysis, etc.

Keyword 0.1. Neutrosophic set, Single-valued neutrosophic set, Interval neutrosophic Set, Similarity measure, Hybrid vector similarity measure, Multi-attribute decision making.

1 Introduction

Multi attribute decision making (MADM) has received much attention to the researchers as it has caught great acceptance in the areas of operations research, social economics, and management science, etc. We encounter MADM problems under various situations, where the number of feasible alternatives and actions need to be selected based on a set of predefined attributes. Lots of research work have been done on

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MADM problems, where the ratings of alternatives and/or attribute values are expressed in terms of crisp numbers [1], interval numbers [2], fuzzy numbers [3], interval-valued fuzzy numbers [4], intuitionistic fuzzy numbers [5], interval-valued intuitionistic fuzzy numbers [6], grey numbers [7, 8], etc. However, in realistic situations, due to time pressure, complexity of the problem, lack of information processing capabilities, poor knowledge of the public domain and information, decision makers cannot provide exact evaluation of decision-parameters involved in MADM problems. In such situation, preference information of alternatives with respect to the attributes provided by the decision makers may be imprecise or incomplete in nature.

Imprecise or incomplete type of information can be dealt with neutrosophic sets (NSs), originally developed by Smarandache [9, 10]. NSs are characterized by truth, indeterminacy, and falsity membership functions which are independent in nature. In MADM context, the ratings of the alternatives provided by the decision maker can be expressed with NSs. These NSs can handle indeterminate and inconsistent information quite well, whereas, intuitionistic fuzzy sets and fuzzy sets can only handle incomplete or partial information. The application of neutrosophic set in MADM problems is recently an attractive and interesting topic to the researchers [11, 12, 13, 14]. From scientific and engineering point of view, Wang et al. [15] proposed single-valued neutrosophic set (SVNS) and offered some basic definitions regarding to the set theoretic operators. However, in reality sometimes the truth, the indeterminacy, and the falsity degree of a certain statement can be easily defined by interval numbers instead of crisp values. Wang et al. [16] proposed interval neutrosophic set (INS) and provided some definitions relating to set theoretic operators. As an important part of the modern decision science, some methods have been developed for MADM problems in single-valued neutrosophic set or interval neutrosophic set environment, for example, weighted aggregation operators [17, 18, 19, 20, 21, 22], TOPSIS method [23, 24], outranking method [25, 26], grey relational analysis method [27, 28, 29], inclusion measures [30], subset-hood measure [31], maximizing deviation method [32], etc.

However, as an effective method and a wide range of applications in various fields, similarity measure [33, 34, 35, 36, 37] can be used as a fruitful tool to deal with MADM problems, in which largest weighted similarity measure value between positive ideal alternative and alternatives determines the best alternative. Majumdar and Samanta [38] defined some similarity measures between two SVNSs with the help of distance measure, matching function, and membership grades of neutrosophic sets. Ye [39] proposed improved correlation coefficient of SVNS and studied some of its properties, and then extended it to a correlation coefficient between INSs. Broumi and Smarandache [40] defined Hausdorff distance measure between two neutrosophic sets and provided some similarity measures based on these distances. They also proposed the similarity measure between two neutrosophic sets by using set theoretic approach, and matching function in the same discussion. Ye [41] developed some similarity measures of INSs and applied them to multi-criteria decision making problems. Furthermore, Ye [42] proposed another similarity measure called vector similarity measure of SVNSs and INSs by considering the SVNSs as a three dimensional vector elements. Similarly, Broumi and Smarandache [43], extended the concept of cosine similarity measure of SVNSs into INSs and applied it to pattern recognition. Ye [44] developed the improved cosine similarity measure by using the vector concept and used it to medical diagnosis.

In the paper, we propose hybrid vector similarity measures for both SVNSs and INSs by extending the concept of variation coefficient similarity method [45] to neutrosophic environment and establish some of their basic properties. We also present the application of these proposed similarity measures to MADM under SVNSs and INSs. In order to do so, the rest of the paper is organized as follows: Section 2 presents the preliminaries of neutrosophic sets, SVNSs, and INSs. Section 3 represents vector similarity measure of SVNSs and INSs. Section 4 is devoted to develop the hybrid vector similarity measures for SVNSs and INSs. Hybrid vector similarity measure based MADM problems under SVNSs and INSs environment are described in Section 5. Finally in Section 6, two examples are provided to illustrate the MADM problems under SVNSs and INSs environment, and compared the results with other existing methods to demonstrate the effectiveness of the proposed similarity measures.

2 Preliminaries

In this section, we provide a brief overview of the concepts of neutrosophic sets, single-valued neutrosophic sets, interval neutrosophic sets, some vector similarity measures and their some properties.

2.1 Single valued neutrosophic set

Definition 1. [9, 10] Let X be a space of points (objects) with generic element in X denoted by x. Then a neutrosophic set A in X is characterized by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$, and a falsity membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ in X are real standard and non-standard subsets of $]^{-}0, 1^{+}[$ and satisfy the relation

$$^{-}0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^{+}.$$

However, Smarandache [9] introduced the neutrosophic set from philosophical point of view. To deal with science and engineering applications, Wang et al. [15] introduced the concept of SVNS, which is a subclass of the neutrosophic set and provided the following definitions.

Definition 2. [15] Let X be a universal space of points (objects), with a generic element in X denoted by x. A single-valued neutrosophic set $A \subseteq X$ is characterized by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$, and a falsity membership function $F_A(x)$. Then a SVNS A can be denoted by the following form: $A = \left\{\left\langle x, T_A(x), I_A(x), F_A(x)\right\rangle | x \in X\right\}$ where, $T_A(x)$, $I_A(x)$,and $F_A(x)$

belong to the unit interval $[0,1]$ for all $x \in X$. Therefore, the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$ satisfies the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

For convenience, we assume that $A = \langle T_A(x), I_A(x), F_A(x) \rangle$ is the single valued neutrosophic set in X.

Definition 3. [15, 19] Let A and B be two SVNSs defined by $A = \langle T_A(x), I_A(x), F_A(x) \rangle$ and $B = \langle T_B(x), I_B(x), F_B(x) \rangle$ in a universe of discourse X. Then some operational rules are presented as follows:

1. Complement: $A^c = \langle F_A(x), 1 - I_A(x), T_A(x) \rangle$
2. Containment: $A \subseteq B$ if and only if $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$ for all x in X;
3. Equality: $A = B$ if and only if $A \subseteq B$ and $A \supseteq B$
4. Union: $A \cup B = \langle x, T_A(x) \vee T_B(x), I_A(x) \wedge I_B(x), F_A(x) \wedge F_B(x) \rangle$ for all x in X;
5. Intersection: $A \cap B = \langle x, T_A(x) \wedge T_B(x), I_A(x) \vee I_B(x), F_A(x) \vee F_B(x) \rangle$ for all x in X;
6. Addition: $A \oplus B = \{\langle x, T_A(x) + T_B(x) - T_A(x).T_B(x), I_A(x).I_B(x), F_A(x).F_B(x) \rangle | x \in X\}$;
7. Multiplication: $A \otimes B = \left\{ \left\langle \begin{array}{cc} x, T_A(x).T_B(x), & I_A(x) + I_B(x) - I_A(x).I_B(x), \\ & F_A(x) + F_B(x) - F_A(x).F_B(x) \end{array} \right\rangle | x \in X \right\}$.

2.2 Interval neutrosophic set

Definition 4. [16] Let $D[0,1]$ be the set of all closed sub-intervals of the interval $[0,1]$ and let X be an ordinary finite non-empty set. An interval neutrosophic set (INS) $\tilde{A}$ in X is an object of the form

$$\tilde{A} = \left\{ \left\langle x, \tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x), \tilde{F}_{\tilde{A}}(x) \right\rangle | x \in X \right\},$$

where, $\tilde{T}_{\tilde{A}}(x) \in D[0,1]$, $\tilde{I}_{\tilde{A}}(x) \in D[0,1]$, and $\tilde{F}_{\tilde{A}}(x) \in D[0,1]$ with the relation

$$0 \le sup\tilde{T}_{\tilde{A}}(x) + sup\tilde{I}_{\tilde{A}}(x) + sup\tilde{F}_{\tilde{A}}(x) \le 3, \text{ for all } x \in X.$$

Here intervals $\tilde{T}_{\tilde{A}}(x) = \left[T^L_{\tilde{A}}(x), T^U_{\tilde{A}}(x)\right] \subset [0,1]$, $\tilde{I}_{\tilde{A}}(x) = \left[I^L_{\tilde{A}}(x), I^U_{\tilde{A}}(x)\right] \subset [0,1]$, $\tilde{F}_{\tilde{A}}(x) = \left[F^L_{\tilde{A}}(x), F^U_{\tilde{A}}(x)\right] \subset [0,1]$ denote, respectively the degree of truth, indeterminacy, and falsity membership of $x \in X$ in $\tilde{A}$; moreover $T^L_{\tilde{A}}(x) = inf\tilde{T}_{\tilde{A}}(x)$, $T^U_{\tilde{A}}(x) = sup\tilde{T}_{\tilde{A}}(x)$, $I^L_{\tilde{A}}(x) = inf\tilde{I}_{\tilde{A}}(x)$, $I^U_{\tilde{A}}(x) = sup\tilde{I}_{\tilde{A}}(x)$, $F^L_{\tilde{A}}(x) = inf\tilde{F}_{\tilde{A}}(x)$, $F^U_{\tilde{A}}(x) = sup\tilde{F}_{\tilde{A}}(x)$ for every $x \in X$. Thus, the interval neutrosophic set $\tilde{A}$ can be expressed in the following interval format:

$$\tilde{A} = \{\langle x, [T^L_{\tilde{A}}(x), T^U_{\tilde{A}}(x)]\ [I^L_{\tilde{A}}(x), I^U_{\tilde{A}}(x)]\ [F^L_{\tilde{A}}(x), F^U_{\tilde{A}}(x)] \rangle | x \in X\}$$

where, $0 \le supT^U_{\tilde{A}}(x) + supI^U_{\tilde{A}}(x) + supF^U_{\tilde{A}}(x) \le 3$, $T^L_{\tilde{A}}(x) \ge 0$, $I^L_{\tilde{A}}(x) \ge 0$ and $F^L_{\tilde{A}}(x) \ge 0$ for all $x \in X$.

