

Interval Neutrosophic Sets

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Abstract

Neutrosophic set is a part of neutrosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set is a powerful general formal framework that has been recently proposed. However, neutrosophic set needs to be specified from a technical point of view. To this effect, we define the set-theoretic operators on an instance of neutrosophic set, we call it interval neutrosophic set (INS). We prove various properties of INS, which are connected to the operations and relations over INS. Finally, we introduce and prove the convexity of interval neutrosophic sets.

Key words: Neutrosophic set, interval neutrosophic set, set-theoretic operator, convexity

1 Introduction

In this section, we introduce the related works, motivation and the problems that we are facing.

1.1 Related Works and Historical Perspective

The concept of fuzzy sets was introduced by Zadeh in 1965 [7]. Since then fuzzy sets and fuzzy logic have been applied in many real applications to handle uncertainty. The traditional fuzzy set uses one real number $\mu_A(x) \in [0, 1]$ to represent the grade of membership of fuzzy set A defined on universe X . Sometimes $\mu_A(x)$ itself is uncertain and hard to be defined by a crisp value. So

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the concept of interval valued fuzzy sets was proposed [4] to capture the uncertainty of grade of membership. Interval valued fuzzy set uses an interval value $[\mu_A^L(x), \mu_A^U(x)]$ with $0 \leq \mu_A^L(x) \leq \mu_A^U(x) \leq 1$ to represent the grade of membership of fuzzy set A . In some applications such as expert system, belief system and information fusion, we should consider not only the truth-membership supported by the evidence but also the false-membership agansted by the evidence. That is beyond the scope of fuzzy sets and interval valued fuzzy sets. In 1986, Atanassov introduced the intuitionistic fuzzy sets [1] which is a generalization of fuzzy sets and provably equivalent to interval valued fuzzy sets. The intuitionistic fuzzy sets consider both truth-membership and false-membership. Later on, intuitionistic fuzzy sets were extended to the interval valued intuitionistic fuzzy sets [2]. The interval valued intuitionistic fuzzy set uses a pair of intervals $[t^-, t^+]$, $0 \leq t^- \leq t^+ \leq 1$ and $[f^-, f^+]$, $0 \leq f^- \leq f^+ \leq 1$ with $t^+ + f^+ \leq 1$ to describe the degree of true belief and false belief. Because of the restriction that $t^+ + f^+ \leq 1$, intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and inconsistent information which exists commonly in belief systems. For example, when we ask the opinion of an expert about certain statement, he or she may say that the possibility that the statement is true is between 0.5 and 0.7 and the statement is false is between 0.2 and 0.4 and the degree that he or she is not sure is between 0.1 and 0.3. Here is another example, suppose there are 10 votees during a voting process. In time t_1 , three vote "yes", two vote "no" and five are undecided, using neutrosophic notation, it can be expressed as $x(0.3, 0.5, 0.2)$. In time t_2 , three vote "yes", two vote "no", two give up and three are undecided, it then can be expressed as $x(0.3, 0.3, 0.2)$. That is beyond the scope of the intuitionistic fuzzy set. So, the notion of neutrosophic set is more general and overcomes the aforementioned issues.

1.2 Motivation

In neutrosophic set, indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and false-membership are independent. This assumption is very important in information fusion when we try to combine the data from different sensors. Neutrosophy was introduced by Florentin Smarandache in 1980. "It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra" [5]. Neutrosophic set is a powerful general formal framework which generalizes the concept of the classic set, fuzzy set [7], interval valued fuzzy set [4], intuitionistic fuzzy set [1], interval valued intuitionistic fuzzy set [2], paraconsistent set [5], dialetheist set [5], paradoxist set [5], tautological set [5]. A neutrosophic set A defined on universe U . $x = x(T, I, F) \in A$ with T, I and F being the real standard or non-standard subsets of $]0^-, 1^+[$. T is the degree of true membership function in the set A , I is the degree of indeterminate membership function in the set A and F is the degree of false membership function in the set A .

The neutrosophic set generalizes the above mentioned sets from philosophical point of view. From scientific or engineering point of view, the neutrosophic set and set-theoretic operators need to be specified. Otherwise, it will be difficult to apply in the real applications. In this paper, we define the set-theoretic operators on an instance of neutrosophic set called interval neutrosophic set

(INS). We call it as “interval” because it is subclass of neutrosophic set, that is we only consider the subunitary interval of $[0, 1]$.

1.3 Problem Statement

An interval neutrosophic set A defined on universe X , $x = x(T, I, F) \in A$ with T , I and F being the subinterval of $[0, 1]$. Interval neutrosophic set can represent uncertainty, imprecise, incomplete and inconsistent information which exist in real world. The interval neutrosophic set generalizes the following sets:

1. the *classical set*, $I = \emptyset$, $\inf T = \sup T = 0$ or 1 , $\inf F = \sup F = 0$ or 1 and $\sup T + \sup F = 1$.
2. the *fuzzy set*, $I = \emptyset$, $\inf T = \sup T \in [0, 1]$, $\inf F = \sup F \in [0, 1]$ and $\sup T + \sup F = 1$.
3. the *interval valued fuzzy set*, $I = \emptyset$, $\inf T, \sup T, \inf F, \sup F \in [0, 1]$, $\sup T + \inf F = 1$ and $\inf T + \sup F = 1$.
4. the *intuitionistic fuzzy set*, $I = \emptyset$, $\inf T = \sup T \in [0, 1]$, $\inf F = \sup F \in [0, 1]$ and $\sup T + \sup F \leq 1$.
5. the *interval valued intuitionistic fuzzy set*, $I = \emptyset$, $\inf T, \sup T, \inf F, \sup F \in [0, 1]$ and $\sup T + \sup F \leq 1$.
6. the *paraconsistent set*, $I = \emptyset$, $\inf T = \sup T \in [0, 1]$, $\inf F = \sup F \in [0, 1]$ and $\sup T + \sup F > 1$.
7. the *interval valued paraconsistent set*, $I = \emptyset$, $\inf T, \sup T, \inf F, \sup F \in [0, 1]$ and $\inf T + \inf F > 1$.

