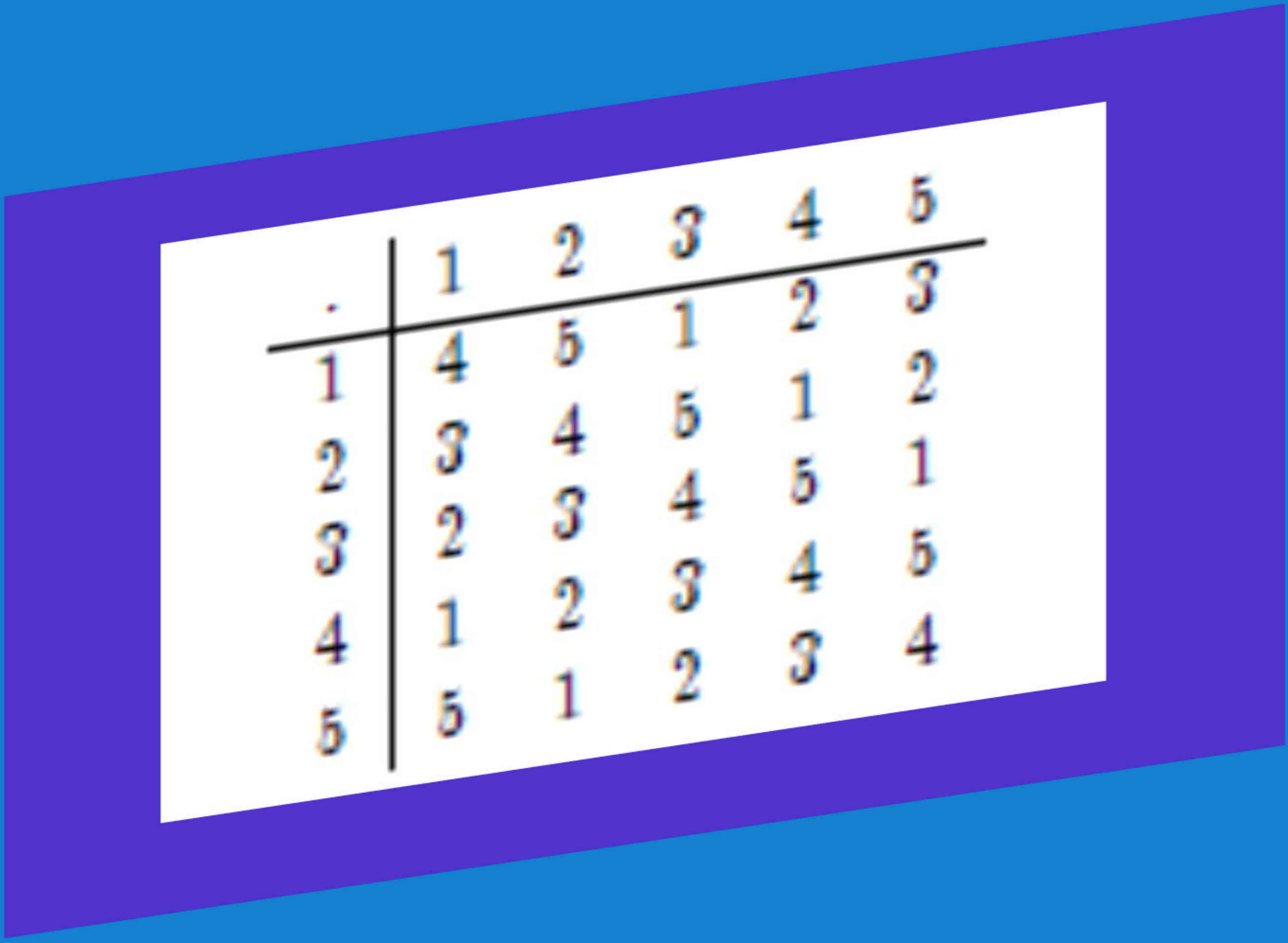
Madad Khan ▌ Florentin Smarandache ▌ Tariq Aziz

Fuzzy Abel Grassmann Groupoids

second updated and enlarged version



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Contents

Preface

Usually the models of real world problems in almost all disciplines like engineering, medical sciences, mathematics, physics, computer science, management sciences, operations research and artificial intelligence are mostly full of complexities and consist of several types of uncertainties while dealing them in several occasion. To overcome these difficulties of uncertainties, many theories have been developed such as rough sets theory, probability theory, fuzzy sets theory, theory of vague sets, theory of soft ideals and the theory of intuitionistic fuzzy sets, theory of neutrosophic sets, Dezert-Smarandache Theory (DSmT), etc. Zadeh introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. Atanassov introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set. He has coined the words "neutrosophy" and "neutrosophic". In 2013 he refined the neutrosophic set to n components: $t_1, t_2, \ldots; i_1, i_2, \ldots; f_1, f_2, \ldots$.

Zadeh discovered the relationships of probability and fuzzy set theory which has appropriate approach to deal with uncertainties. Many authors have applied the fuzzy set theory to generalize the basic theories of Algebra. Mordeson et al. [27] has discovered the grand exploration of fuzzy semigroups, where theory of fuzzy semigroups is explored along with the applications of fuzzy semigroups in fuzzy coding, fuzzy finite state mechanics and fuzzy languages and the use of fuzzification in automata and formal language has widely been explored. Moreover the complete l-semigroups have wide range of applications in the theories of automata, formal languages and programming. It is worth mentioning that some recent investigations of l-semigroups are closely connected with algebraic logic and non-classical logics.

An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. Mostly it works like a commutative semigroup. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e. Moreover $ab = (ba)e$ for all elements a and b of the AG-groupoid. Now our aim is to discover some logical investigations for regular and intra-regular AG-groupoids using the new generalized concept of fuzzy sets. It is therefore concluded that this research work will give a new direction for applications of fuzzy set theory particularly in algebraic logic, non-classical logics, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

To overcome these diffculties of uncertainties, many theories have been developed such as rough sets theory, probability theory, fuzzy sets theory, theory of vague sets, theory of soft ideals and the theory of intuitionistic fuzzy sets,

In [29], Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined in [33]. Bhakat and Das [1, 2] gave the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Davvaz defined $(\in, \in \vee q)$-fuzzy subnearrings and ideals of a near ring in [4]. Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup in [14]. In [37] regular semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals. In [36] semigroups are characterized by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

In chapter one we have introduced the concept of $(\in, \in \vee q_k)$-fuzzy ideals in an AG-groupoid. We have discussed several important features of a right regular AG-groupoid.

In chapter two, we investigate some characterizations of regular and intra-regular Abel-Grassmann's groupoids in terms of $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy quasi-ideals.

In chapter three we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals in an AG-groupoid. We characterize intra-regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals.

In chapter four we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime (semiprime) ideals in AG-groupoids. We characterize intra regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideals.

