Quadratic Model To Solve Equations Of Degree *n* In One Variable

This paper introduces a numeric method to solving equations of degree n in one variable. Because the method is based on a quadratic model, provides, in general, more accuracy than the similar linear model invented by Isaac Newton.

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1. Nomenclature

$y = f(x) =$ polynomial of degree n in one variable
$n =$ degree of the polynomial
$m = m(x) =$ model function (also known as modelling polynomial)
$a =$ coefficient of the model function (real number)
$b =$ coefficient of the model function (real number)
$\alpha =$ approximate solution (real number)
$\alpha_1 =$ approximate solution yielded by the first iteration
$\alpha_2 =$ approximate solution yielded by the second iteration
$\alpha_3 =$ approximate solution yielded by the third iteration
$\alpha_n =$ approximate solution yielded by iteration number n
$P_1 =$ first starting point $P_1(x_1, y_1)$
$x_1 =$ abscissa of P_1
$y_1 =$ ordinate of P_1
$P_2 =$ second starting point $P_2(x_2, y_2)$
$x_2 =$ abscissa of P_2
$y_2 =$ ordinate of P_2

2. The Method

The problem many people encounter many times is to find the values of x that satisfy the equation

Equation to solve $$f(x) = 0 \tag{2.1}$$

Where $f(x)$ is a polynomial of degree n in one variable. The method presented here aims to solving this problem with a relatively high degree of accuracy. The method is based on a model given by the following second degree equation

Model $$m(x) = a x^2 + b x \tag{2.2}$$

Where *a* and *b* are two real numbers (coefficients of the model). Knowing that a particular solution to equation (2.1) falls between two values of the abscissa, let's say between x_1 , and x_2 , we may use the above model to find the solutions to equation (2.1) through a series of iterations (one series of iterations for each solution). The method may be summarized in 5 easy theoretical steps. You don't need to go through these steps every time you want to find a solution to a given equation. The steps are the theoretical framework of this model and the applied method is easier than it seems as the example of the next section shows. The theoretical steps of the method are:

Step 1) Find the values of *f*(*x*) for the two abscissas.

$$y_1 = f(x_1) \tag{2.3}$$
$$y_2 = f(x_2) \tag{2.4}$$

Thus we have two starting points

$$P_1 = (x_{1,} y_1) \tag{2.5}$$
$$P_2 = (x_{2,} y_2) \tag{2.6}$$

Step 2) Fit the model to the two points given in the previous step

$$y_1 = a x_1^2 + b x_1 \tag{2.7}$$
$$y_2 = a x_2^2 + b x_2 \tag{2.8}$$

This is a system of two equations with two unknowns: *a* and *b*

Step 3) Solve the system of equations developed in the previous step. Since the unknowns are the variables[1] *a* and *b*, it is convenient to write the equations as follows

System of two equations with two unknowns

$$x_1^2 a + x_1 b = f(x_1) \tag{2.9}$$
$$x_2^2 a + x_2 b = f(x_2) \tag{2.10}$$

I shall use the Cramer rule to solve the above system of equations, however, you may use any other method you are comfortable with. According to this rule, the value of each unknown (*a* and *b* in this case) is obtained dividing the determinant for each unknown by the determinant of the system. The determinant for a given unknown is obtained from the determinant of the system substituting the column of the coefficients of the given unknown with the column of the independent terms ($f(x_1)$ and $f(x_2)$). Thus, provided the determinant of the system is not zero, the values of the two unknowns may be computed as follows

(1) The coefficients of equations (2.9) and (2.10) must not be confused with the coefficients *a* and *b* of the model equation (2.2). In the above system of equations, *a* and *b* are not coefficients but unknowns.

$$a = \frac{\Delta_a}{\Delta} \tag{2.11}$$

$$b = \frac{\Delta_b}{\Delta} \tag{2.12}$$

Where the system determinant, Δ , is given by the following expression

Determinant for the system

$$\Delta = \begin{vmatrix} x_1^2 & x_1 \\ x_2^2 & x_2 \end{vmatrix} = x_2 x_1^2 - x_1 x_2^2 \tag{2.13}$$

The determinant for the variable a is denoted by Δ_a and is given by

Determinant for variable *a*

$$\Delta_a = \begin{vmatrix} f(x_1) & x_1 \\ f(x_2) & x_2 \end{vmatrix} \tag{2.14}$$

The determinant for the variable b is denoted by Δ_b and is given by

Determinant for variable *b*

$$\Delta_b = \begin{vmatrix} x_1^2 & f(x_1) \\ x_2^2 & f(x_2) \end{vmatrix} \tag{2.15}$$

Therefore we may write

$$a = \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 x_1^2 - x_1 x_2^2} \tag{2.16}$$

$$b = \frac{x_1^2 f(x_2) - x_2^2 f(x_1)}{x_2 x_1^2 - x_1 x_2^2} \tag{2.17}$$

Step 4) Substitute the coefficients a and b [of the model equation (2.2)] with the values given by equations (2.16) and (2.17), respectively. This yields

$$m(x) = \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 x_1^2 - x_1 x_2^2} x^2 + \frac{x_1^2 f(x_2) - x_2^2 f(x_1)}{x_2 x_1^2 - x_1 x_2^2} x \tag{2.18}$$

Step 5) Solve the following equation $m(x) = 0$

$$\frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 x_1^2 - x_1 x_2^2} x^2 + \frac{x_1^2 f(x_2) - x_2^2 f(x_1)}{x_2 x_1^2 - x_1 x_2^2} x = 0 \tag{2.19}$$

The solution is

Approximate solution

$$x = \alpha = \frac{x_2^2 f(x_1) - x_1^2 f(x_2)}{x_2 f(x_1) - x_1 f(x_2)} \tag{2.20}$$

Where I have denoted with the Greek letter α this particular value of x. This is an approximation to one of the solutions to the given equation (2.1). If you are happy with the accuracy provided by this solution, you may stop the process here. However, if you wish to improve the accuracy, you need to perform more iterations. The iteration process is explained next.