The relationship among interval neutrosophic set and other sets is illustrated in Fig 1.

Note that \rightarrow in Fig. 1 such as $a \rightarrow b$ means that b is a generalization of a .

We define the set-theoretic operators on interval neutrosophic set (INS). Various properties of INS are proved, which are connected to the operations and relations over INS.

The rest of paper is organized as follows. Section 2 gives a brief overview of neutrosophic set. Section 3 gives the definition of interval neutrosophic set and set-theoretic operations. Section 4 gives some properties of set-theoretic operations. Section 5 gives the definition of convexity of interval neutrosophic sets and prove some properties of convexity. Section 6 concludes the paper. To maintain a smooth flow throughout the paper, we present the proofs to all theorems in Appendix.

2 Neutrosophic Set

This section gives a brief overview of concepts of neutrosophic set defined in [5]. Here, we use different notations to express the same meaning. Let S_1 and S_2 be two real standard or non-standard subsets, then $S_1 \oplus S_2 = \{x|x = s_1 + s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $\{1^+\} \oplus S_2 = \{x|x = 1^+ + s_2, s_2 \in S_2\}$. $S_1 \ominus S_2 = \{x|x = s_1 - s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $\{1^+\} \ominus S_2 = \{x|x = 1^+ - s_2, s_2 \in S_2\}$. $S_1 \odot S_2 = \{x|x = s_1 \cdot s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$.

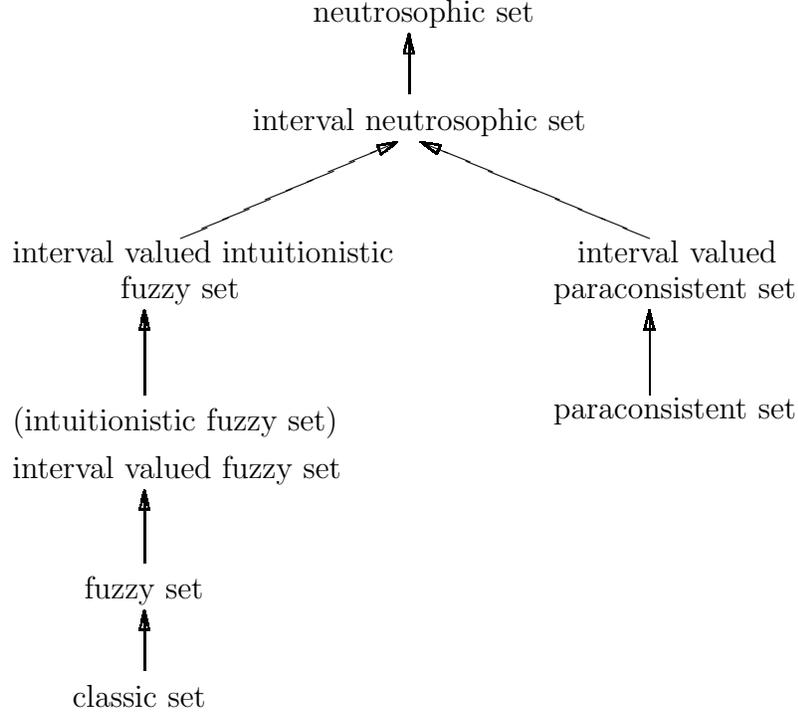


Figure 1: Relationship among interval neutrosophic set and other sets

Definition 1 (Neutrosophic Set) Let X be a space of points (objects), with a generic element in X denoted by x . A neutrosophic set A in X is characterized by a truth-membership function T_A , a indeterminacy-membership function I_A and a false-membership function F_A . $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is

$$T_A : X \rightarrow]0^-, 1^+[, \quad (1)$$

$$I_A : X \rightarrow]0^-, 1^+[, \quad (2)$$

$$F_A : X \rightarrow]0^-, 1^+[. \quad (3)$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2 The complement of a neutrosophic set A is denoted by \bar{A} and is defined by

$$T_{\bar{A}}(x) = \{1^+\} \ominus T_A(x), \quad (4)$$

$$I_{\bar{A}}(x) = \{1^+\} \ominus I_A(x), \quad (5)$$

$$F_{\bar{A}}(x) = \{1^+\} \ominus F_A(x), \quad (6)$$

for all x in X .

Definition 3 (Containment) A neutrosophic set A is contained in the other neutrosophic set B , $A \subseteq B$, if and only if

$$\inf T_A(x) \leq \inf T_B(x) \quad , \quad \sup T_A(x) \leq \sup T_B(x), \quad (7)$$

$$\inf F_A(x) \geq \inf F_B(x) \quad , \quad \sup F_A(x) \geq \sup F_B(x), \quad (8)$$

for all x in X .

Definition 4 (Union) The union of two neutrosophic sets A and B is a neutrosophic set C , written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A and B by

$$T_C(x) = T_A(x) \oplus T_B(x) \ominus T_A(x) \odot T_B(x), \quad (9)$$

$$I_C(x) = I_A(x) \oplus I_B(x) \ominus I_A(x) \odot I_B(x), \quad (10)$$

$$F_C(x) = F_A(x) \oplus F_B(x) \ominus F_A(x) \odot F_B(x), \quad (11)$$

for all x in X .

Definition 5 (Intersection) The intersection of two neutrosophic sets A and B is a neutrosophic set C , written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A and B by

$$T_C(x) = T_A(x) \odot T_B(x), \quad (12)$$

$$I_C(x) = I_A(x) \odot I_B(x), \quad (13)$$

$$F_C(x) = F_A(x) \odot F_B(x), \quad (14)$$

for all x in X .

Definition 6 (Difference) The difference of two neutrosophic sets A and B is a neutrosophic set C , written as $C = A \setminus B$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A and B by

$$T_C(x) = T_A(x) \ominus T_A(x) \odot T_B(x), \quad (15)$$

$$I_C(x) = I_A(x) \ominus I_A(x) \odot I_B(x), \quad (16)$$

$$F_C(x) = F_A(x) \ominus F_A(x) \odot F_B(x), \quad (17)$$

for all x in X .