In chapter five we introduce generalized fuzzy soft ideals in a non-associative algebraic structure namely Abel Grassmann groupoid. We discuss some basic properties concerning these new types of generalized fuzzy ideals in Abel-Grassmann groupoids. Moreover we characterize a regular Abel Grassmann groupoid in terms of its classical and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideals.

1

Generalized Fuzzy Ideals of AG-groupoids

In this chapter, we have introduced the concept of $(\in, \in \vee q)$-fuzzy and $(\in, \in \vee q_k)$-fuzzy ideals in an AG-groupoid. We have discussed several important features of right regular AG-groupoid by using the $(\in, \in \vee q_k)$-fuzzy ideals. We proved that the $(\in, \in \vee q_k)$-fuzzy left (right, two-sided), $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals, and $(\in, \in \vee q_k)$-fuzzy interior ideals coincide in a right regular AG-groupoid.

1.1 Introduction

Fuzzy set theory and its applications in several branches of Science are growing day by day. Since pacific models of real world problems in various fields such as computer science, artificial intelligence, operation research, management science, control engineering, robotics, expert systems and many others, may not be constructed because we are mostly and unfortunately uncertain in many occasions. For handling such difficulties we need some natural tools such as probability theory and theory of fuzzy sets [42] which have already been developed. Associative Algebraic structures are mostly used for applications of fuzzy sets. Mordeson, Malik and Kuroki [27] have discovered the vast field of fuzzy semigroups, where theoretical exploration of fuzzy semigroups and their applications are used in fuzzy coding, fuzzy finite-state machines and fuzzy languages. The use of fuzzification in automata and formal language has widely been explored. Moreover the complete l-semigroups have wide range of applications in the theories of automata, formal languages and programming.

The fundamental concept of fuzzy sets was first introduced by Zadeh [42] in 1965. Given a set X, a fuzzy subset of X is, by definition an arbitrary mapping $f : X \to [0, 1]$ where $[0, 1]$ is the unit interval. Rosenfeld introduced the definition of a fuzzy subgroup of a group [34]. Kuroki initiated the theory of fuzzy bi ideals in semigroups [18]. The thought of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset was defined by Murali [29]. The concept of quasi-coincidence of a fuzzy point to a fuzzy set was introduce in [33]. Jun and Song introduced (α, β)-fuzzy interior ideals in semigroups [14].

In [29], Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined in [33]. Bhakat and Das [1, 2] gave the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Davvaz defined $(\in, \in \vee q)$-fuzzy subnearrings and ideals of a near ring in [4]. Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup in [14]. In [37] regular semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals. In [36] semigroups are characterized by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

In this paper, we have introduced the concept of $(\in, \in \vee q_k)$-fuzzy ideals in a new non-associative algebraic structure, that is, in an AG-groupoid and developed some new results. We have defined regular and intra-regular AG-groupoids and characterized them by $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy quasi-ideals.

An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. Mostly it works like a commutative semigroup. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e. Moreover $ab = (ba)e$ for all elements a and b of the AG-groupoid. Now our aim is to discover some logical investigations for regular and intra-regular AG-groupoids using the new generalized concept of fuzzy sets. It is therefore concluded that this research work will give a new direction for applications of fuzzy set theory particularly in algebraic logic, non-classical logics, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

1.2 Abel Grassmann Groupoids

The concept of a left almost semigroup (LA-semigroup) [16] or an AG-groupoid was first given by M. A. Kazim and M. Naseeruddin in 1972. an AG-groupoid M is a groupoid having the left invertive law,

$$(ab)c = (cb)a, \text{ for all } a,\ b,\ c \in M. \tag{1}$$

In an AG-groupoid M, the following medial law [16] holds,

$$(ab)(cd) = (ac)(bd), \text{ for all } a,\ b,\ c,\ d \in M. \tag{2}$$

The left identity in an AG-groupoid if exists is unique [28]. In an AG-groupoid M with left identity the following paramedial law holds [32],

$$(ab)(cd) = (dc)(ba), \text{ for all } a, b, c, d \in M. \tag{3}$$

If an AG-groupoid M contains a left identity, then,

$$a(bc) = b(ac), \text{ for all } a, b, c \in M. \tag{4}$$

1.3 Preliminaries

Let S be an AG-groupoid. By an AG-subgroupoid of S, we means a non-empty subset A of S such that $A^2 \subseteq A$. A non-empty subset A of an AG-groupoid S is called a left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$) and it is called a two-sided ideal if it is both left and a right ideal of S. A non-empty subset A of an AG-groupoid S is called quasi-ideal of S if $SA \cap AS \subseteq A$. A non-empty subset A of an AG-groupoid S is called a generalized bi-ideal of S if $(AS)A \subseteq A$ and an AG-subgroupoid A of S is called a bi-ideal of S if $(AS)A \subseteq A$. A non-empty subset A of an AG-groupoid S is called an interior ideal of S if $(SA)S \subseteq A$.

If S is an AG-groupoid with left identity e then $S = S^2$. It is easy to see that every one sided ideal of S is quasi-ideal of S. In [31] it is given that $L[a] = a \cup Sa$, $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (aS \cap Sa)$ are principal left ideal, principal two-sided ideal and principal quasi-ideal of S generated by a. Moreover using (1), left invertive law, paramedial law and medial law we get the following equations

$$a\,(Sa) = S(aa) = Sa^2,\ (Sa)a = (aa)S = a^2S \text{ and } (Sa)\,(Sa) = (SS)\,(aa) = Sa^2.$$

To obtain some more useful equations we use medial, paramedial laws and (1), we get

$$\begin{aligned}(Sa)^2 &= (Sa)(Sa) = (SS)a^2 = (aa)(SS) = S((aa)S)\\ &= (SS)((aa)S) = (Sa^2)SS = (Sa^2)S.\end{aligned}$$

Therefore

$$Sa^2 = a^2S = (Sa^2)S. \tag{2}$$

The following definitions are available in [27].

A fuzzy subset f of an AG-groupoid S is called a fuzzy AG-subgroupoid of S if $f(xy) \geq f(x) \wedge f(y)$ for all x, $y \in S$. A fuzzy subset f of an AG-groupoid S is called a fuzzy left (right) ideal of S if $f(xy) \geq f(y)$ ($f(xy) \geq f(x)$) for all x, $y \in S$. A fuzzy subset f of an AG-groupoid S is called a fuzzy two-sided ideal of S if it is both a fuzzy left and a fuzzy right ideal of S. A fuzzy subset f of an AG-groupoid S is called a fuzzy quasi-ideal of S if $f \circ C_S \cap C_S \circ f \subseteq f$. A fuzzy subset f of an AG-groupoid S is called a fuzzy generalized bi-ideal of S if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in S$. A fuzzy AG-subgroupoid f of an AG-groupoid S is called a fuzzy bi-ideal of S if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in S$. A fuzzy AG-subgroupoid f of an AG-groupoid S is called a fuzzy interior ideal of S if $f((xa)y) \geq f(a)$, for all x, a and $y \in S$. Let f be a fuzzy subset of an AG-groupoid S, then f is called a fuzzy prime if $\max\{f(a), f(b)\} \geq f(ab)$, for all $a, b \in S$. f is called a fuzzy semiprime if $f(a) \geq f(a^2)$, for all $a \in S$.