Iteration process) If you want to increase the accuracy of the solution, you need to iterate by calculating a new starting point, P_2, (the other starting point, P_1, remains unaltered). Thus, we repeat the process from step 1 onwards, but now the abscissa x_2 of P_2 will be the value of α calculated in the previous step, and the ordinate y_2 of this point will be calculated with the given polynomial, $f(x)$, as follows

$$y_2 = f(\alpha) \tag{2.21}$$

Thus, the coordinates of the new starting point, P_2, will be

$$P_2 = (\alpha, f(\alpha)) \tag{2.22}$$

In fact you don't need to do all the theoretical steps every time. In the next section I shall illustrate the method through a simple example.

3. Example

Let us consider the following polynomial of third degree

$$y = f(x) = x^3 - x - 4 \tag{3.1}$$

In this example, the equation to solve is $f(x)=0$. Thus, we write

$$x^3 - x - 4 = 0 \tag{3.2}$$

Firstly, we find an interval in which $f(x)=0$. We try the following

$$f(1) = 1^3 - 1 - 4 = -4 \tag{3.3}$$

And we also try the following

$$f(2) = 2^3 - 2 - 4 = 2 \tag{3.4}$$

Because the signs of $f(1)$ and $f(2)$ are different, there must be, at least one root, inside the interval

$$1 < x < 2 \tag{3.5}$$

(In general, you might need to try different values in order to find a suitable interval such as this one). Thus, the abscissas of the two starting points P_1 and P_2 are

$$x_1 = 1 \tag{3.6}$$
$$x_2 = 2 \tag{3.7}$$

, respectively. And the values of the given function for these abscissas are

$$y_1 = f(x_1) = x_1^3 - x_1 - 4 = -4 \tag{3.8}$$
$$y_2 = f(x_2) = x_2^3 - x_2 - 4 = 2 \tag{3.9}$$

, respectively.

First iteration

Now, in order to compute the first approximation to the solution, we use equation (2.20), where, in order to differentiate the iterations, is convenient to use α_1 instead of α . Thus we write

$$\alpha_1 = \frac{x_2^2 f(x_1) - x_1^2 f(x_2)}{x_2 f(x_1) - x_1 f(x_2)} \tag{3.10}$$

Here we substitute the variables with the values we have found. This gives

$$\alpha = \frac{2^2(-4) - 1^2 \times 2}{2(-4) - 1 \times 2} = 1.8 \tag{3.11}$$

This is the first approximation to one of the roots of equation (2.1)

Second iteration

In order to perform the second iteration, we need to compute a new starting point, P_2 . This is done as follows

$$f(\alpha_1) = \alpha_1^3 - \alpha_1 - 4 = \frac{4}{125} = 0.032 \tag{3.12}$$

or

$$f(1.8) = 0.032 \tag{3.13}$$

Thus, we have the abscissas of the two points

$$x_1 = 1 \tag{3.14}$$
$$x_2 = 1.8 \tag{3.15}$$

And the corresponding ordinates

$$f(x_1) = -4 \tag{3.16}$$
$$f(x_2) = 0.032 \tag{3.17}$$

Now, in order to compute the second approximation of the solution, we use equation (2.20), where we shall use α_2 to denote the second iteration

$$\alpha_2 = \frac{x_2^2 f(x_1) - x_1^2 f(x_2)}{x_2 f(x_1) - x_1 f(x_2)} \tag{3.18}$$

Here we substitute the variables with the values we have found. This gives

$$\alpha_2 = \frac{1.8^2(-4) - 1^2 \times 0.032}{1.8(-4) - 1 \times 0.032} = \frac{203}{113} = 1.796\ 460\ 177 \tag{3.19}$$

This is the second approximation to one of the solutions of equation (2.1).

Third iteration

In order to perform the third iteration, we need to compute a new starting point, P_2. This is done as follows

$$f(\alpha_2) = \alpha_2^3 - \alpha_2 - 4 = 0.001\ 200\ 362\ 95 \tag{3.20}$$

Thus we have the coordinates for the two starting points

Abscissas

$$x_1 = 1 \tag{3.21}$$
$$x_2 = \alpha_2 = 0.001\ 2009\ 362\ 95 \tag{3.22}$$

and

Ordinates

$$f(x_1) = -4 \tag{3.23}$$
$$f(x_2) = 0.001\ 200\ 362\ 95 \tag{3.24}$$

Now, in order to compute the third approximation, we use equation (3.20), where we shall use α_3 to denote the third iteration

$$\alpha_3 = \frac{x_2^2 f(x_1) - x_1^2 f(x_2)}{x_2 f(x_1) - x_1 f(x_2)} \tag{3.25}$$

Here we substitute the variables with the values we have found. This gives

$$\alpha_3 = \frac{\left(\frac{203}{113}\right)^2(-4) - 1^2 \times 0.00120036295}{\left(\frac{203}{113}\right)(-4) - 1 \times 0.00120036295} = 1.796\,327\,154 \qquad (3.26)$$

This is the third approximation for the solution to equation (2.1).

If you consider that

$$f(\alpha_3) = \alpha_3^3 - \alpha_3 - 4 = 0.000\,045\,578\,24 \qquad (3.27)$$

is close enough to zero, you are done. In other words, if you are happy with the result of any iteration, you don't need to continue with the iteration process.