For convenience of computation, we assume that $\tilde{A} = \langle \tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x), \tilde{F}_{\tilde{A}}(x)\rangle$ is the interval neutrosophic set in X.

Definition 5. [16] Let $\tilde{A} =\langle \tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x), \tilde{F}_{\tilde{A}}(x)\rangle$ and $\tilde{B} =\langle \tilde{T}_{\tilde{B}}(x), \tilde{I}_{\tilde{B}}(x), \tilde{F}_{\tilde{B}}(x)\rangle$ be two INSs in a universe of discourse X, then the following operations are defined as follows:

1. Complement: $\tilde{A}^c = \left\{\left\langle x, \left[F^L_{\tilde{A}}(x), F^U_{\tilde{A}}(x)\right], \left[1 - I^U_{\tilde{A}}(x), 1 - I^L_{\tilde{A}}(x)\right], \left[T^L_{\tilde{A}}(x), T^U_{\tilde{A}}(x)\right]\right\rangle |x \in X\right\}$;
2. Inclusion: $\tilde{A} \subseteq \tilde{B}$ if and only if $T^L_{\tilde{A}}(x) \le T^L_{\tilde{B}}(x)$, $T^U_{\tilde{A}}(x) \le T^U_{\tilde{B}}(x)$, $I^L_{\tilde{A}}(x) \ge I^L_{\tilde{B}}(x)$, $I^U_{\tilde{A}}(x) \ge I^U_{\tilde{B}}(x)$, $F^L_{\tilde{A}}(x) \ge F^L_{\tilde{B}}(x)$, $F^U_{\tilde{A}}(x) \ge F^U_{\tilde{B}}(x)$ for all $x \in X$;
3. Equality: $\tilde{A} = \tilde{B}$ if and only if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{A} \supseteq \tilde{B}$ for all $x \in X$;
4. Union: $\tilde{A} \cup \tilde{B} = \left\{\left\langle x, \left[T^L_{\tilde{A}}(x) \vee T^L_{\tilde{B}}(x), T^U_{\tilde{A}}(x) \vee T^U_{\tilde{B}}(x)\right], \left[I^L_{\tilde{A}}(x) \wedge I^L_{\tilde{B}}(x), I^U_{\tilde{A}}(x) \wedge I^U_{\tilde{B}}(x)\right], \left[F^L_{\tilde{A}}(x) \wedge F^L_{\tilde{B}}(x), F^U_{\tilde{A}}(x) \wedge F^U_{\tilde{B}}(x)\right]\right\rangle |x \in X\right\}$;
5. Intersection: $\tilde{A} \cap \tilde{B} = \left\{\left\langle x, \left[T^L_{\tilde{A}}(x) \wedge T^L_{\tilde{B}}(x), T^U_{\tilde{A}}(x) \wedge T^U_{\tilde{B}}(x)\right], \left[I^L_{\tilde{A}}(x) \vee I^L_{\tilde{B}}(x), I^U_{\tilde{A}}(x) \vee I^U_{\tilde{B}}(x)\right], \left[F^L_{\tilde{A}}(x) \vee F^L_{\tilde{B}}(x), F^U_{\tilde{A}}(x) \vee F^U_{\tilde{B}}(x)\right]\right\rangle |x \in X\right\}$.

2.3 Vector similarity measures

The vector similarity measure is one of the important tools for the degree of similarity between objects. However, the Jaccard, Dice, and cosine similarity measures are often used for this purpose. In the following discussions, we recall some definitions of the Jaccard [46], Dice [47], and cosine [48] similarity measures between two vectors. Let $X =(x_1, x_2, \ldots, x_n)$ and $Y =(y_1, y_2, \ldots, y_n)$ be two n-dimensional vectors with positive co-ordinates.

Definition 6. [46] The Jaccard similarity measure between two vectors $X =(x_1, x_2, \ldots, x_n)$ and $Y =(y_1, y_2, \ldots, y_n)$ is defined as follows:

$$J(X, Y) = \frac{X.Y}{||X||^2 + ||Y||^2 - X.Y} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i y_i} \tag{1}$$

where, $||X|| = \sqrt{\sum_{i=1}^n x_i^2}$ and $||Y|| = \sqrt{\sum_{i=1}^n y_i^2}$ are the Euclidean norms of X and Y, $X.Y = \sum_{i=1}^n x_i y_i$ is the inner product of the vectors X and Y. Then, this similarity measure satisfies the following properties:

J1 $0 \le J(X, Y) \le 1$;

J2 $J(X, Y) = J(Y, X)$;

J3 $J(X, Y) = 1$ for $X = Y$ i.e. $x_i = y_i (i = 1, 2, \ldots, n)$ for every $x_i \in X$ and $y_i \in Y$.

Definition 7. [47] The Dice similarity measure between two vectors $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ is defined as follows:

$$E(X,Y) = \frac{2X.Y}{||X||^2 + ||Y||^2} = \frac{\sum_{i=1}^{n} 2x_i y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2}. \tag{2}$$

It satisfies the following properties:

E1 $0 \leq E(X,Y) \leq 1$;

E2 $E(X,Y) = E(Y,X)$;

E3 $E(X,Y) = 1$ for $X = Y$ i.e. $x_i = y_i (i = 1, 2, \ldots, n)$ for every $x_i \in X$ and $y_i \in Y$.

Definition 8. [48] The cosine similarity measure between two vectors $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ is the inner product of these two vectors divided by the product of their lengths and is defined as follows:

$$C(X,Y) = \frac{X.Y}{||X||.||Y||} = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2} . \sqrt{\sum_{i=1}^{n} y_i^2}}. \tag{3}$$

It satisfies the following properties:

C1 $0 \leq C(X,Y) \leq 1$;

C2 $C(X,Y) = C(Y,X)$;

C3 $C(X,Y) = 1$ for $X = Y$ i.e. $x_i = y_i (i = 1, 2, \ldots, n)$ for every $x_i \in X$ and $y_i \in Y$.

These three formulas are similar in the sense that they assume values in the interval $[0, 1]$. Jaccard and Dice similarity measure are undefined when $x_i = 0$ and $y_i = 0$ and cosine similarity measure is undefined when $x_i = 0$ or $y_i = 0$ for $i = 1, 2, \ldots, n$.

Definition 9. [45] The variation co-efficient similarity measure between two vectors $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ is defined as follows:

$$\begin{aligned} V(X,Y) &= \lambda \frac{2XY}{||X||^2 + ||Y||^2} + (1-\lambda) \frac{XY}{||X||.||Y||} \\ &= \lambda \frac{\sum_{i=1}^{n} 2x_i y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2} + (1-\lambda) \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2} . \sqrt{\sum_{i=1}^{n} y_i^2}}. \end{aligned} \tag{4}$$

It satisfies the following properties:

V1 $0 \leq V(X,Y) \leq 1$;

V2 $V(X,Y) = V(Y,X)$;

V3 $V(X,Y) = 1$ for $X = Y$ i.e. $x_i = y_i (i = 1, 2, \ldots, n)$ for every $x_i \in X$ and $y_i \in Y$.

3 Vector similarity measures of SVNSs and INSs

3.1 Vector similarity measures of SVNSs

We assume that the triples $\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$ and $\langle T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ represent respectively the coordinates of two SVNSs $A=\{\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle \mid x_i \in X\}$ and $B = \{\langle T_B(x_i), I_B(x_i), F_B(x_i)\rangle \mid x_i \in X\}$ in a three dimensional space. Then the vector similarity measures between SVNSs can be defined as follows.

Definition 10. [39] Let $A=\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$ and $B = \langle T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ be two SVNSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. Then the Jaccard similarity measure between SVNSs A and B in the vector space is defined as follows:

$$Jac(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)}{\left[\begin{array}{r}\left(T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i)\right) + \left(T_B^2(x_i) + I_B^2(x_i) + F_B^2(x_i)\right) \\ -\left(T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)\right)\end{array}\right]} \tag{5}$$

and if $w_i \in [0,1]$ be the weight of each element x_i for $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} w_i = 1$, then the weighted Jaccard similarity measure between SVNSs A and B is defined as follows:

$$Jac_w(A,B) = \sum_{i=1}^{n} w_i \frac{T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)}{\left[\begin{array}{r}\left(T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i)\right) + \left(T_B^2(x_i) + I_B^2(x_i) + F_B^2(x_i)\right) \\ -\left(T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)\right)\end{array}\right]}. \tag{6}$$

Definition 11. [39] Let $A=\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$ and $B = \langle T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ be two SVNSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. Then the Dice similarity measure between SVNSs A and B in the vector space is defined as follows:

$$Dic(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{2\left(T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)\right)}{[(T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i)) + (T_B^2(x_i) + I_B^2(x_i) + F_B^2(x_i))]} \tag{7}$$

and if $w_i \in [0,1]$ be the weight of each element x_i for $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} w_i = 1$, then the weighted Dice similarity measure between SVNSs A and B is defined as follows:

$$Dic_w(A,B) = \sum_{i=1}^{n} w_i \frac{2\left(T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)\right)}{[(T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i)) + (T_B^2(x_i) + I_B^2(x_i) + F_B^2(x_i))]}. \tag{8}$$

Definition 12. [39] Let $A=\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$ and $B = \langle T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ be two SVNSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. Then the cosine similarity measure between SVNSs A and B in the vector space is defined as follows:

$$Cos(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{(T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)+F_A(x_i)F_B(x_i))}{\left[\sqrt{(T_A^2(x_i)+I_A^2(x_i)+F_A^2(x_i))}.\sqrt{(T_B^2(x_i)+I_B^2(x_i)+F_B^2(x_i))}\right]} \tag{9}$$

and if $w_i \in [0,1]$ be the weight of each element x_i for $i=1,2,\ldots,n$ such that $\sum_{i=1}^{n} w_i = 1$, then the weighted cosine similarity measure between SVNSs A and B is defined as follows:

$$Cos_w(A,B)=\sum_{i=1}^{n} w_i\frac{(T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)+F_A(x_i)F_B(x_i))}{\left[\sqrt{(T_A^2(x_i)+I_A^2(x_i)+F_A^2(x_i))}.\sqrt{(T_B^2(x_i)+I_B^2(x_i)+F_B^2(x_i))}\right]}. \tag{10}$$

Eq. (6), Eq.(8), and Eq.(10) satisfy the following properties:

P1. $0 \le Jac_w(A,B) \le 1$; $0 \le Dic_w(A,B) \le 1$; $0 \le Cos_w(A,B) \le 1$;

P2. $Jac_w(A,B) = Jac_w(B,A)$; $Dic_w(A,B) = Dic_w(B,A)$; and $Cos_w(A,B) = Cos_w(B,A)$;

P3. $Jac_w(A,B) = 1$; $Dic_w(A,B) = 1$; $Cos_w(A,B) = 1$ if $B = A$ i.e. $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$ for every $x_i (i = 1,2,\ldots,n)$ in X.