Let f and g be any two fuzzy subsets of an AG-groupoid S, then the product $f \circ g$ is defined by,

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c)\}, \text{ if there exist } b, c \in S, \text{ such that } a = bc. \\ 0, \quad \text{otherwise.} \end{cases}$$

The symbols $f \cap g$ and $f \cup g$ will means the following fuzzy subsets of S

$$(f \cap g)(x) = \min\{f(x),\ g(x)\} = f(x) \wedge g(x),\ \text{for all } x \text{ in } S$$

and

$$(f \cup g)(x) = \max\{f(x),\ g(x)\} = f(x) \vee g(x),\ \text{for all } x \text{ in } S.$$

Let f be a fuzzy subset of an AG-groupoid S and $t \in (0, 1]$. Then $x_t \in f$ means $f(x) \geq t$, $x_t q f$ means $f(x) + t > 1$, $x_t \alpha \vee \beta f$ means $x_t \alpha f$ or $x_t \beta f$, where α, β denotes any one of $\in$, q, $\in \vee q$, $\in \wedge q$. $x_t \alpha \wedge \beta f$ means $x_t \alpha f$ and $x_t \beta f$, $x_t \overline{\alpha} f$ means $x_t \alpha f$ does not holds. Generalizing the concept of $x_t q f$, Jun [13, 14] defined $x_t q_k f$, where $k \in [0, 1)$, as $f(x) + t + k > 1$. $x_t \in \vee q_k f$ if $x_t \in f$ or $x_t q_k f$.

Let f and g be any two fuzzy subsets of an AG-groupoid S, then for $k \in [0, 1)$, the product $f \circ_k g$ is defined by,

$$(f \circ_k g)(a) = \begin{cases} \bigvee\limits_{a=bc} \left\{f(b) \wedge g(c) \wedge \frac{1-k}{2}\right\}, \text{ if there exist } b, c \in S, \text{ such that } a = bc. \\ 0, \quad \text{otherwise.} \end{cases}$$

The symbols $f \wedge g$ and $f \vee g$ will means the following fuzzy subsets of an AG-groupoid S.

$(f \wedge g)(x) = \min\{f(x), g(x)\}$ for all x in S.

$(f \vee g)(x) = \max\{f(x), g(x)\}$ for all x in S.

Definition 1 *A fuzzy subset f of an AG-groupoid S is called fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $k \in [0, 1)$ such that $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$.*

Definition 2 *A fuzzy subset f of an AG-groupoid S is called fuzzy left (right) ideal of S if for all $x, y \in S$ and $k \in [0, 1)$ such that $f(xy) \geq \min\{f(y), \frac{1-k}{2}\}$ $(f(xy) \geq \min\{f(x), \frac{1-k}{2}\})$.*

A fuzzy subset f of an AG-groupoid S is called fuzzy ideal if it is fuzzy left as well as fuzzy right ideal of S.

Definition 3 *A fuzzy subset f of an AG-groupoid S is called fuzzy quasi ideal of S, if*
$f(a) \geq \min\{(f \circ \varsigma)(a), (\varsigma \circ f)(a), \frac{1-k}{2}\}$. *where ς is the fuzzy subset of S mapping every element of S on* 1.

Definition 4 *A fuzzy subset f is called a fuzzy generalized bi-ideal of S if $f((xa)y) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$, for all x, a and $y \in S$. A fuzzy AG-subgroupoid f of S is called a fuzzy bi-ideal of S if $f((xa)y) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$, for all $x, a, y \in S$ and $k \in [0,1)$.*

Definition 5 *An $(\in, \in \vee q_k)$-fuzzy subset f of an AG-groupoid S is called prime if for all $a, b \in S$ and $t \in (0,1]$, it satisfies,*
$(ab)_t \in f$ *implies that* $a_t \in \vee q_k f$ *or* $b_t \in \vee q_k f$.

Theorem 6 *An $(\in, \in \vee q_k)$-fuzzy ideal f of an AG-groupoid S is prime if for all $a, b \in S$, it satisfies,*
$\max\{f(a), f(b)\} \geq \min\{f(ab), \frac{1-k}{2}\}$.

Proof. It is straightforward. ■

Definition 7 *A fuzzy subset f of an AG-groupoid S is called $(\in, \in \vee q_k)$-fuzzy semiprime if it satisfies,*
$a_t^2 \in f$ *this implies that* $a_t \in \vee q_k f$ *for all* $a \in S$ *and* $t \in (0,1]$.

Theorem 8 *An $(\in, \in \vee q_k)$-fuzzy ideal f of an AG-groupoid S is called semiprime if for any $a \in S$ and $k \in [0,1)$, if it satisfies,*
$f(a) \geq \min\{f(a^2), \frac{1-k}{2}\}$.

Proof. It is easy. ■

Definition 9 *For a fuzzy subset F of an AG-groupoid M and $t \in (0,1]$, the crisp set $U(F;t) = \{x \in M$ such that $F(x) \geq t\}$ is called a level subset of F.*

Definition 10 *A fuzzy subset F of an AG-groupoid M of the form*

$$F(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Lemma 11 *A fuzzy subset F of an AG-groupoid M is a fuzzy interior ideal of M if and only if $U(F;t)\ (\neq \emptyset)$ is an interior ideal of M.*

Definition 12 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q)$-fuzzy interior ideal of M if for all $t, r \in (0,1]$ and $x, a, y \in M$.*
(A1) $x_t \in F$ *and* $y_r \in F$ *implies that* $(xy)_{\min\{t,r\}} \in \vee qF$.
(A2) $a_t \in F$ *implies* $((xa)y)_t \in \vee qF$.

Definition 13 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q)$-fuzzy bi-ideal of M if for all $t, r \in (0,1]$ and $x, y, z \in M$.*
$(B1)$ $x_t \in F$ and $y_r \in F$ implies that $(xy)_{\min\{t,r\}} \in \vee qF$.
$(B2)$ $x_t \in F$ and $z_r \in F$ implies $((xy)z)_{\min\{t,r\}} \in \vee qF$.

Theorem 14 *For a fuzzy subset F of an AG-groupoid M. The conditions $(B1)$ and $(B2)$ of Definition 5, are equivalent to the following,*
$(B3)$ $(\forall x, y \in M)F(xy) \geq \min\{F(x), F(y), 0.5\}$
$(B4)$ $(\forall x, y, z \in M)F((xy)z) \geq \min\{F(x), F(y), 0.5\}$.

Proof. It is similar to proof of theorem **??**. ■

Definition 15 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M if*
(i) $F(xy) \geq \min\{F(x), F(y), 0.5\}$, for all $x, y \in M$.
(ii) $F((xa)(yz)) \geq \min\{F(x), F(y), F(z), 0.5\}$, for all $x, a, y, z \in M$.