Definition 7 (Cartesian Product) Let A be the neutrosophic set defined on universe E_1 and B be the neutrosophic set defined on universe E_2 . If $x(T_A^1, I_A^1, F_A^1) \in A$ and $y(T_A^2, I_A^2, F_A^2) \in B$, then the cartesian product of two neutrosophic sets A and B is defined by

$$(x(T_A^1, I_A^1, F_A^1), y(T_A^2, I_A^2, F_A^2)) \in A \times B \quad (18)$$

3 Interval Neutrosophic Set

In this section, we present the notion of *interval neutrosophic set* (INS). Interval neutrosophic set (INS) is an instance of neutrosophic set which can be used in real scientific and engineering applications.

Definition 8 (Interval Neutrosophic Set) *Let X be a space of points (objects), with a generic element in X denoted by x . An interval neutrosophic set (INS) A in X is characterized by truth-membership function T_A , indeterminacy-membership function I_A and false-membership function F_A . For each point x in X , $T_A(x), I_A(x), F_A(x) \subseteq [0, 1]$.*

An interval neutrosophic set (INS) in R^1 is illustrated in Fig. 2.

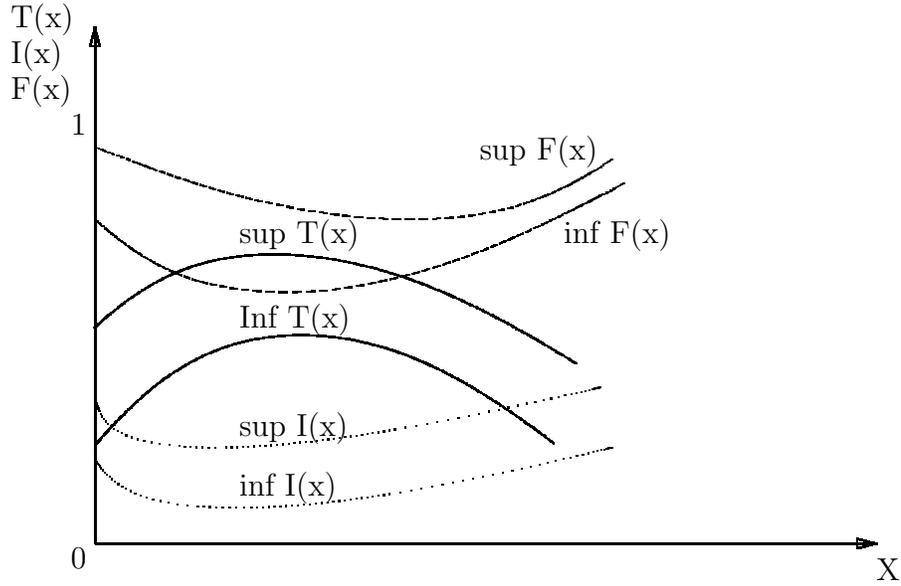


Figure 2: Illustration of interval neutrosophic set in R^1

When X is continuous, an INS A can be written as

$$A = \int_X \langle T(x), I(x), F(x) \rangle / x, \quad x \in X \quad (19)$$

When X is discrete, an INS A can be written as

$$A = \sum_{i=1}^n \langle T(x_i), I(x_i), F(x_i) \rangle / x_i, \quad x_i \in X \quad (20)$$

Consider parameters such as capability, trustworthiness and price of semantic Web services. These parameters are commonly used to define quality of service of semantic Web services. In this section, we will use the evaluation of quality of service of semantic Web services [6] as running example to illustrate every set-theoretic operation on interval neutrosophic set.

Example 1 Assume that $X = [x_1, x_2, x_3]$. x_1 is capability, x_2 is trustworthiness and x_3 is price. The values of x_1, x_2 and x_3 are in $[0, 1]$. They are obtained from the questionnaire of some domain experts, their option could be degree of good, degree of indeterminacy and degree of poor. A is an interval neutrosophic set of X defined by

$$A = \langle [0.2, 0.4], [0.3, 0.5], [0.3, 0.5] \rangle / x_1 + \langle [0.5, 0.7], [0, 0.2], [0.2, 0.3] \rangle / x_2 + \langle [0.6, 0.8], [0.2, 0.3], [0.2, 0.3] \rangle / x_3.$$

B is an interval neutrosophic set of X defined by

$$B = \langle [0.5, 0.7], [0.1, 0.3], [0.1, 0.3] \rangle / x_1 + \langle [0.2, 0.3], [0.2, 0.4], [0.5, 0.8] \rangle / x_2 + \langle [0.4, 0.6], [0, 0.1], [0.3, 0.4] \rangle / x_3.$$

Definition 9 An interval neutrosophic set A is empty if and only if its $\inf T_A(x) = \sup T_A(x) = 0$, $\inf I_A(x) = \sup I_A(x) = 1$ and $\inf F_A(x) = \sup F_A(x) = 0$, for all x in X .

We now present the set-theoretic operators on interval neutrosophic set.

Definition 10 (Complement) The complement of an interval set A is denoted by \bar{A} and is defined by

$$T_{\bar{A}}(x) = F_A(x), \quad (21)$$

$$\inf I_{\bar{A}}(x) = 1 - \sup I_A(x), \quad (22)$$

$$\sup I_{\bar{A}}(x) = 1 - \inf I_A(x), \quad (23)$$

$$F_{\bar{A}}(x) = T_A(x), \quad (24)$$

for all x in X .

Example 2 Let A be the interval neutrosophic set defined in Example 1. Then, $\bar{A} = \langle [0.3, 0.5], [0.5, 0.7], [0.2, 0.4] \rangle / x_1 + \langle [0.2, 0.3], [0.8, 1.0], [0.5, 0.7] \rangle / x_2 + \langle [0.2, 0.3], [0.7, 0.8], [0.6, 0.8] \rangle / x_3$.