Jaccard and Dice similarity measures between two SVNSs A=$\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$ and $B = \langle T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ are undefined for A=$\langle 0,0,0\rangle$ and B=$\langle 0,0,0\rangle$ that is when $T_A = I_A = F_A = 0$ and $T_B = I_B = F_B = 0$ for all $i = 1,2,\ldots,n$. Similarly, the cosine similarity measure is undefined for A=$\langle 0,0,0\rangle$ or B=$\langle 0,0,0\rangle$ that is when $T_A = I_A = F_A = 0$ or $T_B = I_B = F_B = 0$ for all $i = 1,2,\ldots,n$. In this case, the similarity measure values $Jac_w(A,B)$, $Dic_w(A,B)$ and $Cos_w(A,B)$ of SVNSs A and B are assumed to be zero.

3.2 Vector similarity measure of INSs

Let $\tilde{A}$=$\left\langle \tilde{T}_{\tilde{A}}(x_i), \tilde{I}_{\tilde{A}}(x_i), \tilde{F}_{\tilde{A}}(x_i)\right\rangle$ and $\tilde{B}$=$\left\langle \tilde{T}_{\tilde{B}}(x_i), \tilde{I}_{\tilde{B}}(x_i), \tilde{F}_{\tilde{B}}(x_i)\right\rangle$ be two INSs in a universe of discourse X. We consider the triples $\left\langle \Delta\tilde{T}_{\tilde{A}}(x_i), \Delta\tilde{I}_{\tilde{A}}(x_i), \Delta\tilde{F}_{\tilde{A}}(x_i)\right\rangle$ and $\left\langle \Delta\tilde{T}_{\tilde{B}}(x_i), \Delta\tilde{I}_{\tilde{B}}(x_i), \Delta\tilde{I}_{\tilde{B}}(x_i)\right\rangle$ as the representations of $\tilde{A}$ and $\tilde{B}$ in a three dimensional vector space, where for all $x_i \in X (i = 1,2,\ldots,n)$:

$$2\Delta\tilde{T}_{\tilde{A}}(x_i) = \left[T_{\tilde{A}}^L(x_i) + T_{\tilde{A}}^U(x_i)\right], \quad 2\Delta\tilde{I}_{\tilde{A}}(x_i) = \left[I_{\tilde{A}}^L(x_i) + I_{\tilde{A}}^U(x_i)\right], \quad 2\Delta\tilde{F}_{\tilde{A}}(x_i) = \left[F_{\tilde{A}}^L(x_i) + F_{\tilde{A}}^U(x_i)\right],$$
$$2\Delta\tilde{T}_{\tilde{B}}(x_i) = \left[T_{\tilde{B}}^L(x_i) + T_{\tilde{B}}^U(x_i)\right], \quad 2\Delta\tilde{I}_{\tilde{B}}(x_i) = \left[I_{\tilde{B}}^L(x_i) + I_{\tilde{B}}^U(x_i)\right], \quad 2\Delta\tilde{F}_{\tilde{B}}(x) = \left[F_{\tilde{B}}^L(x_i) + F_{\tilde{B}}^U(x_i)\right].$$

Then the vector similarity measures between INSs can be defined as follows.

Definition 13. [43] Let $\tilde{A}$=$\left\langle \tilde{T}_{\tilde{A}}(x_i), \tilde{I}_{\tilde{A}}(x_i), \tilde{F}_{\tilde{A}}(x_i)\right\rangle$ and $\tilde{B}$=$\left\langle \tilde{T}_{\tilde{B}}(x_i), \tilde{I}_{\tilde{B}}(x_i), \tilde{F}_{\tilde{B}}(x_i)\right\rangle$ be two INSs in a universe of discourse X =$\{x_1, x_2, \ldots, x_n\}$. Then the cosine similarity measure between $\tilde{A}$ and $\tilde{B}$ in the vector space is defined as follows:

$$Cos(\tilde{A},\tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\Delta\tilde{T}_{\tilde{A}}(x_i)\Delta\tilde{T}_{\tilde{B}}(x_i)+\Delta\tilde{I}_{\tilde{A}}(x_i)\Delta\tilde{I}_{\tilde{B}}(x_i)+\Delta\tilde{F}_{\tilde{A}}(x_i)\Delta\tilde{F}_{\tilde{B}}(x_i)}{\sqrt{\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{A}}(x_i)\right)^2}.\sqrt{\left(\Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2}}. \tag{11}$$

If $w_i \in [0,1]$ be the weight of each element x_i for $i = 1,2,\ldots,n$ such that $\sum_{i=1}^{n} w_i = 1$, then the weighted cosine similarity measure between $\tilde{A}$ and $\tilde{B}$ is defined as follows:

$$Cos_w(\tilde{A},\tilde{B}) = \sum_{i=1}^{n} w_i \frac{\Delta\tilde{T}_{\tilde{A}}(x_i)\Delta\tilde{T}_{\tilde{B}}(x_i)+\Delta\tilde{I}_{\tilde{A}}(x_i)\Delta\tilde{I}_{\tilde{B}}(x_i)+\Delta\tilde{F}_{\tilde{A}}(x_i)\Delta\tilde{F}_{\tilde{B}}(x_i)}{\left[\begin{array}{l}\sqrt{\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{A}}(x_i)\right)^2} \\ \qquad \times \sqrt{\left(\Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2}\end{array}\right]}. \tag{12}$$

Eq.(12) satisfies the following properties:

C1. $0 \le Cos_w(\tilde{A},\tilde{B}) \le 1$;

C2. $Cos_w(\tilde{A},\tilde{B}) = Cos_w(\tilde{B},\tilde{A})$;

C3. $Cos_w(\tilde{A},\tilde{B}) = 1$ if $\tilde{A} = \tilde{B}$ i.e. when $T^L_{\tilde{A}}(x_i)= T^L_{\tilde{B}}(x_i)$, $I^L_{\tilde{A}}(x_i)= I^L_{\tilde{B}}(x_i)$, $F^L_{\tilde{A}}(x_i)= F^L_{\tilde{B}}(x_i)$, $T^U_{\tilde{A}}(x_i)= T^U_{\tilde{B}}(x_i)$, $I^U_{\tilde{A}}(x_i)= I^U_{\tilde{B}}(x_i)$ and $F^U_{\tilde{A}}(x_i)= F^U_{\tilde{B}}(x_i)$ for $i = 1,2,\ldots,n$.

Definition 14. Let $\tilde{A}=\left\langle \tilde{T}_{\tilde{A}}(x_i),\tilde{I}_{\tilde{A}}(x_i),\tilde{F}_{\tilde{A}}(x_i)\right\rangle$ and $\tilde{B}=\left\langle \tilde{T}_{\tilde{B}}(x_i),\tilde{I}_{\tilde{B}}(x_i),\tilde{F}_{\tilde{B}}(x_i)\right\rangle$ be two INSs in a universe of discourse $X = \{x_1,x_2,\ldots,x_n\}$. Then the Dice similarity measure between INSs $\tilde{A}$ and $\tilde{B}$ in the vector space is defined as follows:

$$Dic(\tilde{A},\tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\frac{2\Big(\Delta\tilde{T}_{A}(x_i)\Delta\tilde{T}_{B}(x_i)+\Delta\tilde{I}_{A}(x_i)\Delta\tilde{I}_{B}(x_i)+\Delta\tilde{F}_{A}(x_i)\Delta\tilde{F}_{B}(x_i)\Big)}{\left[\begin{array}{l}\Big(\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{A}}(x_i)\right)^2\Big) \\ \qquad + \Big(\left(\Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2\Big)\end{array}\right]}, \tag{13}$$

and if $w_i \in [0,1]$ be the weight of each element x_i for $i = 1,2,\ldots,n$ such that $\sum_{i=1}^{n} w_i = 1$, then the weighted Dice similarity measure between $\tilde{A}$ and $\tilde{B}$ is defined as follows:

$$Dic_w(\tilde{A},\tilde{B}) = \sum_{i=1}^{n} w_i \frac{2\Big(\Delta\tilde{T}_{\tilde{A}}(x_i)\Delta\tilde{T}_{\tilde{B}}(x_i)+\Delta\tilde{I}_{\tilde{A}}(x_i)\Delta\tilde{I}_{\tilde{B}}(x_i)+\Delta\tilde{F}_{\tilde{A}}(x_i)\Delta\tilde{F}_{\tilde{B}}(x_i)\Big)}{\left[\begin{array}{l}\Big(\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{A}}(x_i)\right)^2\Big) \\ \qquad + \Big(\left(\Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2\Big)\end{array}\right]} \tag{14}$$

Proposition 3.1. *The Dice similarity measure $Dic_w(\tilde{A},\tilde{B})$ between $\tilde{A}$ and $\tilde{B}$ satisfies the following properties*

D1. $0 \le Dic_w(\tilde{A},\tilde{B}) \le 1$;

D2. $Dic_w(\tilde{A},\tilde{B}) = Dic_w(\tilde{B},\tilde{A})$*;*

D3. $Dic_w(\tilde{A},\tilde{B}) = 1$ *if* $\tilde{A} = \tilde{B}$ *i.e. when* $T^L_{\tilde{A}}(x_i)= T^L_{\tilde{B}}(x_i)$*,* $I^L_{\tilde{A}}(x_i)= I^L_{\tilde{B}}(x_i)$*,* $F^L_{\tilde{A}}(x_i)= F^L_{\tilde{B}}(x_i)$*,* $T^U_{\tilde{A}}(x_i)= T^U_{\tilde{B}}(x_i)$*,* $I^U_{\tilde{A}}(x_i)= I^U_{\tilde{B}}(x_i)$ *and* $F^U_{\tilde{A}}(x_i)= F^U_{\tilde{B}}(x_i)$ *for* $i = 1, 2, \ldots, n$*.*