Example 16 *Let $M = \{1, 2, 3\}$ be a right regular modular groupoid and "$\cdot$" be any binary operation defined as follows:*

$\cdot$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Let F be a fuzzy subset of M such that

$$F(1) = 0.6, \qquad F(2) = 0.3, \qquad F(3) = 0.2.$$

Then we can see easily $F(1 \cdot 3) \geq F(3) \wedge 0.5$ that is F is an $(\in, \in \vee q)$-fuzzy left ideal but F is not an $(\in, \in \vee q)$-fuzzy right ideal.

Definition 17 *A fuzzy subset f is called $(\in, \in \vee q_k)$-fuzzy quasi-ideal of AG-groupoid S, if*

$$f(x) \geq \left\{(f \circ S)(x) \wedge (S \circ f)(x) \wedge \frac{1-k}{2}\right\} \text{ for all } x \in S.$$

Now we are going to develop $(\in, \in \vee q_k)$-fuzzy $(1,2)$ ideals in AG-groupoids.

Definition 18 *Let S be an AG-groupoid, and f be an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S. Then f is an $(\in, \in \vee q_k)$-fuzzy $(1,2)$ ideal of S, if for all x, a, y, $z \in S$ and t, r, $s \in (0,1]$, we have $x_t \in f$, $y_r \in f$ and $z_s \in f \Longrightarrow ((xa)(yz))_{(t \wedge r) \wedge s} \in \vee q_k f$.*

Theorem 19 *Let f be a non-zero (α, β)- fuzzy $(1,2)$ ideal of S. Then the set $f_0 = \{x \in S \mid f(x) > 0\}$ is $(1,2)$ ideal of S.*

Proof. Let $x,\ y \in f_0 \subseteq S$, then $f(x) > 0$ and $f(y) > 0$. Assume that $f(xy) = 0$. If $\alpha \in \{\in, \in \vee q\}$ then $x_{f(x)}\alpha f$ and $y_{f(y)}\alpha f$ but $f(xy) = 0 < f(x) \wedge f(y)$ and $f(xy) + \min\{f(x), f(y)\} \leq 0 + 1 = 1$. So $(xy)_{f(x)\wedge f(y)}\overline{\beta} f$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Note that $x_1 q f$ and $y_1 q f$ but $(xy)_{1\wedge 1} = (xy)_1\overline{\beta} f$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $f(xy) > 0$, that is $xy \in f_0$. Thus f_0 is an AG-subgroupoid of S.

Let $x,\ y,\ z \in f_0$ and $a \in S$, then $f(x) > 0$, $f(y) > 0$ and $f(z) > 0$. Assume that $f((xa)(yz)) = 0$. If $\alpha \in \{\in, \in \vee q\}$ then $x_{f(x)}\alpha f$, $y_{f(y)}\alpha f$ and $z_{f(z)}\alpha f$ but $f((xa)(yz)) = 0 < \min\{f(x), f(y), f(z)\}$ and $f((xa)(yz)) + \min\{f(x), f(y), f(z)\} \leq 0 + 1 = 1$. So $((xa)(yz))_{f(x)\wedge f(y)\wedge f(z)}\overline{\beta} f$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Note that $x_1 q f$, $y_1 q f$ and $z_1 q f$ but $((xa)(yz))_{1\wedge 1\wedge 1} = ((xa)(yz))_1\overline{\beta} f$ for every$\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $f((xa)(yz)) > 0$, that is, $(xa)(yz) \in f_0$. Consequently, f_0 is an $(1,2)$ ideal of S. ∎

Theorem 20 *For a fuzzy subset f of an AG-groupoid S, the following are equivalent,*

(i) f is a fuzzy $(1,2)$ ideal of S

(ii) f is an $(\in,\in)$-fuzzy $(1,2)$ ideal.

Proof. $(i) \Longrightarrow (ii)$

Let $x, y \in S$ and $t, r \in (0,1]$ be such that $x_t,\ y_r \in f$. Then $f(x) \geq t$, and $f(y) \geq r$. Now by definition $f(xy) \geq f(x) \wedge f(y) \geq t \wedge r$, implies that $(xy)_{t\wedge r} \in f$. Now let $x,\ a,\ y,\ z \in S$ and $t, r, s \in (0,1]$ be such that $x_t,\ y_r,\ z_s \in f$. Then.$f(x) \geq t$ $f(y) \geq r$ and $f(z) \geq s$. Now by definition $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z) \geq t \wedge r \wedge s$, which implies that $((xa)(yz))_{\min\{t,r,s\}} \in f$. Therefore f is an $(\in,\in)$-fuzzy $(1,2)$ ideal of S.

$(ii) \Longrightarrow (i)$

Let $x, y \in S$. Since $x_{f(x)} \in f$ and $y_{f(y)} \in f$, since f is an $(\in,\in)$-fuzzy $(1,2)$ ideal, so $(xy)_{f(x)\wedge f(y)} \in f$, it follows that $f(xy) \geq f(x) \wedge f(y)$, and let $x,\ a,\ y,\ z \in S$. Since $x_{f(x)} \in f$, $y_{f(y)} \in f$and $z_{f(z)} \in f$ and f is an $(\in,\in)$-fuzzy $(1,2)$ ideal so $((xa)(yz))_{f(x)\wedge f(y)\wedge f(z)} \in f$, it follows that $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z)$, so f is a fuzzy $(1,2)$ ideal of S. ∎

1.4 $(\in, \in \vee q_k)$-fuzzy Ideals in AG-groupoids

Theorem 21 *Let A be a $(1,2)$ ideal of S and let f be a fuzzy subset of S such that,*

$$f(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) f is a $(q, \in \vee q_k)$-fuzzy subsemigroup of S.

(2) f is an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S.

Proof. (1) Let x, a, y, $z \in S$ and t, r, $s \in (0,1]$, be such that $x_t q f$, $y_r q f$ and $z_s q f$. Then $x, y, z \in A$, $f(x) + t > 1$ $f(y) + r > 1$ and $f(z) + s > 1$. Since A is an $(1,2)$ ideal of S, we have $(xa)(yz) \in A$ for $a \in S$. Thus $f((xa)(yz)) \geq \frac{1-k}{2}$. If $\min\{t,r,s\} \leq \frac{1-k}{2}$, then $f((xa)(yz)) \geq \min\{t,r,s\}$ and so $((xa)(yz))_{\min\{t,r,s\}} \in f$. If $\min\{t,r,s\} > \frac{1-k}{2}$, then $f((xa)(yz)) + \min\{t,r,s\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $((xa)(yz))_{\min\{t,r,s\}} \in \vee q_k f$. Hence f is a $(q, \in \vee q)$-fuzzy $(1,2)$ ideal of S.