Definition 11 (Containment) An interval neutrosophic set A is contained in the other interval neutrosophic set B , $A \subseteq B$, if and only if

$$\inf T_A(x) \leq \inf T_B(x) \quad , \quad \sup T_A(x) \leq \sup T_B(x), \quad (25)$$

$$\inf I_A(x) \geq \inf I_B(x) \quad , \quad \sup I_A(x) \geq \sup I_B(x), \quad (26)$$

$$\inf F_A(x) \geq \inf F_B(x) \quad , \quad \sup F_A(x) \geq \sup F_B(x), \quad (27)$$

for all x in X .

Note that by the definition of containment, X is partial order not linear order. For example, let A and B be the interval neutrosophic sets defined in Example 1. Then, A is not contained in B and B is not contained in A .

Definition 12 Two interval neutrosophic sets A and B are equal, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$

Definition 13 (Union) *The union of two interval neutrosophic sets A and B is an interval neutrosophic set C , written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A and B by*

$$\inf T_C(x) = \max(\inf T_A(x), \inf T_B(x)), \quad (28)$$

$$\sup T_C(x) = \max(\sup T_A(x), \sup T_B(x)), \quad (29)$$

$$\inf I_C(x) = \min(\inf I_A(x), \inf I_B(x)), \quad (30)$$

$$\sup I_C(x) = \min(\sup I_A(x), \sup I_B(x)), \quad (31)$$

$$\inf F_C(x) = \min(\inf F_A(x), \inf F_B(x)), \quad (32)$$

$$\sup F_C(x) = \min(\sup F_A(x), \sup F_B(x)), \quad (33)$$

for all x in X .

Example 3 *Let A and B be the interval neutrosophic sets defined in Example 1. Then, $A \cup B = \langle [0.5, 0.7], [0.1, 0.3], [0.1, 0.3] \rangle / x_1 + \langle [0.5, 0.7], [0, 0.2], [0.2, 0.3] \rangle / x_2 + \langle [0.6, 0.8], [0, 0.1], [0.2, 0.3] \rangle / x_3$.*

The intuition behind the union operator is that if one of elements in A and B is true then it is true in $A \cup B$, only both are indeterminate and false in A and B then it is indeterminate and false in $A \cup B$. The other operators should be understood similarly.

Theorem 1 *$A \cup B$ is the smallest interval neutrosophic set containing both A and B .*

Definition 14 (Intersection) *The intersection of two interval neutrosophic sets A and B is an interval neutrosophic set C , written as $C = A \cap B$, whose truth-membership, indeterminacy-membership functions and false-membership functions are related to those of A and B by*

$$\inf T_C(x) = \min(\inf T_A(x), \inf T_B(x)), \quad (34)$$

$$\sup T_C(x) = \min(\sup T_A(x), \sup T_B(x)), \quad (35)$$

$$\inf I_C(x) = \max(\inf I_A(x), \inf I_B(x)), \quad (36)$$

$$\sup I_C(x) = \max(\sup I_A(x), \sup I_B(x)), \quad (37)$$

$$\inf F_C(x) = \max(\inf F_A(x), \inf F_B(x)), \quad (38)$$

$$\sup F_C(x) = \max(\sup F_A(x), \sup F_B(x)), \quad (39)$$

for all x in X .

Example 4 *Let A and B be the interval neutrosophic sets defined in Example 1. Then, $A \cap B = \langle [0.2, 0.4], [0.3, 0.5], [0.3, 0.5] \rangle / x_1 + \langle [0.2, 0.3], [0.2, 0.4], [0.5, 0.8] \rangle / x_2 + \langle [0.4, 0.6], [0.2, 0.3], [0.3, 0.4] \rangle / x_3$.*

Theorem 2 *$A \cap B$ is the largest interval neutrosophic set contained in both A and B .*

Definition 15 (Difference) *The difference of two interval neutrosophic sets A and B is an interval neutrosophic set C , written as $C = A \setminus B$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A and B by*

$$\inf T_C(x) = \min(\inf T_A(x), \inf F_B(x)), \quad (40)$$

$$\sup T_C(x) = \min(\sup T_A(x), \sup F_B(x)), \quad (41)$$

$$\inf I_C(x) = \max(\inf I_A(x), 1 - \sup I_B(x)), \quad (42)$$

$$\sup I_C(x) = \max(\sup I_A(x), 1 - \inf I_B(x)), \quad (43)$$

$$\inf F_C(x) = \max(\inf F_A(x), \inf T_B(x)), \quad (44)$$

$$\sup F_C(x) = \max(\sup F_A(x), \sup T_B(x)), \quad (45)$$

for all x in X .

Example 5 *Let A and B be the interval neutrosophic sets defined in Example 1. Then, $A \setminus B = \langle [0.1, 0.3], [0.7, 0.9], [0.5, 0.7] \rangle / x_1 + \langle [0.5, 0.7], [0.6, 0.8], [0.2, 0.3] \rangle / x_2 + \langle [0.3, 0.4], [0.9, 1.0], [0.4, 0.6] \rangle / x_3$.*

Theorem 3 $A \subseteq B \leftrightarrow \bar{B} \subseteq \bar{A}$

Definition 16 (Addition) *The addition of two interval neutrosophic sets A and B is an interval neutrosophic set C , written as $C = A + B$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A and B by*

$$\inf T_C(x) = \min(\inf T_A(x) + \inf T_B(x), 1), \quad (46)$$

$$\sup T_C(x) = \min(\sup T_A(x) + \sup T_B(x), 1), \quad (47)$$

$$\inf I_C(x) = \min(\inf I_A(x) + \inf I_B(x), 1), \quad (48)$$

$$\sup I_C(x) = \min(\sup I_A(x) + \sup I_B(x), 1), \quad (49)$$

$$\inf F_C(x) = \min(\inf F_A(x) + \inf F_B(x), 1), \quad (50)$$

$$\sup F_C(x) = \min(\sup F_A(x) + \sup F_B(x), 1), \quad (51)$$

for all x in X .