Proof. D1. It is obvious that $Dic_w(\tilde{A},\tilde{B}) \geq 0$ for all real values of $\Delta\tilde{T}_{\tilde{A}}(x_i)$, $\Delta\tilde{I}_{\tilde{A}}(x_i)$, $\Delta\tilde{F}_{\tilde{A}}(x_i)$, $\Delta\tilde{T}_{\tilde{B}}(x_i)$, $\Delta\tilde{I}_{\tilde{B}}(x_i)$, and $\Delta\tilde{F}_{\tilde{B}}(x_i)$ for $i = 1, 2, \ldots, n$. Now consider the expression

$$\begin{aligned}
&\left[\begin{array}{l} \left(\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\right)^2 + \left(\Delta\tilde{I}_{\tilde{A}}(x_i)\right)^2 + \left(\Delta\tilde{F}_{\tilde{A}}(x_i)\right)^2\right) \\ +\left(\left(\Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2 + \left(\Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2 + \left(\Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2\right) \end{array}\right] - 2\left(\begin{array}{r} \Delta\tilde{T}_{\tilde{A}}(x_i)\Delta\tilde{T}_{\tilde{B}}(x_i) + \Delta\tilde{I}_{\tilde{A}}(x_i)\Delta\tilde{I}_{\tilde{B}}(x_i) \\ +\Delta\tilde{F}_{\tilde{A}}(x_i)\Delta\tilde{F}_{\tilde{B}}(x_i) \end{array}\right) \\
&= \left(\Delta\tilde{T}_{\tilde{A}}(x_i) - \Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2 + \left(\Delta\tilde{I}_{\tilde{A}}(x_i) - \Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2 + \left(\Delta\tilde{F}_{\tilde{A}}(x_i) - \Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2. \qquad (15)
\end{aligned}$$

It is obviously greater than zero for any real value of $\Delta\tilde{T}_{\tilde{A}}(x_i)$, $\Delta\tilde{I}_{\tilde{A}}(x_i)$, $\Delta\tilde{F}_{\tilde{A}}(x_i)$, $\Delta\tilde{T}_{\tilde{B}}(x_i)$, $\Delta\tilde{I}_{\tilde{B}}(x_i)$, and $\Delta\tilde{F}_{\tilde{A}}(x_i)$ for $i = 1, 2, \ldots, n$. Therefore the first property i.e the inequality $0 \leq Dic_w(\tilde{A},\tilde{B}) \leq 1$ holds good for all values of $x_i(i = 1, 2, \ldots, n)$.

D2. Symmetry of Eq. (14) validates the property $D2$.

D3. We see that if $T^L_{\tilde{A}}(x_i)= T^L_{\tilde{B}}(x_i)$, $I^L_{\tilde{A}}(x_i)= I^L_{\tilde{B}}(x_i)$, $F^L_{\tilde{A}}(x_i)= F^L_{\tilde{B}}(x_i)$, $T^U_{\tilde{A}}(x_i)= T^U_{\tilde{B}}(x_i)$, $I^U_{\tilde{A}}(x_i)= I^U_{\tilde{B}}(x_i)$ and $F^U_{\tilde{A}}(x_i)= F^U_{\tilde{B}}(x_i)$ for $i = 1, 2, \ldots, n$ then from Eq. (14), we have $Dic_w(\tilde{A},\tilde{B}) = 1$.

□

However, Dice similarity measure between two INSs $\tilde{A}=\left\langle\tilde{T}_{\tilde{A}}(x_i), \tilde{I}_{\tilde{A}}(x_i), \tilde{F}_{\tilde{A}}(x_i)\right\rangle$ and $\tilde{B}=\left\langle\tilde{T}_{\tilde{B}}(x_i), \tilde{I}_{\tilde{B}}(x_i), \tilde{F}_{\tilde{B}}(x_i)\right\rangle$ is undefined for $\Delta\tilde{T}_{\tilde{A}}= \Delta\tilde{I}_{\tilde{A}}= \Delta\tilde{F}_{\tilde{A}}=\tilde{0}$ and $\Delta\tilde{T}_{\tilde{B}}= \Delta\tilde{I}_{\tilde{B}}= \Delta\tilde{F}_{\tilde{B}}=\tilde{0}$. Similarly, the cosine similarity is undefined for $\Delta\tilde{T}_{\tilde{A}}= \Delta\tilde{I}_{\tilde{A}}= \Delta\tilde{F}_{\tilde{A}}=\tilde{0}$ or $\Delta\tilde{T}_{\tilde{B}}= \Delta\tilde{I}_{\tilde{B}}= \Delta\tilde{F}_{\tilde{B}}=\tilde{0}$. In this case, the similarity measure values $Dic_w(\tilde{A},\tilde{B})$ and $Cos_w(\tilde{A},\tilde{B})$ of IVNSs $\tilde{A}$ and $\tilde{B}$ are also assumed to be zero.

4 Hybrid vector similarity measures of neutrosophic sets

In the following two subsections, we propose two co-efficient parameter depended vector similarity measures for both SVNSs and INSs.

4.1 Hybrid vector similarity measure of SVNSs

Definition 15. Let $A=\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$ and $B = \langle T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ be two SVNSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, and $w_i \in [0, 1]$ be the weight of each element x_i for $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} w_i = 1$. Then, the hybrid vector similarity measure (HVSM) of SVNSs in the vector space is defined as follows:

$$Hyb(A,B)=\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\frac{2\left(T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)+F_A(x_i)F_B(x_i)\right)}{[(T_A^2(x_i)+I_A^2(x_i)+F_A^2(x_i))+(T_B^2(x_i)+I_B^2(x_i)+F_B^2(x_i))]}\\ \quad+(1-\lambda)\sum_{i=1}^{n}\frac{T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)+F_A(x_i)F_B(x_i}{\left[\sqrt{(T_A^2(x_i)+I_A^2(x_i)+F_A^2(x_i))}\sqrt{(T_B^2(x_i)+I_B^2(x_i)+F_B^2(x_i))}\right]}\end{array}\right] \tag{16}$$

and if $w_i \in [0,1]$ be the weight of each element x_i for $i=1,2,\ldots,n$ such that $\sum_{i=1}^{n} w_i = 1$, then the weighted hybrid vector similarity measure of SVNSs is defined as follows:

$$Hyb_w(A,B)=\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\frac{2\left(T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)+F_A(x_i)F_B(x_i)\right)}{[(T_A^2(x_i)+I_A^2(x_i)+F_A^2(x_i))+(T_B^2(x_i)+I_B^2(x_i)+F_B^2(x_i))]}\\ \quad+(1-\lambda)\sum_{i=1}^{n}w_i\frac{T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)+F_A(x_i)F_B(x_i)}{\left[\sqrt{(T_A^2(x_i)+I_A^2(x_i)+F_A^2(x_i))}\sqrt{(T_B^2(x_i)+I_B^2(x_i)+F_B^2(x_i))}\right]}\end{array}\right]. \tag{17}$$

.

Proposition 4.1. *The weighted hybrid vector similarity measure (WHVSM) of SVNSs A and B is denoted by $Hyb_w(A,B)$, satisfies the following properties:*

H1. $0 \le Hyb_w(A,B) \le 1$;

H2. $Hyb_w(A,B) = Hyb_w(B,A)$;

H3. $Hyb_w(A,B) = 1$ *if* $A = B$ *i.e. when* $T_A(x_i)$=$T_B(x_i)$, $I_A(x_i)$=$I_B(x_i)$, *and* $F_A(x_i)$=$F_B(x_i)$, *for* $i = 1,2,\ldots,n$.

Proof. H1. From Dice and cosine similarity measures of SVNSs defined in Eq. (8) and Eq. (10), we have $0 \le Dic_w(A,B) \le 1$ and $0 \le Cos_w(A,B) \le 1$ for all $i = 1,2,\ldots,n$. Now from Eq. (17), the HVSM can be written as follows:

$$\begin{aligned}Hyb_w(A,B) &= \lambda Dic_w(A,B) + (1-\lambda)Cos_w(A,B)\\ &\le \lambda + (1-\lambda) = 1.\end{aligned} \tag{18}$$

Because $Dic_w(A,B) \ge 0$ and $Cos_w(A,B) \ge 0$, the HVSM $Hyb_w(A,B) \ge 0$ for any values of $\lambda \in [0,1]$. This proves the first property of $Hyb_w(A,B)$ i.e. $0 \le Hyb_w(A,B) \le 1$.

H2. Symmetry of Eq. (17) validates the property $H2$.

H3. If $T_A(x_i)$=$T_B(x_i)$, $I_A(x_i)$=$I_B(x_i)$, and $F_A(x_i)$=$F_B(x_i)$, for $i = 1,2,\ldots,n$, then the value of $Dic_w(A,B) = 1$ and $Cos_w(A,B) = 1$. Therefore from Eq. (18), the value of $Hyb_w(A,B) = 1$

This completes the proof. □

Hybrid vector similarity measure value between two SVNSs $A=\langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$ and $B = \langle T_B(x_i), I_B(x_i), F_B(x_i)\rangle$ is assumed to be zero for $A=\langle 0,0,0\rangle$ and $B=\langle 0,0,0\rangle$.