(2) Let $x, a, y, z \in S$ and t, r, $s \in (0,1]$ be such that $x_t \in f$, $y_r \in f$, and $z_s \in f$. Then $f(x) \geq t > 0$ $f(y) \geq r > 0$ and $f(z) \geq s > 0$. Thus $f(x) \geq \frac{1-k}{2}$ $f(y) \geq \frac{1-k}{2}$ and $f(z) \geq \frac{1-k}{2}$, this implies that $x, y, z \in A$. Since A is an $(1,2)$ ideal of S, we have $(xa)(yz) \in A$. Thus $f((xa)(yz)) \geq \frac{1-k}{2}$. If $\min\{t,r,s\} \leq \frac{1-k}{2}$, then $f((xa)(yz)) \geq \min\{t,r,s\}$ and so $((xa)(yz))_{\min\{t,r,s\}} \in f$. If $\min\{t,r,s\} > \frac{1-k}{2}$, then $f(xy)+\min\{t,r,s\}+k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $((xa)(yz))_{\min\{t,r,s\}} \in \vee q_k f$. Hence f is a $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of S. ■

Lemma 22 *Let f be a fuzzy subset of AG-groupoid S, then f is an $(\in, \in \vee q_k)$-fuzzy $(1,2)$ ideal of S if and only if*

(i) $f(xy) \geq \min\{f(x),\ f(y),\ \frac{1-k}{2}\}$, for all $x, y \in S$,

(ii) $f((xa)(yz)) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$, for all $x, a, y, z \in S$.

Proof. Let f be $(\in, \in \vee q_k)$-fuzzy $(1,2)$ ideal of S, then $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$, for all $x, y \in S$ is automatically satisfied. On contrary suppose that there exists $x, a, y, z \in S$ such that $f((xa)(yz)) < \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$. Choose $t \in (0,1]$ such that $f((xa)(yz)) < t \leq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$ then $f((xa)(yz))+t+k < \frac{1-k}{2}+\frac{1-k}{2}+k = 1$, so $((xa)(yz))_{\min\{t,t,t\}} \overline{\in \vee q_k} f$, which is contradiction. Hence $f((xa)(yz)) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\}$, for all $x, a, y, z \in S$.

Conversely suppose that (i) and (ii) holds. From (i) and it is clear that f is $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S. Now let $x_t \in f$, $y_r \in f$ and $z_s \in f$ for $t, r, s \in (0.1]$, then $f(x) \geq t$, $f(y) \geq r$ and $f(z) \geq s$. Now $f((xa)(yz)) \geq \min\{f(x), f(y), f(z), \frac{1-k}{2}\} \geq \min\{t, r, s, \frac{1-k}{2}\}$. If $\min\{t,r,s\} > \frac{1-k}{2}$, then $f((xa)(yz)) \geq \frac{1-k}{2}$. So $f((xa)(yz)) + \min\{t,r,s\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, which implies that $((xa)(yz))_{\min\{t,r,s\}} q_k f$. If $\min\{t,r,s\} \leq \frac{1-k}{2}$, then $f((xa)(yz)) \geq \min\{t,r,s\}$. So $((xa)(yz))_{\min\{t,r,s\}} \in f$. Thus $((xa)(yz))_{\min\{t,r,s\}} \in \vee q_k f$. Therefore f is an $(\in, \in \vee q_k)$-fuzzy $(1,2)$-ideal of S. ■

Proposition 23 *Let f be an $(\in, \in \vee q_k)$-fuzzy $(1,2)$ ideal of S, then f_k is fuzzy $(1,2)$ ideal of S.*

Proof. Let f be an $(\in, \in \vee q_k)$-fuzzy $(1,2)$ ideal of S, then for all $x, a, y, z \in S$, we have $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2}$. This implies that $f((xa)(yz)) \wedge \frac{1-k}{2} \geq f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2}$. So $f_k((xa)(yz)) \geq f_k(x) \wedge f_k(y) \wedge f_k(z)$. Thus f_k is fuzzy $(1,2)$ ideal of S. ■

Lemma 24 *For a fuzzy subset f of an AG-groupoid S, the following conditions are true.*

(i) f_k is a fuzzy left (right) ideal of S if and only if $S \circ_k f \leq f_k$ $(f \circ_k S \leq f_k)$.
(ii) f_k is a fuzzy AG-subgroupoid of S if and only if $f \circ_k f \leq f_k$.

Lemma 25 *Let A be a non-empty subset of an AG-groupoid S. Then the following properties holds.*

(i) A is an AG-subgroupoid of S if and only if $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S.
(ii) A is a left (right, two-sided) ideal of S if and only if C_A is an $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S.

Lemma 26 *For any non-empty subsets A and B of an AG-groupoid S, the following conditions are true.*

(i) $C_A \circ_k C_B = (C_{AB})_k$
(ii) $C_A \wedge_k C_B = (C_{A\cap B})_k$

Lemma 27 *Let f and g be fuzzy subset of AG-groupoid S. Then the following holds,*
(i) $(f \wedge_k g) = (f_k \wedge g_k)$
(ii) $(f \vee_k g) = (f_k \vee g_k)$
(iii) $(f \circ_k g) = (f_k \circ g_k)$
(iv) $f_k(x) = f(x) \wedge \frac{1-k}{2}$.

Proof. It is easy. ■

Lemma 28 *Let A and B be non-empty subsets of a AG-groupoid S, then the following holds.*
(i) $(C_A \wedge_k C_B) = (C_{A\cap B})_k$
(ii) $(C_A \vee_k C_B) = (C_{A\cup B})_k$
(i) $(C_A \circ_k C_B) = (C_{AB})_k$.

Definition 29 *Let f and g be any two fuzzy subsets of an AG-groupoid S, then the product $f \circ_k g$ is defined by,*

$$(f \circ_k g)(a) = \begin{cases} \bigvee\limits_{a=bc} \left\{ f(b) \wedge g(c) \wedge \frac{1-k}{2} \right\}, \text{ if there exists } b,\ c \in S, \text{ such that } a = bc. \\ 0, \quad \text{otherwise.} \end{cases}$$

Example 30 *Let $S = \{a, b, c, d, e\}$ be an AG-groupoid with left identity d with the following multiplication table.*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

Note that S is non-commutative as $ed \neq de$ and also S is non-associative because $(cc)d \neq c(cd)$.

Clearly S is right regular because, $a = a^2d$, $b = b^2c$, $c = c^2c$, $d = d^2d$, $e = e^2e$.

Define a fuzzy subset f of S as follows: $f(a) = 1$ and $f(b) = f(c) = f(d) = f(e) = 0$, then clearly f is a $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S.

It is easy to observe that every $(\in, \in \vee q_k)$-fuzzy two sided ideal of an AG-groupoid S is a $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S but the converse is not true in general which is discussed in the following.

Let us define a fuzzy subset f of S as follows: $f(a) = 1$, $f(b) = 0$ and $f(c) = f(d) = f(e) = 0.5$, then f is a $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S but it is not a $(\in, \in \vee q_k)$-fuzzy two sided ideal of S because $f(db) \not\geq f(d) \wedge \frac{1-k}{2}$ or $f(bd) \not\geq f(d) \wedge \frac{1-k}{2}$.