Example 6 *Let A and B be the interval neutrosophic sets defined in Example 1. Then, $A + B = \langle [0.7, 1.0], [0.4, 0.8], [0.4, 0.8] \rangle / x_1 + \langle [0.7, 1.0], [0.2, 0.6], [0.7, 1.0] \rangle / x_2 + \langle [1.0, 1.0], [0.2, 0.4], [0.5, 0.7] \rangle / x_3$.*

Definition 17 (Cartesian product) *The cartesian product of two interval neutrosophic sets A defined on universe X_1 and B defined on universe X_2 is an interval neutrosophic set C , written as $C = A \times B$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A and B by*

$$\inf T_C(x, y) = \inf T_A(x) + \inf T_B(y) - \inf T_A(x) \cdot \inf T_B(y), \quad (52)$$

$$\sup T_C(x, y) = \sup T_A(x) + \sup T_B(y) - \sup T_A(x) \cdot \sup T_B(y), \quad (53)$$

$$\inf I_C(x, y) = \inf I_A(x) \cdot \sup I_B(y), \quad (54)$$

$$\sup I_C(x, y) = \sup I_A(x) \cdot \sup I_B(y), \quad (55)$$

$$\inf F_C(x, y) = \inf F_A(x) \cdot \inf F_B(y), \quad (56)$$

$$\sup F_C(x, y) = \sup F_A(x) \cdot \sup F_B(y), \quad (57)$$

for all x in X_1 , y in X_2 .

Example 7 Let A and B be the interval neutrosophic sets defined in Example 1. Then, $A \times B = \langle [0.6, 0.82], [0.03, 0.15], [0.03, 0.15] \rangle / x_1 + \langle [0.6, 0.79], [0, 0.08], [0.1, 0.24] \rangle / x_2 + \langle [0.76, 0.92], [0, 0.03], [0.03, 0.12] \rangle / x_3$.

Definition 18 (Scalar multiplication) The scalar multiplication of interval neutrosophic set A is an interval neutrosophic set B , written as $B = a \cdot A$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A by

$$\inf T_B(x) = \min(\inf T_A(x) \cdot a, 1), \quad (58)$$

$$\sup T_B(x) = \min(\sup T_A(x) \cdot a, 1), \quad (59)$$

$$\inf I_B(x) = \min(\inf I_A(x) \cdot a, 1), \quad (60)$$

$$\sup I_B(x) = \min(\sup I_A(x) \cdot a, 1), \quad (61)$$

$$\inf F_B(x) = \min(\inf F_A(x) \cdot a, 1), \quad (62)$$

$$\sup F_B(x) = \min(\sup F_A(x) \cdot a, 1), \quad (63)$$

for all x in X , $a \in R^+$.

Definition 19 (Scalar division) The scalar division of interval neutrosophic set A is an interval neutrosophic set B , written as $B = a \cdot A$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A by

$$\inf T_B(x) = \min(\inf T_A(x)/a, 1), \quad (64)$$

$$\sup T_B(x) = \min(\sup T_A(x)/a, 1), \quad (65)$$

$$\inf I_B(x) = \min(\inf I_A(x)/a, 1), \quad (66)$$

$$\sup I_B(x) = \min(\sup I_A(x)/a, 1), \quad (67)$$

$$\inf F_B(x) = \min(\inf F_A(x)/a, 1), \quad (68)$$

$$\sup F_B(x) = \min(\sup F_A(x)/a, 1), \quad (69)$$

for all x in X , $a \in R^+$.

Now we will define two operators: truth-favorite (Δ) and false-favorite (∇) to remove the indeterminacy in the interval neutrosophic sets and transform it into interval valued intuitionistic fuzzy sets or interval valued paraconsistent sets. These two operators are unique on interval neutrosophic sets.

Definition 20 (Truth-favorite) The truth-favorite of interval neutrosophic set A is an interval neutrosophic set B , written as $B = \Delta A$, whose truth-membership and false-membership functions are related to those of A by

$$\inf T_B(x) = \min(\inf T_A(x) + \inf I_A(x), 1), \quad (70)$$

$$\sup T_B(x) = \min(\sup T_A(x) + \sup I_A(x), 1), \quad (71)$$

$$\inf I_B(x) = 0, \quad (72)$$

$$\sup I_B(x) = 0, \quad (73)$$

$$\inf F_B(x) = \inf F_A(x), \quad (74)$$

$$\sup F_B(x) = \sup F_A(x), \quad (75)$$

for all x in X .

Example 8 Let A be the interval neutrosophic set defined in Example 1. Then, $\Delta A = \langle [0.5, 0.9], [0, 0], [0.3, 0.5] \rangle / x_1 + \langle [0.5, 0.9], [0, 0], [0.2, 0.3] \rangle / x_2 + \langle [0.8, 1.0], [0, 0], [0.2, 0.3] \rangle / x_3$.

The purpose of truth-favorite operator is to evaluate the maximum of degree of truth-membership of interval neutrosophic set.

Definition 21 (False-favorite) The false-favorite of interval neutrosophic set A is an interval neutrosophic set B , written as $B = \nabla A$, whose truth-membership and false-membership functions are related to those of A by

$$\inf T_B(x) = \inf T_A(x), \quad (76)$$

$$\sup T_B(x) = \sup T_A(x), \quad (77)$$

$$\inf I_B(x) = 0, \quad (78)$$

$$\sup I_B(x) = 0, \quad (79)$$

$$\inf F_B(x) = \min(\inf F_A(x) + \inf I_A(x), 1), \quad (80)$$

$$\sup F_B(x) = \min(\sup F_A(x) + \sup I_A(x), 1), \quad (81)$$

for all x in X .

Example 9 Let A be the interval neutrosophic set defined in Example 1. Then, $\nabla A = \langle [0.2, 0.4], [0, 0], [0.6, 1.0] \rangle / x_1 + \langle [0.5, 0.7], [0, 0], [0.2, 0.5] \rangle / x_2 + \langle [0.6, 0.8], [0, 0], [0.4, 0.6] \rangle / x_3$.

The purpose of false-favorite operator is to evaluate the maximum of degree of false-membership of interval neutrosophic set.