4.2 Hybrid vector similarity measure of INSs

Definition 16. Let $\tilde{A}=\left\langle \tilde{T}_{\tilde{A}}(x_i), \tilde{I}_{\tilde{A}}(x_i), \tilde{F}_{\tilde{A}}(x_i)\right\rangle$ and $\tilde{B}=\left\langle \tilde{T}_{\tilde{B}}(x_i), \tilde{I}_{\tilde{B}}(x_i), \tilde{F}_{\tilde{B}}(x_i)\right\rangle$ be two INSs in a universe of discourse $X=\{x_1, x_2, \ldots, x_n\}$. Then the hybrid vector similarity measure between $\tilde{A}$ and $\tilde{B}$ in the vector space is defined as follows:

$$H(\tilde{A},\tilde{B})=\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\Delta\tilde{T}_{\tilde{B}}(x_i)+\Delta\tilde{I}_{\tilde{A}}(x_i)\Delta\tilde{I}_{\tilde{B}}(x_i)+\Delta\tilde{F}_{\tilde{A}}(x_i)\Delta\tilde{F}_{\tilde{B}}(x_i)\right)}{\left[\begin{array}{l}\left(\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{A}}(x_i)\right)^2\right)\\ \qquad+\left(\left(\Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2\right)\end{array}\right]}\\ \quad+(1-\lambda)\sum_{i=1}^{n}\dfrac{\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\Delta\tilde{T}_{\tilde{B}}(x_i)+\Delta\tilde{I}_{\tilde{A}}(x_i)\Delta\tilde{I}_{\tilde{B}}(x_i)+\Delta\tilde{F}_{\tilde{A}}(x_i)\Delta\tilde{F}_{\tilde{B}}(x_i)\right)}{\left[\begin{array}{l}\sqrt{\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{A}}(x_i)\right)^2}\\ \qquad\times\sqrt{\left(\Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2}\end{array}\right]}\end{array}\right], \tag{19}$$

where for any $x_i \in X (i=1,2,\ldots,n)$,

$$2\Delta\tilde{T}_{\tilde{A}}(x_i)=\left[T^L_{\tilde{A}}(x_i)+T^U_{\tilde{A}}(x_i)\right], \quad 2\Delta\tilde{I}_{\tilde{A}}(x_i)=\left[I^L_{\tilde{A}}(x_i)+I^U_{\tilde{A}}(x_i)\right], \qquad 2\Delta\tilde{F}_{\tilde{A}}(x_i)=\left[F^L_{\tilde{A}}(x_i)+F^U_{\tilde{A}}(x_i)\right],$$
$$2\Delta\tilde{T}_{\tilde{B}}(x_i)=\left[T^L_{\tilde{B}}(x_i)+T^U_{\tilde{B}}(x_i)\right], \quad 2\Delta\tilde{I}_{\tilde{B}}(x_i)=\left[I^L_{\tilde{B}}(x_i)+I^U_{\tilde{B}}(x_i)\right], \qquad 2\Delta\tilde{F}_{\tilde{B}}(x)=\left[F^L_{\tilde{B}}(x_i)+F^U_{\tilde{B}}(x_i)\right].$$

If $w_i \in [0,1]$ be the weight of the element x_i for $i=1,2,\ldots,n$ such that $\sum_{i=1}^{n} w_i = 1$, then, the weighted hybrid vector similarity measure(WHVSM) between $\tilde{A}$ and $\tilde{B}$ in the vector space is defined as follows:

$$H_w(\tilde{A},\tilde{B})=\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\dfrac{2\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\Delta\tilde{T}_{\tilde{B}}(x_i)+\Delta\tilde{I}_{\tilde{A}}(x_i)\Delta\tilde{I}_{\tilde{B}}(x_i)+\Delta\tilde{F}_{\tilde{A}}(x_i)\Delta\tilde{F}_{\tilde{B}}(x_i)\right)}{\left[\begin{array}{l}\left(\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{A}}(x_i)\right)^2\right)\\ \qquad+\left(\left(\Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2\right)\end{array}\right]}\\ \quad+(1-\lambda)\sum_{i=1}^{n}w_i\dfrac{\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\Delta\tilde{T}_{\tilde{B}}(x_i)+\Delta\tilde{I}_{\tilde{A}}(x_i)\Delta\tilde{I}_{\tilde{B}}(x_i)+\Delta\tilde{F}_{\tilde{A}}(x_i)\Delta\tilde{F}_{\tilde{B}}(x_i)\right)}{\left[\begin{array}{l}\sqrt{\left(\Delta\tilde{T}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{A}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{A}}(x_i)\right)^2}\\ \qquad\times\sqrt{\left(\Delta\tilde{T}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{I}_{\tilde{B}}(x_i)\right)^2+\left(\Delta\tilde{F}_{\tilde{B}}(x_i)\right)^2}\end{array}\right]}\end{array}\right]. \tag{20}$$

Proposition 4.2. *The weighted hybrid vector similarity measure of two INSs $\tilde{A}$ and $\tilde{B}$ is denoted by $H_w(\tilde{A},\tilde{B})$, satisfies the following properties:*

H1. $0 \le H_w(\tilde{A},\tilde{B}) \le 1$;

H2. $H_w(\tilde{A},\tilde{B}) = H_w(\tilde{B},\tilde{A})$;

H3. $H_w(\tilde{A},\tilde{B}) = 1$ *if* $\tilde{A} = \tilde{B}$ *i.e. when* $T^L_{\tilde{A}}(x_i)= T^L_{\tilde{B}}(x_i)$, $I^L_{\tilde{A}}(x_i)= I^L_{\tilde{B}}(x_i)$, $F^L_{\tilde{A}}(x_i)= F^L_{\tilde{B}}(x_i)$, $T^U_{\tilde{A}}(x_i)= T^U_{\tilde{B}}(x_i)$, $I^U_{\tilde{A}}(x_i)= I^U_{\tilde{B}}(x_i)$ *and* $F^U_{\tilde{A}}(x_i)= F^U_{\tilde{B}}(x_i)$ *for* $i = 1, 2, \ldots, n$.

Proof. H1. Dice and cosine similarity measure of two INSs $\tilde{A}$ and $\tilde{B}$ lie in the unit interval i.e.

$$0 \le Dic_w(\tilde{A},\tilde{B}) \le 1; \quad 0 \le Cos_w(\tilde{A},\tilde{B}) \le 1$$

for all values of $x_i(i = 1, 2, \ldots, n)$. Now, according to Eqs. (14) and (12), the WHVSM of $\tilde{A}$ and $\tilde{B}$ can be written as follows:

$$\begin{aligned} H_w(\tilde{A},\tilde{B}) &= \lambda Dic_w(\tilde{A},\tilde{B}) + (1-\lambda)Cos_w(\tilde{A},\tilde{B}) \\ &\le \lambda + (1-\lambda) = 1. \end{aligned} \tag{21}$$

On the other hand, for all real values of $\tilde{T}_{\tilde{A}}(x_i)$, $\tilde{I}_{\tilde{A}}(x_i)$, $\tilde{F}_{\tilde{A}}(x_i)$, $\tilde{T}_{\tilde{B}}(x_i)$, $\tilde{I}_{\tilde{B}}(x_i)$ and $\tilde{F}_{\tilde{B}}(x_i)$, the WHVSM $H_w(\tilde{A},\tilde{B}) \ge 0$. Therefore, $0 \le H_w(\tilde{A},\tilde{B}) \le 1$.

H2. Symmetry of Eq. (20) validates the property $H2$.

H3. If $T^L_{\tilde{A}}(x_i)= T^L_{\tilde{B}}(x_i)$, $I^L_{\tilde{A}}(x_i)= I^L_{\tilde{B}}(x_i)$, $F^L_{\tilde{A}}(x_i)= F^L_{\tilde{B}}(x_i)$, $T^U_{\tilde{A}}(x_i)= T^U_{\tilde{B}}(x_i)$, $I^U_{\tilde{A}}(x_i)= I^U_{\tilde{B}}(x_i)$ and $F^U_{\tilde{A}}(x_i)= F^U_{\tilde{B}}(x_i)$ for $i = 1, 2, \ldots, n$, then the value of $Dic_w(\tilde{A},\tilde{B}) = 1$ and $Cos_w(\tilde{A},\tilde{B}) = 1$. Therefore from Eq. (20), the value of $H_w(\tilde{A},\tilde{B}) = 1$

This completes the proof. □

However, for $\Delta\tilde{T}_{\tilde{A}}= \Delta\tilde{I}_{\tilde{A}}= \Delta\tilde{F}_{\tilde{A}}=\tilde{0}$ and $\Delta\tilde{T}_{\tilde{B}}= \Delta\tilde{I}_{\tilde{B}}= \Delta\tilde{F}_{\tilde{B}}=\tilde{0}$ the hybrid vector similarity measure between two INSs $\tilde{A}=\left\langle \tilde{T}_{\tilde{A}}(x_i), \tilde{I}_{\tilde{A}}(x_i), \tilde{F}_{\tilde{A}}(x_i)\right\rangle$ and $\tilde{B}=\left\langle \tilde{T}_{\tilde{B}}(x_i), \tilde{I}_{\tilde{B}}(x_i), \tilde{F}_{\tilde{B}}(x_i)\right\rangle$ is undefined and then its value assumed to be zero.

5 Hybrid vector similarity measure based multi-attribute decision making under neutrosophic environment

In the following subsection, we apply the weighted hybrid vector similarity measure to multi attribute decision making under neutrosophic environment.

5.1 Multi-attribute decision making with single valued neutrosophic information

Consider a MADM problem of m alternatives and n attributes, where all the attribute values are characterized by single valued neutrosophic sets. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a finite set of alternatives, $C =$

$\{C_1, C_2, \ldots, C_n\}$ be the set of attributes and $W=(w_1, w_2, \ldots, w_n)^T$ be the weight vector of the attributes $C_j (j = 1, 2, \ldots, n)$ such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Let $D = (d_{ij})_{m\times n}$ be the decision matrix in which the rating values of the alternatives $A_i (i = 1, 2, \ldots, m)$ over the attributes $C_j (j = 1, 2, \ldots, n)$ are presented with the single valued neutrosophic element of the form $d_{ij} = \langle T_{ij}, I_{ij}, F_{ij}\rangle$. In this decision matrix, T_{ij} indicates the degree of membership that the alternative A_i satisfies the attribute C_j, I_{ij} indicates the degree of indeterminacy for the alternative A_i with respect to attribute C_j and F_{ij} indicates the degree of non-membership for the alternative A_i with respect to the attribute C_j such that

$$T_{ij} \in [0, 1], I_{ij} \in [0, 1], F_{ij} \in [0, 1], 0 \leq T_{ij} + I_{ij} + F_{ij} \leq 3$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Assume that the characteristic of the alternative $A_i (i = 1, 2, \ldots, m)$ are represented by SVNSs that are shown in the following pattern:

$$\begin{aligned} A_i &= (d_{i1}, d_{i2}, \cdots, d_{in}), \text{ for } i = 1, 2, \ldots, m; \\ &= \{\langle T_{i1}, I_{i1}, F_{i1}\rangle, \langle T_{i2}, I_{i2}, F_{i2}\rangle, \cdots, \langle T_{in}, I_{in}, F_{in}\rangle\}. \end{aligned} \tag{22}$$

Step 1. Determination of the SVNS based relative positive ideal solution

In multi-attribute decision-making environment, the concept of ideal point is used to identify the best alternative properly in the decision set.