Definition 31 *An element a of an AG-groupoid S is called a right regular if there exists $x \in S$ such that $a = a^2x$ and S is called right regular if every element of S is right regular.*

An AG-groupoid S considered in Example 30 is right regular because, $a = a^2d$, $b = b^2c$, $c = c^2c$, $d = d^2d$, $e = e^2e$.

Example 32 *Let $S = \{a, b, c, d, e\}$ be a right regular AG-groupoid with left identity c in the following multiplication table.*

$\cdot$	a	b	c	d	e
a	b	a	a	a	a
b	a	b	b	b	b
c	a	b	c	d	e
d	a	b	e	c	d
e	a	b	d	e	c

Theorem 33 *Let S be an AG-groupoid with left identify and let f be any fuzzy subset of S, then S is right regular if $f_k(x) = f_k(x^2)$ holds for all x in S.*

Proof. Assume that S is an AG-groupoid with left identify. Clearly x^2S is a subset of S and therefore its characteristic function C_{x^2S} is a fuzzy subset of

S let $x \in S$. Now by given assumption $(C_{x^2S})_k(x^2) = (C_{x^2S})_k(x)$ holds for all $x \in S$. As $x^2 \in x^2S$ therefore $(C_{x^2S})_k(x^2) = \frac{1-k}{2} \Longrightarrow (C_{x^2S})_k(x) = \frac{1-k}{2}$ which implies that $x \in x^2S$. Thus S is right regular. ■

The converse is not true in general. For this, let us consider a right regular AG-groupoid S in Example 32. Define a fuzzy subset f of S as follows: $f(a) = 0.6$, $f(b) = 0$ and $f(c) = f(d) = f(e) = 0.9$, then it is easy to see that $f_k(a) \neq f_k(a^2)$ for $a \in S$.

Lemma 34 *A fuzzy subset f of a right regular AG-groupoid S is an $(\in, \in \vee q_k)$-fuzzy left ideal of S if and only if it is an $(\in, \in \vee q_k)$-fuzzy right ideal of S.*

Proof. Let S be a right regular AG-groupoid and let $a \in S$, then there exists $x \in S$ such that $a = a^2x$. Let f be a $(\in, \in \vee q_k)$-fuzzy left ideal of S, then by using (1), we have

$$\begin{aligned} f(ab) &= f(((aa)x)b) = f(((xa)a)b) \\ &= f((ba)(xa)) \geq \left\{ f(xa) \wedge \frac{1-k}{2} \right\} \\ &\geq \left\{ f(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \right\} = f(a) \wedge \frac{1-k}{2}. \end{aligned}$$

Similarly we can show that every $(\in, \in \vee q_k)$-fuzzy right ideal of S is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. ■

Example 35 *Let us consider an AG-groupoid $S = \{a, b, c, d, e\}$ with left identity d in the following Cayley's table.*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

Note that S is not right regular because for $c \in S$ there does not exists $x \in S$ such that $c = c^2x$.

Definition 36 *The symbols $f \wedge_k g$ and $f \vee_k g$ will means the following fuzzy subsets of S*

$$\begin{aligned} (f \wedge_k g)(x) &= \min\left\{ f(x),\ g(x), \frac{1-k}{2} \right\},\ \textit{for all } x \textit{ in } S. \\ (f \vee_k g)(x) &= \max\left\{ f(x),\ g(x), \frac{1-k}{2} \right\},\ \textit{for all } x \textit{ in } S. \end{aligned}$$

Lemma 37 *If f is a $(\in, \in \vee q_k)$-fuzzy interior ideal of a right regular AG-groupoid S with left identity, then $f_k(ab) = f_k(ba)$ holds for all a, b in S.*

Proof. Let f be a $(\in, \in \vee q_k)$-fuzzy interior ideal of a right regular AG-groupoid S with left identity and let $a \in S$, then $a = a^2x$ for some x in S. Then we have

$$\begin{aligned}
f_k(a) &= f(a) \wedge \frac{1-k}{2} = f((aa)x) \wedge \frac{1-k}{2} \\
&= f((xa)a) \wedge \frac{1-k}{2} = f((xa)((aa)x)) \wedge \frac{1-k}{2} \\
&= f((aa)((xa)x)) \wedge \frac{1-k}{2} = f((ea^2)((xa)x)) \wedge \frac{1-k}{2} \\
&\geq f(a^2) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} = f_k(a^2) = f(aa) \wedge \frac{1-k}{2} \\
&= f(a((aa)x)) \wedge \frac{1-k}{2} = f((aa)(ax)) \wedge \frac{1-k}{2} \\
&= f((xa)(aa)) \wedge \frac{1-k}{2} = f((xa)a^2) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}$$

Which implies that $f_k(a) = f_k(a^2)$ for all a in S.
Now we have

$$\begin{aligned}
f_k(ab) &= f(ab) \wedge \frac{1-k}{2} = f((ab)^2) \wedge \frac{1-k}{2} \\
&= f((ab)(ab)) \wedge \frac{1-k}{2} = f((ba)(ba)) \wedge \frac{1-k}{2} \\
&= f((e(ba))(ba)) \wedge \frac{1-k}{2} \geq f(ba) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \\
&= f(ba) \wedge \frac{1-k}{2} = f(b((aa)x)) \wedge \frac{1-k}{2} \\
&= f((aa)(bx)) \wedge \frac{1-k}{2} = f((ab)(ax)) \wedge \frac{1-k}{2} \\
&= f((e(ab))(ax)) \wedge \frac{1-k}{2} \geq f(ab) \wedge \frac{1-k}{2} \\
&= f_k(ab).
\end{aligned}$$

Therefore $f_k(ab) = f_k(ba)$ holds for all a, b in S. ■

The converse is not true in general. for this, let us define a fuzzy subset f of a right regular AG-groupoid S in Example 30 as follows: $f(a) = 0$, $f(b) = 0.2, f(c) = 0.6, f(d) = 0.4$ and $f(e) = 0.6$, then it is easy to see that $f(ab) = f(ba)$ holds for all a and b in S but f is not a fuzzy interior ideal of S because $f((ab)c) \ngeq f(b) \wedge \frac{1-k}{2}$.

Theorem 38 *Let S be an AG-groupoid with left identity and. Let f be any*

$(\in, \in \vee q_k)$-fuzzy interior ideal of S, then $f_k(ab) = f_k(ba)$ holds for all a, b in S if S right regular.

Proof. Assume that S is a right regular AG-groupoid with left identity and let f be a fuzzy interior ideal of S, then by using Lemma 37, $f_k(ab) = f_k(ba)$ holds for all a, b in S. ∎

The converse is not true in general. For this, let us consider an AG-groupoid S in Example 35 with left identity d. Let us define a fuzzy subset f of S as follows: $f(a) = f(b) = f(c) = 0.4$, $f(d) = 0.2$ and $f(e) = 0.5$, then it is easy to see that f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S such that $f_k(ab) = f_k(ba)$ holds for all a and b in S but S is not right regular.

Note that S itself is a fuzzy subset such that $S(x) = 1$ for all $x \in S$.