Theorem 4 For every two interval neutrosophic sets A and B :

1. $\Delta(A \cup B) \subseteq \Delta A \cup \Delta B$
2. $\Delta A \cap \Delta B \subseteq \Delta(A \cap B)$
3. $\nabla A \cup \nabla B \subseteq \nabla(A \cup B)$
4. $\nabla(A \cap B) \subseteq \nabla A \cap \nabla B$

4 Properties of Set-theoretic Operators

In this section, we will give some properties of set-theoretic operators defined on interval neutrosophic sets as in Section 3. The proof of these properties is left for the readers.

Property 1 (Commutativity) $A \cup B = B \cup A$, $A \cap B = B \cap A$, $A + B = B + A$, $A \times B = B \times A$

Property 2 (Associativity) $A \cup (B \cup C) = (A \cup B) \cup C$,
 $A \cap (B \cap C) = (A \cap B) \cap C$,
 $A + (B + C) = (A + B) + C$,
 $A \times (B \times C) = (A \times B) \times C$.

Property 3 (Distributivity) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Property 4 (Idempotency) $A \cup A = A$, $A \cap A = A$, $\Delta \Delta A = \Delta A$, $\nabla \nabla A = \nabla A$.

Property 5 $A \cap \Phi = \Phi$, $A \cup X = X$, where $\inf T_\Phi = \sup T_\Phi = 0$, $\inf I_\Phi = \sup I_\Phi = \inf F_\Phi = \sup F_\Phi = 1$ and $\inf T_X = \sup T_X = 1$, $\inf I_X = \sup I_X = \inf F_X = \sup F_X = 0$.

Property 6 $\Delta(A + B) = \Delta A + \Delta B$, $\nabla(A + B) = \nabla A + \nabla B$.

Property 7 $A \cup \Psi = A$, $A \cap X = A$, where $\inf T_\Phi = \sup T_\Phi = 0$, $\inf I_\Phi = \sup I_\Phi = \inf F_\Phi = \sup F_\Phi = 1$ and $\inf T_X = \sup T_X = 1$, $\inf I_X = \sup I_X = \inf F_X = \sup F_X = 0$.

Property 8 (Absorption) $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

Property 9 (DeMorgan's Laws) $\overline{A \cup B} = \bar{A} \cap \bar{B}$, $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Property 10 (Involution) $\overline{\bar{A}} = A$

Here, we notice that by the definitions of complement, union and intersection of interval neutrosophic set, interval neutrosophic set satisfies the most properties of class set, fuzzy set and intuitionistic fuzzy set. Same as fuzzy set and intuitionistic fuzzy set, it does not satisfy the principle of middle exclude.

5 Convexity of Interval Neutrosophic Set

We assume that X is a real Euclidean space E^n for correctness.

Definition 22 (Convexity) *An interval neutrosophic set A is convex if and only if*

$$\inf T_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\inf T_A(x_1), \inf T_A(x_2)), \quad (82)$$

$$\sup T_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\sup T_A(x_1), \sup T_A(x_2)), \quad (83)$$

$$\inf I_A(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\inf I_A(x_1), \inf I_A(x_2)), \quad (84)$$

$$\sup I_A(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\sup I_A(x_1), \sup I_A(x_2)), \quad (85)$$

$$\inf F_A(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\inf F_A(x_1), \inf F_A(x_2)), \quad (86)$$

$$\sup F_A(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\sup F_A(x_1), \sup F_A(x_2)), \quad (87)$$

for all x_1 and x_2 in X and all λ in $[0, 1]$.

Fig. 2 is an illustration of convex interval neutrosophic set.

Theorem 5 *If A and B are convex, so is their intersection.*

Definition 23 (Strongly Convex) *An interval neutrosophic set A is strongly convex if for any two distinct points x_1 and x_2 , and any λ in the open interval $(0, 1)$,*

$$\inf T_A(\lambda x_1 + (1 - \lambda)x_2) > \min(\inf T_A(x_1), \inf T_A(x_2)), \quad (88)$$

$$\sup T_A(\lambda x_1 + (1 - \lambda)x_2) > \min(\sup T_A(x_1), \sup T_A(x_2)), \quad (89)$$

$$\inf I_A(\lambda x_1 + (1 - \lambda)x_2) < \max(\inf I_A(x_1), \inf I_A(x_2)), \quad (90)$$

$$\sup I_A(\lambda x_1 + (1 - \lambda)x_2) < \max(\sup I_A(x_1), \sup I_A(x_2)), \quad (91)$$

$$\inf F_A(\lambda x_1 + (1 - \lambda)x_2) < \max(\inf F_A(x_1), \inf F_A(x_2)), \quad (92)$$

$$\sup F_A(\lambda x_1 + (1 - \lambda)x_2) < \max(\sup F_A(x_1), \sup F_A(x_2)), \quad (93)$$

for all x_1 and x_2 in X and all λ in $[0, 1]$.

Theorem 6 *If A and B are strongly convex, so is their intersection.*

6 Conclusions and Future Works

In this paper, we have presented an instance of neutrosophic set called interval neutrosophic set (INS). The interval neutrosophic set is a generalization of classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy sets, interval valued intuitionistic fuzzy set, interval type-2 fuzzy set [3] and paraconsistent set. The notions of inclusion, union, intersection, complement, relation, and composition have been defined on interval neutrosophic set. Various properties of set-theoretic operators have been proved. In the future, we will create the logic inference system based on interval neutrosophic set and apply the theory to solve practical applications in areas such as expert system, data mining, question-answering system, bioinformatics and database, etc.