Definition 17. Let H be the collection of two types of attribute namely benefit type attribute (P) and cost type attribute (L) in the MADM problems. The relative positive ideal neutrosophic solution (RPINS) $A^* = (d_1^*, d_2^*, \cdots, d_n^*)$ is the solution of decision matrix $D = (d_{ij})_{m\times n}$ where, every component of has the following form:

1. $d_j^* = \left\langle T_j^*, I_j^*, F_j^*\right\rangle = \left\langle \max_i\{T_{ij}\}, \min_i\{I_{ij}\}, \min_i\{F_{ij}\}\right\rangle$ for benefit type attribute(P) and (23)
2. $d_j^* = \left\langle T_j^*, I_j^*, F_j^*\right\rangle = \left\langle \min_i\{T_{ij}\}, \max_i\{I_{ij}\}, \max_i\{F_{ij}\}\right\rangle$ for cost type attribute(L). (24)

Step 2. Calculation of WHVSM between the ideal alternative and each alternative

According to the Eq.(17), the WHVSM between the ideal alternative A^* and the alternative $A_i (i = 1, 2, \ldots, m)$ is

$$Hyb_w(A^*, A_i) = \left[\begin{array}{l} \lambda \sum_{j=1}^{n} w_j \dfrac{2\left(T_j^* T_{ij} + I_j^* I_{ij} + F_j^* F_{ij}\right)}{\left[\left((T_j^*)^2 + (I_j^*)^2 + (F_j^*)^2\right) + \left((T_{ij})^2 + (I_{ij})^2 + (F_{ij})^2\right)\right]} \\ + (1-\lambda) \sum_{j=1}^{n} w_j \dfrac{\left(T_j^* T_{ij} + I_j^* I_{ij} + F_j^* F_{ij}\right)}{\left[\sqrt{\left((T_j^*)^2 + (I_j^*)^2 + (F_j^*)^2\right)}.\sqrt{\left((T_{ij})^2 + (I_{ij})^2 + (F_{ij})^2\right)}\right]} \end{array} \right], \tag{25}$$

where, RPINS A^* is determined according to the nature of benefit type and cost type attributes defined in Eqs. (23) and (24).

Step 3. Ranking of the alternatives

According to the values obtained from Eq.(25), the ranking order of all the alternatives can be easily determined. Ranking of alternatives is done according to the decreasing order WHVSM.

5.2 Multi-attribute decision making with interval neutrosophic information

Similar to SVNSs, consider $D = (\tilde{d}_{ij})_{m\times n}$ be an interval neutrosophic decision matrix, where all the attribute values are represented by INSs $\tilde{d}_{ij} = \left\langle \tilde{T}_{ij}, \tilde{I}_{ij}, \tilde{F}_{ij} \right\rangle$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Assume that the membership degree $\tilde{T}_{ij}$ indicates that the alternative A_i satisfies the attribute C_j, $\tilde{I}_{ij}$ indicates the degree of indeterminacy for the alternative A_i with respect to attribute C_j, and the membership degree $\tilde{F}_{ij}$ indicates that the alternative A_i does not satisfy the attribute C_j. Let $\tilde{T}_{ij}=\left[T_{ij}^L, T_{ij}^U\right]$, $\tilde{I}_{ij}=\left[I_{ij}^L, I_{ij}^U\right]$, and $\tilde{F}_{ij}=\left[F_{ij}^L, F_{ij}^U\right]$ be the representation of INSs such that

$$\left[T_{ij}^L, T_{ij}^U\right] \subseteq [0,1], \left[I_{ij}^L, I_{ij}^U\right] \subseteq [0,1], \left[F_{ij}^L, F_{ij}^U\right] \subseteq [0,1], \text{ and } 0 \leq T_{ij}^U + I_{ij}^U + F_{ij}^U \leq 3$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Similar to SVNSs, Assume that the characteristic of the alternative $A_i (i = 1, 2, \ldots, m)$ are presented by INSs shown as:

$$\begin{aligned} A_i &= (\tilde{d}_{i1}, \tilde{d}_{i2}, \cdots, \tilde{d}_{in}), \text{ for } i = 1, 2, \ldots, m; \\ &= \left\{ \left\langle \tilde{T}_{i1}, \tilde{I}_{i1}, \tilde{F}_{i1} \right\rangle, \left\langle \tilde{T}_{i2}, \tilde{I}_{i2}, \tilde{F}_{i2} \right\rangle, \cdots, \left\langle \tilde{T}_{in}, \tilde{I}_{in}, \tilde{F}_{in} \right\rangle \right\}. \end{aligned} \tag{26}$$

Step 1. Determination of the INS based relative positive ideal solution

Definition 18. Let H be the collection of two types of attributes namely benefit type attribute (P) and cost type attribute (L) in the INS based MADM problems. The relative positive ideal interval valued neutrosophic solution (RPIINS) $A^*=(\tilde{d}_1^*, \tilde{d}_2^*, \cdots, \tilde{d}_n^*)$ is the solution of decision matrix $D=(\tilde{d}_{ij})_{m\times n}$ where, every component has the following form:

1. The RPIINS of the benefit type attribute C_j is defined by $\tilde{d}_j^* = \left\langle \tilde{T}_j^*, \tilde{I}_j^*, \tilde{F}_j^* \right\rangle$ where,

$$\left\langle \tilde{T}_j^*, \tilde{I}_j^*, \tilde{F}_j^* \right\rangle = \langle [\max_i\{T_{ij}^L\}, \max_i\{T_{ij}^U\}], [\min_i\{I_{ij}^L\}, \min_i\{I_{ij}^U\}], [\min_i\{F_{ij}^L\}, \min_i\{F_{ij}^U\}] \rangle \text{ for } j \in P. \tag{27}$$

2. The RPIINS of the cost type attribute C_j is defined by $\tilde{d}_j^* = \left\langle \tilde{T}_j^*, \tilde{I}_j^*, \tilde{F}_j^* \right\rangle$ where,

$$\left\langle \tilde{T}_j^*, \tilde{I}_j^*, \tilde{F}_j^* \right\rangle = \langle [\min_i\{T_{ij}^L\}, \min_i\{T_{ij}^U\}], [\max_i\{I_{ij}^L\}, \max_i\{I_{ij}^U\}], [\max_i\{F_{ij}^L\}, \max_i\{F_{ij}^U\}] \rangle \text{ for } j \in L. \tag{28}$$

Step 2. Calculation of WHVSM between the ideal alternative and each alternative

According to the Eq. (20), the WHVSM between ideal alternative A^* and alternative $A_i (i = 1, 2, \ldots, m)$ is

$$H_w(A^*, A_i) = \left[\begin{array}{l} \lambda \sum_{j=1}^{n} w_j \dfrac{2\left(\Delta\tilde{T}_j^* \Delta\tilde{T}_{ij} + \Delta\tilde{I}_j^* \Delta\tilde{I}_{ij} + \Delta\tilde{F}_j^* \Delta\tilde{F}_{ij}\right)}{\left[\left((\Delta\tilde{T}_j^*)^2 + (\Delta\tilde{I}_j^*)^2 + (\Delta\tilde{F}_j^*)^2\right) + \left((\Delta\tilde{T}_{ij})^2 + (\Delta\tilde{I}_{ij})^2 + (\Delta\tilde{F}_{ij})^2\right)\right]} \\ + (1-\lambda) \sum_{j=1}^{n} w_j \dfrac{\left(\Delta\tilde{T}_j^* \Delta\tilde{T}_{ij} + \Delta\tilde{I}_j^* \Delta\tilde{I}_{ij} + \Delta\tilde{F}_j^* \Delta\tilde{F}_{ij}\right)}{\left[\sqrt{(\Delta\tilde{T}_j^*)^2 + (\Delta\tilde{I}_j^*)^2 + (\Delta\tilde{F}_j^*)^2} . \sqrt{(\Delta\tilde{T}_{ij})^2 + (\Delta\tilde{I}_{ij})^2 + (\Delta\tilde{F}_{ij})^2}\right]} \end{array} \right], \quad (29)$$

where RPIINS A^* is determined according to benefit type and cost type attributes defined in Eqs. (27) and (28).

Step 3. Ranking the alternatives

According to the values obtained from Eq. (29), the ranking order of all the alternatives can be easily determined based on the decreasing order of WHVSM.

6 Illustrative examples

In this section, two MADM related examples in neutrosophic environment are provided to demonstrate the applicability and effectiveness of the proposed approach.

6.1 Example 1

Consider a decision-making problem [11], in which an investment company wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; and (4) A_4 is an arms company. The investment company must take a decision based on the following three criteria: (1) C_1 is the risk analysis; (2) C_2 is the growth analysis; and (3) C_3 is the environmental impact analysis. The four possible alternatives are to be evaluated under the criteria/attributes by the SVNS assessments provided by the decision maker. These assessment values are provided by the following SVNSs based decision matrix $D=(d_{ij})_{4\times3}$ shown in Table 1.

Table 1: Single valued neutrosophic set based decision matrix

	C_1	C_2	C_3
A_1	$\langle 0.4, 0.2, 0.3 \rangle$	$\langle 0.4, 0.2, 0.3 \rangle$	$\langle 0.2, 0.2, 0.5 \rangle$
A_2	$\langle 0.6, 0.1, 0.2 \rangle$	$\langle 0.6, 0.1, 0.2 \rangle$	$\langle 0.5, 0.2, 0.2 \rangle$
A_3	$\langle 0.3, 0.2, 0.3 \rangle$	$\langle 0.5, 0.2, 0.3 \rangle$	$\langle 0.5, 0.3, 0.2 \rangle$
A_4	$\langle 0.7, 0.0, 0.1 \rangle$	$\langle 0.6, 0.1, 0.2 \rangle$	$\langle 0.4, 0.3, 0.2 \rangle$

The known weight information is given as

$$W = \{w_1, w_2, w_3\}^T = \{0.35, 0.25, 0.40\}^T \text{ such that } \sum_{j=1}^{3} w_j = 1. \tag{30}$$

Step 1. Determination of the Type of attribute

The first two attributes i.e. C_1 and C_2 are here considered as the benefit type attribute and C_3 is considered as the cost type attribute.