Lemma 39 *For any fuzzy subset f of a right regular AG-groupoid S, $S \circ_k f = f_k$.*

Proof. It is simple. ∎

Note that for any two fuzzy subsets f and g of S, $f \subseteq g$ means that $f(x) \leq g(x)$ for all x in S.

Lemma 40 *In a right regular AG-groupoid S, $f \circ_k S = f_k$ and $S \circ_k f = f_k$ holds for every $(\in, \in \vee q_k)$-fuzzy two-sided ideal f of S.*

Proof. Let S be a right regular AG-groupoid. Now for every $a \in S$ there exists $x \in S$ such that $a = a^2 x$. Then by using (1), we have $a = (aa)x = (xa)a$, therefore

$$\begin{aligned}
(f \circ_k 1)(a) &= (f \circ 1)(a) \wedge \frac{1-k}{2} = \bigvee_{a=(xa)a} \{f(xa) \wedge S(a)\} \wedge \frac{1-k}{2} \\
&\geq f(xa) \wedge 1(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \geq f(a) \wedge 1 \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}$$

It is easy to observe from Lemma 39 that $S \circ_k f = f_k$ holds for every fuzzy two-sided ideal f of S. ∎

Lemma 41 *In a right regular AG-groupoid S, $S \circ S = S$.*

Proof. It is simple. ∎

Theorem 42 *In a right regular AG-groupoid S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S.

(ii) f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

Proof. $(i) \Longrightarrow (ii)$ is simple.

$(ii) \Longrightarrow (i)$: Let S be a right regular AG-groupoid with left identity, then for $b \in S$ there exists $y \in S$ such that $b = b^2 y$ and let f be a $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, then we have

$$\begin{aligned} f(ab) &= f(a((bb)y)) = f((bb)(ay)) = f(((ay)b)b) = f(((ay)((bb)y))b) \\ &= f(((ay)((yb)b))b) = f(((a(yb))(yb))b) = f(((by)((yb)a))b) \\ &= f(((yb)((by)a))b) = f(((a(by))(by))b) = f((b((a(by))y))b) \\ &\geq f(b) \wedge f(b) \wedge \frac{1-k}{2} = f(b) \wedge \frac{1-k}{2}. \end{aligned}$$

Which shows that f is a $(\in, \in \vee q_k)$-fuzzy left ideal of S. Now we have

$$\begin{aligned} f(ab) &= f(((aa)x)b) = f(((xa)a)b) = f((ba)(xa)) = f((ax)(ab)) \\ &= f(((ab)x)a) = f(((((aa)x)b)x)a) = f(((xb)((aa)x))a) \\ &= f(((aa)((xb)x))a) = f(((a(xb))(ax))a) \\ &= f((a((a(xb))x))a) \geq f(a) \wedge f(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge \frac{1-k}{2}. \end{aligned}$$

Which shows that f is a $(\in, \in \vee q_k)$-fuzzy right ideal of S and therefore f is a $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S. ■

Theorem 43 *In a right regular LA-semigroup S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$-ideal of S.
(ii) f is an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Assume that f is an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$-ideal of a right regular LA-semigroup S with left identity and let $a \in S$, then there exists $y \in S$ such that $a = a^2 y$. Now we have

$$\begin{aligned} f(xa) &= f(x((aa)y)) = f((aa)(xy)) = f((((aa)y)a)(xy)) \\ &= f(((ay)(aa))(xy)) = f(((aa)(ya))(xy)) \\ &= f(((xy)(ya))(aa)) = f(((ay)(yx))a^2) \\ &= f(((((aa)y)y)(yx))a^2) = f((((yy)(aa))(yx))a^2) \\ &= f((((aa)y^2)(yx))a^2) = f((((yx)y^2)(aa))a^2) \\ &= f((a(((yx)y^2)a))(aa)) \geq f(a) \wedge f(a) \wedge f(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge \frac{1-k}{2}. \end{aligned}$$

This shows that f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S and f is an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S.

$(ii) \Longrightarrow (i)$ is obvious. ■

Lemma 44 *In a right regular AG-groupoid S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.
(ii) f is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S.

Proof. $(i) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$: Let S be a right regular AG-groupoid with left identity and let $a \in S$, then there exists $x \in S$ such that $a = a^2x$. Let f be a $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, then

$$\begin{aligned} f(ab) &= f(((aa)x)b) = f(((aa)(ex))b) = f(((xe)(aa))b) \\ &= f((a((xe)a))b) \geq f(a) \wedge f(b) \wedge \frac{1-k}{2}. \end{aligned}$$

Which shows that f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. ∎

Lemma 45 *Every $(\in, \in \vee q_k)$-fuzzy right ideal of an AG-groupoid S with left identity becomes an $(\in, \in \vee q_k)$-fuzzy left ideal of S.*

Proof. Let S be an AG-groupoid with left identity and let f be an $(\in, \in \vee q_k)$-fuzzy right ideal. Now

$$f(ab) = f((ea)b) = f((ba)e) \geq f(b) \wedge \frac{1-k}{2}.$$

Therefore f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. ∎

The converse is not true in general. For this, let us define a fuzzy subset f of an AG-groupoid S in Example 35 as follows: $f(a) = 0.8, f(b) = 0.5$, $f(c) = 0$, $f(d) = 0.3$ and $f(e) = 0.6$, then it is easy to observe that f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S but it is not an $(\in, \in \vee q_k)$-fuzzy right ideal of S, because $f(bd) \ngeq f(b) \wedge \frac{1-k}{2}$.

Theorem 46 *In a right regular AG-groupoid S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S.
(ii) f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

Proof. Let S be a right regular AG-groupoid with left identity then for any a, b, x and $y \in S$ there exists a', b', x' and $y' \in S$ such that $a = a^2a'$, $b = b^2b'$, $x = x^2x'$ and $y = y^2y'$.

$(i) \Longrightarrow (ii)$: Let f be a $(\in, \in \vee q_k)$-fuzzy interior ideal of S, then

$$\begin{aligned} f((xa)y) &= f((((xx)x')a)y) = f((((x'x)x)a)y) = f(((ax)(x'x))y) \\ &= f(((xx')(xa))y) = f((((xa)x')x)y) \geq f(x) \wedge \frac{1-k}{2}. \end{aligned}$$

Again we have

$$\begin{aligned} f((xa)y) &= f((xa)((yy)y')) = f((yy)((xa)y')) \\ &= f((((xa)y')y)y) \geq f(y) \wedge \frac{1-k}{2}. \end{aligned}$$