Appendix

Theorem 1 *$A \cup B$ is the smallest interval neutrosophic set containing both A and B .*

Proof Let $C = A \cup B$. $\inf T_C = \max(\inf T_A, \inf T_B)$, $\inf T_C \geq \inf T_A$, $\inf T_C \geq \inf T_B$. $\sup T_C = \max(\sup T_A, \sup T_B)$, $\sup T_C \geq \sup T_A$, $\sup T_C \geq \sup T_B$. $\inf I_C = \min(\inf I_A, \inf I_B)$, $\inf I_C \leq \inf I_A$, $\inf I_C \leq \inf I_B$, $\sup I_C = \min(\sup I_A, \sup I_B)$, $\sup I_C \leq \sup I_A$, $\sup I_C \leq \sup I_B$, $\inf F_C = \min(\inf F_A, \inf F_B)$, $\inf F_C \leq \inf F_A$, $\inf F_C \leq \inf F_B$. $\sup F_C = \min(\sup F_A, \sup F_B)$, $\sup F_C \leq \sup F_A$, $\sup F_C \leq \sup F_B$. That means C contains both A and B .

Furthermore, if D is any extended vague set containing both A and B , then $\inf T_D \geq \inf T_A$, $\inf T_D \geq \inf T_B$, so $\inf T_D \geq \max(\inf T_A, \inf T_B) = \inf T_C$. $\sup T_D \geq \sup T_A$, $\sup T_D \geq \sup T_B$, so $\sup T_D \geq \max(\sup T_A, \sup T_B) = \sup T_C$. $\inf I_D \leq \inf I_A$, $\inf I_D \leq \inf I_B$, so $\inf I_D \leq \min(\inf I_A, \inf I_B) = \inf I_C$. $\sup I_D \leq \sup I_A$, $\sup I_D \leq \sup I_B$, so $\sup I_D \leq \min(\sup I_A, \sup I_B) = \sup I_C$. $\inf F_D \leq \inf F_A$, $\inf F_D \leq \inf F_B$, so $\inf F_D \leq \min(\inf F_A, \inf F_B) = \inf F_C$. $\sup F_D \leq \sup F_A$, $\sup F_D \leq \sup F_B$, so $\sup F_D \leq \min(\sup F_A, \sup F_B) = \sup F_C$. That implies $C \subseteq D$.

Theorem 2 $A \cap B$ is the largest interval neutrosophic set contained in both A and B .

Proof The proof is analogous to the proof of theorem 1.

Theorem 3 $A \subseteq B \leftrightarrow \bar{B} \subseteq \bar{A}$

Proof $A \subseteq B \Leftrightarrow \inf T_A \leq \inf T_B, \sup T_A \leq \sup T_B, \inf I_A \geq \inf I_B, \sup I_A \geq \sup I_B, \inf F_A \geq \inf F_B, \sup F_A \geq \sup F_B \Leftrightarrow \inf F_B \leq \inf F_A, \sup F_B \leq \sup F_A, 1 - \sup I_B \geq 1 - \sup I_A, 1 - \inf I_B \geq 1 - \inf I_A, \inf T_B \geq \inf T_A, \sup T_B \geq \sup T_A \Leftrightarrow \bar{B} \subseteq \bar{A}$.

Theorem 4 For every two interval neutrosophic sets A and B :

1. $\Delta(A \cup B) \subseteq \Delta A \cup \Delta B$
2. $\Delta A \cap \Delta B \subseteq \Delta(A \cap B)$
3. $\nabla A \cup \nabla B \subseteq \nabla(A \cup B)$
4. $\nabla(A \cap B) \subseteq \nabla A \cap \nabla B$

Proof We now prove the first identity. Let $C = A \cup B$.

$\inf T_C(x) = \max(\inf T_A(x), \inf T_B(x)),$
 $\sup T_C(x) = \max(\sup T_A(x), \sup T_B(x)),$
 $\inf I_C(x) = \min(\inf I_A(x), \inf I_B(x)),$
 $\sup I_C(x) = \min(\sup I_A(x), \sup I_B(x)),$
 $\inf F_C(x) = \min(\inf F_A(x), \inf F_B(x)),$
 $\sup F_C(x) = \min(\sup F_A(x), \sup F_B(x)).$
 $\inf T_{\Delta C}(x) = \min(\inf T_C(x) + \inf I_C(x), 1),$
 $\sup T_{\Delta C}(x) = \min(\sup T_C(x) + \sup T_C(x), 1),$
 $\inf I_{\Delta C}(x) = \sup I_{\Delta C}(x) = 0,$
 $\inf F_{\Delta C}(x) = \inf I_C(x),$
 $\sup F_{\Delta C}(x) = \sup I_C(x).$
 $\inf T_{\Delta A}(x) = \min(\inf T_A(x) + \inf I_A(x), 1),$
 $\sup T_{\Delta A}(x) = \min(\sup T_A(x) + \sup I_A(x), 1),$
 $\inf I_{\Delta A}(x) = \sup I_{\Delta A}(x) = 0,$
 $\inf F_{\Delta A}(x) = \inf I_A(x),$
 $\sup F_{\Delta A}(x) = \sup I_A(x).$
 $\inf T_{\Delta B}(x) = \min(\inf T_B(x) + \inf I_B(x), 1),$
 $\sup T_{\Delta B}(x) = \min(\sup T_B(x) + \sup I_B(x), 1),$
 $\inf I_{\Delta B}(x) = \sup I_{\Delta B}(x) = 0,$
 $\inf F_{\Delta B}(x) = \inf I_B(x),$
 $\sup F_{\Delta B}(x) = \sup I_B(x).$
 $\inf T_{\Delta A \cup \Delta B}(x) = \max(\inf T_{\Delta A}(x), \inf T_{\Delta B}(x)),$
 $\sup T_{\Delta A \cup \Delta B}(x) = \max(\sup T_{\Delta A}(x), \sup T_{\Delta B}(x)),$
 $\inf I_{\Delta A \cup \Delta B}(x) = \sup I_{\Delta A \cup \Delta B}(x) = 0,$
 $\inf F_{\Delta A \cup \Delta B}(x) = \min(\inf F_{\Delta A}(x), \inf F_{\Delta B}(x)),$
 $\sup F_{\Delta A \cup \Delta B}(x) = \min(\sup F_{\Delta A}(x), \sup F_{\Delta B}(x)).$
 Because,
 $\inf T_{\Delta(A \cup B)} \leq \inf T_{\Delta A \cup \Delta B},$

$\sup T_{\Delta(A \cup B)} \leq \sup T_{\Delta A \cup \Delta B}$,
 $\inf I_{\Delta(A \cup B)} = \inf T_{\Delta A \cup \Delta B} = 0$,
 $\sup I_{\Delta(A \cup B)} = \sup T_{\Delta A \cup \Delta B} = 0$,
 $\inf F_{\Delta(A \cup B)} = \inf F_{\Delta A \cup \Delta B}$,
 $\sup F_{\Delta(A \cup B)} = \sup T_{\Delta A \cup \Delta B}$,
 so, $\Delta(A \cup B) \subseteq \Delta A \cup \Delta B$. The other identities can be proved in a similar manner.