Step 2. Determination of the relative neutrosophic positive ideal solution

From Eq. (27)and Eq. (28), the relative positive ideal neutrosophic solution for the given matrix D=$(d_{ij})_{4\times3}$ shown in Table 1 can be obtained as

$$A^* = [\langle 0.7, 0.0, 0.1\rangle, \langle 0.6, 0.1, 0.2\rangle, \langle 0.2, 0.3, 0.5\rangle]. \tag{31}$$

Step 3. Determination of the weighted hybrid vector similarity measure

The weighted hybrid vector similarity measure is determined by using Eq. (25), Eq. (30) and Eq. (31) and the results obtained for different values of λ are shown in the Table 2.

Table 2: Results of SVNS based WHVSM for different values of λ

Similarity measure	Values	Measure Value	Ranking order
$Hyb_w(A^*, A_i)$	$\lambda = 0.1$	$Hyb_w(A^*, A_1) = 0.9036$ $Hyb_w(A^*, A_2) = 0.9019$ $Hyb_w(A^*, A_3) = 0.7912$ $Hyb_w(A^*, A_4) = 0.9433$	$A_4 \succ A_1 \succ A_2 \succ A_3$
$Hyb_w(A^*, A_i)$	$\lambda = 0.25$	$Hyb_w(A^*, A_1) = 0.9014$ $Hyb_w(A^*, A_2) = 0.9015$ $Hyb_w(A^*, A_3) = 0.7942$ $Hyb_w(A^*, A_4) = 0.9429$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Hyb_w(A^*, A_i)$	$\lambda = 0.50$	$Hyb_w(A^*, A_1) = 0.8978$ $Hyb_w(A^*, A_2) = 0.9010$ $Hyb_w(A^*, A_3) = 0.7892$ $Hyb_w(A^*, A_4) = 0.9421$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Hyb_w(A^*, A_i)$	$\lambda = 0.75$	$Hyb_w(A^*, A_1) = 0.8941$ $Hyb_w(A^*, A_2) = 0.9003$ $Hyb_w(A^*, A_3) = 0.7841$ $Hyb_w(A^*, A_4) = 0.9413$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Hyb_w(A^*, A_i)$	$\lambda = 0.90$	$Hyb_w(A^*, A_1) = 0.8919$ $Hyb_w(A^*, A_2) = 0.8999$ $Hyb_w(A^*, A_3) = 0.7811$ $Hyb_w(A^*, A_4) = 0.9409$	$A_4 \succ A_2 \succ A_1 \succ A_3$

Step 4. Ranking the alternatives

According to the different values of λ , the results presented in the Table 2, reflect that A_4 is the best alternative.

6.2 Example 2

Consider the same decision making problem described in Example 1. Here, we consider that the evaluations of the alternatives $\mathrm{A}_i (i = 1, 2, 3, 4)$ over the attributes $C_j (j = 1, 2, 3)$ are expressed in terms of the interval neutrosophic sets. These evaluations are provided in the decision matrix $D = (\tilde{d}_{ij})_{4\times 3}$ shown in Table 3.

Table 3: Interval valued neutrosophic set based decision matrix

	C_1	C_2	C_3
A_1	$\langle[0.4, 0.5], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.4, 0.6], [0.1, 0.3], [0.2, 0.4]\rangle$	$\langle[0.7, 0.9], [0.2, 0.3], [0.4, 0.5]\rangle$
A_2	$\langle[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]\rangle$	$\langle[0.3, 0.6], [0.3, 0.5], [0.8, 0.9]\rangle$
A_3	$\langle[0.3, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.4, 0.5], [0.2, 0.4], [0.7, 0.9]\rangle$
A_4	$\langle[0.7, 0.8], [0.0, 0.1], [0.1, 0.2]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2], [0.1, 0.3]\rangle$	$\langle[0.6, 0.7], [0.3, 0.4], [0.8, 0.9]\rangle$

The weight information of the attributes is considered same as defined in Example 1.

Step 1. Determination of the relative neutrosophic positive ideal solution

Considering C_1 and C_2 as the benefit type attributes and C_3 as the cost type attribute, we determine the relative positive ideal neutrosophic solution by the Eqs.(27) and (28) as:

$$A^* = \{\langle[0.7, 0.8], [0.0, 0.1], [0.1, 0.2]\rangle, \langle[0.6, 0.7], [0.1, 0.2], [0.1, 0.3]\rangle, \langle[0.3, 0.5], [0.3, 0.5], [0.8, 0.9]\rangle\}. \quad (32)$$

Step 2. Determination of the weighted hybrid vector similarity measure

By using Eqs.(29), (30), and (32), we can determine the WHVSM $H_w(A^*, A_i)$ between ideal alternative A^* and each alternative for different values of λ. Table 4 shows the result.

Table 4: Results of INS based HVSM for different values of λ

Similarity measure	Values	Measure Value	Ranking order
$H_w(A^*, A_i)$	$\lambda = 0.1$	$H_w(A^*, A_1) = 0.84293$ $H_w(A^*, A_2) = 0.99020$ $H_w(A^*, A_3) = 0.93000$ $H_w(A^*, A_4) = 0.99041$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$H_w(A^*, A_i)$	$\lambda = 0.25$	$H_w(A^*, A_1) = 0.84553$ $H_w(A^*, A_2) = 0.99005$ $H_w(A^*, A_3) = 0.92730$ $H_w(A^*, A_4) = 0.99013$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$H_w(A^*, A_i)$	$\lambda = 0.50$	$H_w(A^*, A_1) = 0.84985$ $H_w(A^*, A_2) = 0.98980$ $H_w(A^*, A_3) = 0.92280$ $H_w(A^*, A_4) = 0.98965$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$H_w(A^*, A_i)$	$\lambda = 0.75$	$H_w(A^*, A_1) = 0.85417$ $H_w(A^*, A_2) = 0.98955$ $H_w(A^*, A_3) = 0.91830$ $H_w(A^*, A_4) = 0.98917$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$H_w(A^*, A_i)$	$\lambda = 0.90$	$H_w(A^*, A_1) = 0.85677$ $H_w(A^*, A_2) = 0.98940$ $H_w(A^*, A_3) = 0.91560$ $H_w(A^*, A_4) = 0.98889$	$A_2 \succ A_4 \succ A_3 \succ A_1$

Step 3. Ranking the alternatives

According to the different values of λ , the results presented in the Table 4 reflects that A_4 is the best alternative.

6.3 Comparison of hybrid vector similarity measure method with other existing methods for MADM

In this section, we first compare the results of hybrid vector similarity measure with other existing similarity measures for MADM problem. The comparison results according to the Example 1 are presented in Table 5. Similarly, the comparison results for the Example 2 are presented in Table 6. Table 5 shows that our

Table 5: Comparison of HVSM for SVNSs with different similarity measures

Similarity Measure Method	Measure value	Ranking order
$Jac_w(A^*, A_i)$ [42]	$Jac_w(A^*, A_1) = 0.8975$ $Jac_w(A^*, A_2) = 0.8979$ $Jac_w(A^*, A_3) = 0.7689$ $Jac_w(A^*, A_4) = 0.9281$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Dic_w(A^*, A_i)$ [42]	$Dic_w(A^*, A_1) = 0.8975$ $Dic_w(A^*, A_2) = 0.8979$ $Dic_w(A^*, A_3) = 0.7689$ $Dic_w(A^*, A_4) = 0.9281$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Cos_w(A^*, A_i)$ [42]	$Cos_w(A^*, A_1) = 0.8975$ $Cos_w(A^*, A_2) = 0.8979$ $Cos_w(A^*, A_3) = 0.7689$ $Cos_w(A^*, A_4) = 0.9281$	$A_4 \succ A_1 \succ A_2 \succ A_3$
Improved cosine similarity measure $WSC_2(A^*, A_i)$ [44]	$WSC_2(A^*, A_1) = 0.9691$ $WSC_2(A^*, A_2) = 0.9761$ $WSC_2(A^*, A_3) = 0.9401$ $WSC_2(A^*, A_4) = 0.9804$	$A_4 \succ A_2 \succ A_1 \succ A_3$

result for the selection of best alternative agree with Ye's vector similarity measure method [42] as well as improved cosine similarity measure method [44] for SVNSs. We see from Table 6 that the selection for the best alternative according to our proposed method, the result is same as [42, 43, 44] for INSs. Finally, we compare the proposed method with other existing methods [39, 31, 41, 24] and present the results in Table 7. We also observe that the ranking order of the four alternatives for the Example 1 and Example 2 are same as the results given in Table 7.

7 Conclusions

In this paper, we have proposed hybrid vector similarity measures and weighted hybrid vector similarity measures for both single valued and interval neutrosophic sets and proved some of their basic properties. Then, we have compared the proposed similarity measures with the existing similarity measures for MADM problems. Two numerical examples, one for SVNSs and another for INSs have been provided to check the

Table 6: Comparison of HVSM for INSs with existing similarity measures

Similarity Measure Method	Measure value	Ranking order
$Jac_w(A^*, A_i)$ [42]	$Jac_w(A^*, A_1) = 0.7579$ $Jac_w(A^*, A_2) = 0.9773$ $Jac_w(A^*, A_3) = 0.8646$ $Jac_w(A^*, A_4) = 0.9768$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$Dic_w(A^*, A_i)$ [42]	$Dic_w(A^*, A_1) = 0.8594$ $Dic_w(A^*, A_2) = 0.9884$ $Dic_w(A^*, A_3) = 0.9224$ $Dic_w(A^*, A_4) = 0.9880$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$Dic_w(A^*, A_i)$ Proposed	$Dic_w(A^*, A_1) = 0.8585$ $Dic_w(A^*, A_2) = 0.9893$ $Dic_w(A^*, A_3) = 0.9138$ $Dic_w(A^*, A_4) = 0.9887$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$Cos_w(A^*, A_i)$ [42]	$Cos_w(A^*, A_1) = 0.8676$ $Cos_w(A^*, A_2) = 0.9894$ $Cos_w(A^*, A_3) = 0.9276$ $Cos_w(A^*, A_4) = 0.9896$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$Cos_w(A^*, A_i)$ [43]	$Cos_w(A^*, A_1) = 0.8412$ $Cos_w(A^*, A_2) = 0.9903$ $Cos_w(A^*, A_3) = 0.9318$ $Cos_w(A^*, A_4) = 0.9906$	$A_4 \succ A_2 \succ A_3 \succ A_1$
Improved cosine similarity measure $WSC_2(A^*, A_i)$ [44]	$WSC_2(A^*, A_1) = 0.9252$ $WSC_2(A^*, A_2) = 0.9955$ $WSC_2(A^*, A_3) = 0.9704$ $WSC_2(A^*, A_4) = 0.9951$	$A_2 \succ A_4 \succ A_3 \succ A_1$

Table 7: Comparison of HVSM method with other existing methods

Different methods for MADM	Types of sets	Ranking order
Improved correlation coefficient [39]	SVNSs INSs	$A_2 \succ A_4 \succ A_3 \succ A_1$ $A_2 \succ A_4 \succ A_3 \succ A_1$
Subset-hood measure method [31]	SVNSs	$A_2 \succ A_4 \succ A_3 \succ A_1$
Hamming distance measure Euclidean distance measure [41]	INSs	$A_4 \succ A_2 \succ A_3 \succ A_1$ $A_2 \succ A_4 \succ A_3 \succ A_1$
Liu's TOPSIS method [24]	INSs	$A_4 \succ A_2 \succ A_3 \succ A_1$

validity and effectiveness of the proposed approach in MADM problem. However, we hope that the proposed hybrid vector similarity measures for single valued as well as interval neutrosophic sets can be used in the field of practical decision making, medical diagnosis, pattern recognition, data mining, clustering analysis, etc.