Theorem 5 *If A and B are convex, so is their intersection.*

Proof Let $C = A \cap B$, then
 $\inf T_C(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\inf T_A(\lambda x_1 + (1 - \lambda)x_2), \inf T_B(\lambda x_1 + (1 - \lambda)x_2))$,
 $\sup T_C(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\sup T_A(\lambda x_1 + (1 - \lambda)x_2), \sup T_B(\lambda x_1 + (1 - \lambda)x_2))$,
 $\inf I_C(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\inf I_A(\lambda x_1 + (1 - \lambda)x_2), \inf I_B(\lambda x_1 + (1 - \lambda)x_2))$,
 $\sup I_C(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\sup I_A(\lambda x_1 + (1 - \lambda)x_2), \sup I_B(\lambda x_1 + (1 - \lambda)x_2))$,
 $\inf F_C(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\inf F_A(\lambda x_1 + (1 - \lambda)x_2), \inf F_B(\lambda x_1 + (1 - \lambda)x_2))$,
 $\sup F_C(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\sup F_A(\lambda x_1 + (1 - \lambda)x_2), \sup F_B(\lambda x_1 + (1 - \lambda)x_2))$,
 Since A and B are convex: $\inf T_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\inf T_A(x_1), \inf T_A(x_2))$,
 $\sup T_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\sup T_A(x_1), \sup T_A(x_2))$, $\inf I_A(\lambda x_1 + (1 - \lambda)x_2) \leq$
 $\max(\inf I_A(x_1), \inf I_A(x_2))$, $\sup I_A(\lambda x_1 + (1 - \lambda)x_2) \leq$
 $\max(\sup I_A(x_1), \sup I_A(x_2))$, $\inf F_A(\lambda x_1 + (1 - \lambda)x_2) \leq$
 $\max(\inf F_A(x_1), \inf F_A(x_2))$, $\sup F_A(\lambda x_1 + (1 - \lambda)x_2) \leq$
 $\max(\sup F_A(x_1), \sup F_A(x_2))$,
 $\inf T_B(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\inf T_B(x_1), \inf T_B(x_2))$, $\sup T_B(\lambda x_1 + (1 - \lambda)x_2) \geq$
 $\min(\sup T_B(x_1), \sup T_B(x_2))$, $\inf I_B(\lambda x_1 + (1 - \lambda)x_2) \leq$
 $\max(\inf I_B(x_1), \inf I_B(x_2))$, $\sup I_B(\lambda x_1 + (1 - \lambda)x_2) \leq$
 $\max(\sup I_B(x_1), \sup I_B(x_2))$,
 $\inf F_B(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\inf F_B(x_1), \inf F_B(x_2))$,
 $\sup F_B(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\sup F_B(x_1), \sup F_B(x_2))$,
 Hence,
 $\inf T_C(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\min(\inf T_A(x_1), \inf T_A(x_2))$
 $, \min(\inf T_B(x_1), \inf T_B(x_2))) = \min(\min(\inf T_A(x_1), \inf T_B(x_1))$,
 $\min(\inf T_A(x_2), \inf T_B(x_2))) = \min(\inf T_C(x_1), \inf T_C(x_2))$, $\sup T_C(\lambda x_1 + (1 -$
 $\lambda)x_2) \geq \min(\min(\sup T_A(x_1), \sup T_A(x_2))$,
 $\min(\sup T_B(x_1), \sup T_B(x_2))) = \min(\min(\sup T_A(x_1), \sup T_B(x_1))$,
 $\min(\sup T_A(x_2), \sup T_B(x_2))) = \min(\sup T_C(x_1), \sup T_C(x_2))$,
 $\inf I_C(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\max(\inf I_A(x_1)$,
 $\inf I_A(x_2)), \max(\inf I_B(x_1), \inf I_B(x_2))) = \max(\max(\inf I_A(x_1)$,
 $\inf I_B(x_1)), \max(\inf I_A(x_2), \inf I_B(x_2))) = \max(\inf I_C(x_1), \inf I_C(x_2))$,
 $\sup I_C(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\max(\sup I_A(x_1), \sup I_A(x_2))$
 $, \max(\sup I_B(x_1), \sup I_B(x_2))) = \max(\max(\sup I_A(x_1), \sup I_B(x_1))$,
 $\max(\sup I_A(x_2), \sup I_B(x_2))) = \max(\sup I_C(x_1), \sup I_C(x_2))$,
 $\inf F_C(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\max(\inf F_A(x_1), \inf F_A(x_2))$,
 $\max(\inf F_B(x_1), \inf F_B(x_2))) = \max(\max(\inf F_A(x_1), \inf F_B(x_1))$,
 $\max(\inf F_A(x_2), \inf F_B(x_2))) =$
 $\max(\inf F_C(x_1), \inf F_C(x_2))$, $\sup F_C(\lambda x_1 + (1 - \lambda)x_2)$
 $\leq \max(\max(\sup F_A(x_1), \sup F_A(x_2))$,
 $\max(\sup F_B(x_1), \sup F_B(x_2))) = \max(\max(\sup F_A(x_1), \sup F_B(x_1))$,
 $\max(\sup F_A(x_2), \sup F_B(x_2))) = \max(\sup F_C(x_1), \sup F_C(x_2))$.

Theorem 6 *If A and B are strongly convex, so is their intersection.*

Proof The proof is analogous to the proof of Theorem 5.

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