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Table 1: Single valued neutrosophic set based decision matrix

	C_1	C_2	C_3
A_1	$\langle 0.4, 0.2, 0.3 \rangle$	$\langle 0.4, 0.2, 0.3 \rangle$	$\langle 0.2, 0.2, 0.5 \rangle$
A_2	$\langle 0.6, 0.1, 0.2 \rangle$	$\langle 0.6, 0.1, 0.2 \rangle$	$\langle 0.5, 0.2, 0.2 \rangle$
A_3	$\langle 0.3, 0.2, 0.3 \rangle$	$\langle 0.5, 0.2, 0.3 \rangle$	$\langle 0.5, 0.3, 0.2 \rangle$
A_4	$\langle 0.7, 0.0, 0.1 \rangle$	$\langle 0.6, 0.1, 0.2 \rangle$	$\langle 0.4, 0.3, 0.2 \rangle$

Table 2: Results of SVNS based WHVSM for different values of λ

Similarity measure	Values	Measure Value	Ranking order
$Hyb_w(A^*, A_i)$	$\lambda = 0.1$	$Hyb_w(A^*, A_1) = 0.9036$ $Hyb_w(A^*, A_2) = 0.9019$ $Hyb_w(A^*, A_3) = 0.7912$ $Hyb_w(A^*, A_4) = 0.9433$	$A_4 \succ A_1 \succ A_2 \succ A_3$
$Hyb_w(A^*, A_i)$	$\lambda = 0.25$	$Hyb_w(A^*, A_1) = 0.9014$ $Hyb_w(A^*, A_2) = 0.9015$ $Hyb_w(A^*, A_3) = 0.7942$ $Hyb_w(A^*, A_4) = 0.9429$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Hyb_w(A^*, A_i)$	$\lambda = 0.50$	$Hyb_w(A^*, A_1) = 0.8978$ $Hyb_w(A^*, A_2) = 0.9010$ $Hyb_w(A^*, A_3) = 0.7892$ $Hyb_w(A^*, A_4) = 0.9421$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Hyb_w(A^*, A_i)$	$\lambda = 0.75$	$Hyb_w(A^*, A_1) = 0.8941$ $Hyb_w(A^*, A_2) = 0.9003$ $Hyb_w(A^*, A_3) = 0.7841$ $Hyb_w(A^*, A_4) = 0.9413$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Hyb_w(A^*, A_i)$	$\lambda = 0.90$	$Hyb_w(A^*, A_1) = 0.8919$ $Hyb_w(A^*, A_2) = 0.8999$ $Hyb_w(A^*, A_3) = 0.7811$ $Hyb_w(A^*, A_4) = 0.9409$	$A_4 \succ A_2 \succ A_1 \succ A_3$

Table 3: Interval neutrosophic information based decision matrix

	C_1	C_2	C_3
A_1	$\langle[0.4, 0.5], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.4, 0.6], [0.1, 0.3], [0.2, 0.4]\rangle$	$\langle[0.7, 0.9], [0.2, 0.3], [0.4, 0.5]\rangle$
A_2	$\langle[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]\rangle$	$\langle[0.3, 0.6], [0.3, 0.5], [0.8, 0.9]\rangle$
A_3	$\langle[0.3, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.4, 0.5], [0.2, 0.4], [0.7, 0.9]\rangle$
A_4	$\langle[0.7, 0.8], [0.0, 0.1], [0.1, 0.2]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2], [0.1, 0.3]\rangle$	$\langle[0.6, 0.7], [0.3, 0.4], [0.8, 0.9]\rangle$

Table 4: Results of INS based HVSM for different values of λ

Similarity measure	Values	Measure Value	Ranking order
$H_w(A^*, A_i)$	$\lambda = 0.1$	$H_w(A^*, A_1) = 0.84293$ $H_w(A^*, A_2) = 0.99020$ $H_w(A^*, A_3) = 0.93000$ $H_w(A^*, A_4) = 0.99041$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$H_w(A^*, A_i)$	$\lambda = 0.25$	$H_w(A^*, A_1) = 0.84553$ $H_w(A^*, A_2) = 0.99005$ $H_w(A^*, A_3) = 0.92730$ $H_w(A^*, A_4) = 0.99013$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$H_w(A^*, A_i)$	$\lambda = 0.50$	$H_w(A^*, A_1) = 0.84985$ $H_w(A^*, A_2) = 0.98980$ $H_w(A^*, A_3) = 0.92280$ $H_w(A^*, A_4) = 0.98965$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$H_w(A^*, A_i)$	$\lambda = 0.75$	$H_w(A^*, A_1) = 0.85417$ $H_w(A^*, A_2) = 0.98955$ $H_w(A^*, A_3) = 0.91830$ $H_w(A^*, A_4) = 0.98917$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$H_w(A^*, A_i)$	$\lambda = 0.90$	$H_w(A^*, A_1) = 0.85677$ $H_w(A^*, A_2) = 0.98940$ $H_w(A^*, A_3) = 0.91560$ $H_w(A^*, A_4) = 0.98889$	$A_2 \succ A_4 \succ A_3 \succ A_1$

Table 5: Comparison of HVSM for SVNSs with different similarity measures

Similarity Measure Method	Measure value	Ranking order
$Jac_w(A^*, A_i)$	$Jac_w(A^*, A_1) = 0.8975$ $Jac_w(A^*, A_2) = 0.8979$ $Jac_w(A^*, A_3) = 0.7689$ $Jac_w(A^*, A_4) = 0.9281$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Dic_w(A^*, A_i)$	$Dic_w(A^*, A_1) = 0.8975$ $Dic_w(A^*, A_2) = 0.8979$ $Dic_w(A^*, A_3) = 0.7689$ $Dic_w(A^*, A_4) = 0.9281$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$Cos_w(A^*, A_i)$	$Cos_w(A^*, A_1) = 0.8975$ $Cos_w(A^*, A_2) = 0.8979$ $Cos_w(A^*, A_3) = 0.7689$ $Cos_w(A^*, A_4) = 0.9281$	$A_4 \succ A_1 \succ A_2 \succ A_3$
Improved cosine similarity measure $WSC_2(A^*, A_i)$	$WSC_2(A^*, A_1) = 0.9691$ $WSC_2(A^*, A_2) = 0.9761$ $WSC_2(A^*, A_3) = 0.9401$ $WSC_2(A^*, A_4) = 0.9804$	$A_4 \succ A_2 \succ A_1 \succ A_3$

Table 6: Comparison of HVSM for INSs different existing similarity measures

Similarity Measure Method	Measure value	Ranking order
$Jac_w(A^*, A_i)$	$Jac_w(A^*, A_1) = 0.7579$ $Jac_w(A^*, A_2) = 0.9773$ $Jac_w(A^*, A_3) = 0.8646$ $Jac_w(A^*, A_4) = 0.9768$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$Dic_w(A^*, A_i)$	$Dic_w(A^*, A_1) = 0.8594$ $Dic_w(A^*, A_2) = 0.9884$ $Dic_w(A^*, A_3) = 0.9224$ $Dic_w(A^*, A_4) = 0.9880$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$Dic_w(A^*, A_i)$ Proposed	$Dic_w(A^*, A_1) = 0.8585$ $Dic_w(A^*, A_2) = 0.9893$ $Dic_w(A^*, A_3) = 0.9138$ $Dic_w(A^*, A_4) = 0.9887$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$Cos_w(A^*, A_i)$	$Cos_w(A^*, A_1) = 0.8676$ $Cos_w(A^*, A_2) = 0.9894$ $Cos_w(A^*, A_3) = 0.9276$ $Cos_w(A^*, A_4) = 0.9896$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$Cos_w(A^*, A_i)$	$Cos_w(A^*, A_1) = 0.8412$ $Cos_w(A^*, A_2) = 0.9903$ $Cos_w(A^*, A_3) = 0.9318$ $Cos_w(A^*, A_4) = 0.9906$	$A_4 \succ A_2 \succ A_3 \succ A_1$
Improved cosine similarity measure $WSC_2(A^*, A_i)$	$WSC_2(A^*, A_1) = 0.9252$ $WSC_2(A^*, A_2) = 0.9955$ $WSC_2(A^*, A_3) = 0.9704$ $WSC_2(A^*, A_4) = 0.9951$	$A_2 \succ A_4 \succ A_3 \succ A_1$

Table 7: Comparison of HVSM method with other existing methods

Different methods for MADM	Types of sets	Ranking order
Improved correlation coefficient	SVNSs	$A_2 \succ A_4 \succ A_3 \succ A_1$
	INSs	$A_2 \succ A_4 \succ A_3 \succ A_1$
Subset-hood measure method	SVNSs	$A_2 \succ A_4 \succ A_3 \succ A_1$
Hamming distance measure	INSs	$A_4 \succ A_2 \succ A_3 \succ A_1$
Euclidean distance measure		$A_2 \succ A_4 \succ A_3 \succ A_1$
Liu's TOPSIS method	INSs	$A_4 \succ A_2 \succ A_3 \succ A_1$