Which implies that $f((xa)y) \geq f(x) \wedge f(y) \wedge \frac{1-k}{2}$. Now

$$f(ab) = f(((aa)a^{'})b) = f(((a^{'}a)a)b) = f((ba)(a^{'}a)) \geq f(a) \wedge \frac{1-k}{2}$$

and

$$f(ab) = f(a((bb)b^{'})) = f((bb)(ab^{'})) \geq f(b) \wedge \frac{1-k}{2}.$$

Thus f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.
$(ii) \Longrightarrow (i)$: Let f be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, then

$$\begin{aligned} f((xa)y) &= f((x((aa)a^{'}))y) = f(((aa)(xa^{'}))y) = f(((ax)(aa^{'}))y) \\ &= f((y(aa^{'}))(ax)) = f((a(ya^{'}))(ax)) = f(((ax)(ya^{'}))a) \\ &= f(((a^{'}y)(xa))a) = f(((a^{'}y)(x((aa)a^{'})))a) \\ &= f(((a^{'}y)((aa)(xa^{'})))a) = f(((a^{'}y)((a^{'}x)(aa)))a) \\ &= f(((a^{'}y)(a((a^{'}x)a)))a) = f((a((a^{'}y)((a^{'}x)a)))a) \\ &\geq f(a) \wedge f(a) \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2}. \end{aligned}$$

Which shows that f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S. ■

Theorem 47 *In a right regular AG-groupoid S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.
(ii) f is an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$ ideal of S.
Proof. $(i) \Longrightarrow (ii)$: Let S be a right regular AG-groupoid with left identity and let $x, a, y, z \in S$, then there exists $x^{'} \in S$ such that $x = x^2x^{'}$. Let f be

an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, then

$$\begin{aligned}
f((xa)(yz)) &= f((zy)(ax)) = f(((ax)y)z) \geq f((ax)y) \wedge f(z) \wedge \frac{1-k}{2} \\
&\geq f(ax) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \\
&= f(ax) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f(a((xx)x^{'})) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f((xx)(ax^{'})) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f(((ax^{'})x)x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f(((ax^{'})((xx)x^{'}))x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f(((ax^{'})((xx)(ex^{'})))x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f(((ax^{'})((x^{'}e)(xx)))x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f(((ax^{'})(x((x^{'}e)x)))x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f(x((ax^{'})((x^{'}e)x))x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&\geq f(x) \wedge f(x) \wedge \frac{1-k}{2} \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2}.
\end{aligned}$$

Which shows that f is an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$ ideal of S.

$(ii) \Longrightarrow (i)$: Again let S be a right regular AG-groupoid with left identity, then for any a, b, x and $y \in S$ there exists $a^{'}$, $b^{'}$, $x^{'}$ and $y^{'} \in S$ such that $a = a^2a^{'}$, $b = b^2b^{'}$, $x = x^2x^{'}$ and $y = y^2y^{'}$. Let f be a $(\in, \in \vee q_k)$-fuzzy $(1, 2)$ ideal of S, then

$$\begin{aligned}
f((xa)y) &= f((xa)((yy)y^{'})) = f((yy)((xa)y^{'})) = f((y^{'}(xa))(yy)) = f((x(y^{'}a))(yy)) \\
&\geq f(x) \wedge f(y) \wedge f(y) \wedge \frac{1-k}{2} \geq f(x) \wedge f(y) \wedge \frac{1-k}{2}.
\end{aligned}$$

Now

$$\begin{aligned}
f(ab) &= f(a((bb)b^{'})) = f((bb)(ab^{'})) = f((b^{'}a)(bb)) = f(b^{'}((aa)a^{'}))(bb) \\
&= f((aa)(b^{'}a^{'}))(bb) = f((a^{'}b^{'})(aa))(bb) \geq f(a((a^{'}b^{'})a))(bb) \wedge \frac{1-k}{2} \\
&= f(a) \wedge f(b) \wedge f(b) \wedge \frac{1-k}{2} = f(a) \wedge f(b) \wedge \frac{1-k}{2}.
\end{aligned}$$

Which shows that f is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S. ■

Theorem 48 *In a right regular AG-groupoid S with left identity, the following statements are equivalent.*

(i) f is an $(\in,\in\vee q_k)$-fuzzy $(1,2)$ ideal of S.
(ii) f is an $(\in,\in\vee q_k)$-fuzzy interior ideal of S.

Proof. $(i)\Longrightarrow(ii)$: Let S be a right regular AG-groupoid with left identity and let $x,a,y,z\in S$, then there exists $a^{'}\in S$ such that $a=a^2a^{'}$. Let f be a $(\in,\in\vee q_k)$-fuzzy $(1,2)$ ideal of S, then

$$\begin{aligned}
f((xa)(yz)) &= f((x((aa)a^{'}))(yz))=f(((aa)(xa^{'}))(yz))=f((((xa^{'})a)a)(yz))\\
&= f((((xa^{'})((aa)a^{'}))a)(yz))=f((((aa)((xa^{'})a^{'}))a)(yz))\\
&= f(((yz)a)((aa)((xa^{'})a^{'})))=f((aa)(((yz)a)((xa^{'})a^{'})))\\
&= f(((aa)(a^{'}(xa^{'})))(a(yz)))=f(((a(yz))(a^{'}(xa^{'})))(aa))\\
&= f(((((aa)a^{'})(yz))(a^{'}(xa^{'})))(aa))=f(((((a^{'}a)a)(yz))(a^{'}(xa^{'})))(aa))\\
&= f((((xa^{'})a^{'})((yz)((a^{'}a)a)))(aa))=f((((xa^{'})a^{'})((yz)((ae)(aa^{'}))))(aa))\\
&= f((((xa^{'})a^{'})((yz)(a((ae)a^{'}))))(aa))\\
&= f((((xa^{'})a^{'})(a((yz)((ae)a^{'}))))(aa))\\
&= f((a(((xa^{'})a^{'})((yz)((ae)a^{'}))))(aa))\\
&\geq f(a)\wedge f(a)\wedge f(a)=f(a)\wedge\frac{1-k}{2}.
\end{aligned}$$

Which shows that f is an $(\in,\in\vee q_k)$-fuzzy interior ideal of S.

$(ii)\Longrightarrow(i)$: Again let S be a right regular AG-groupoid with left identity and let $x,a,y,z\in S$, then there exists $x^{'}$and $z^{'}\in S$ such that $x=x^2x^{'}$ and $z=z^2z$. Now

$$f((xa)(yz))=f((zy)(ax))\geq f(y)\wedge\frac{1-k}{2}.$$

Now

$$\begin{aligned}
f((xa)(yz)) &= f((((xx)x^{'})a)(yz))=f(((ax^{'})(xx))(yz))\\
&= f(((xx)(x^{'}a))(yz))=f((((x^{'}a)x)x)(yz))\\
&\geq f(x)\wedge\frac{1-k}{2}.
\end{aligned}$$

Now by using (4), we have

$$\begin{aligned}
f((xa)(yz)) &= f((xa)(y(((zz)z^{'}))))=f((xa)((zz)(yz^{'})))\\
&= f((zz)((xa)(yz^{'})))\geq f(z)\wedge\frac{1-k}{2}.
\end{aligned}$$

Thus we get $f((xa)(yz))\geq f(x)\wedge f(y)\wedge f(z)\wedge\frac{1-k}{2}$.

Let a and $b \in S$ then there exists $a^{'}$ and $b^{'} \in S$ such that $a = a^2a^{'}$ and $b = b^2b^{'}$. Now

$$f(ab) = f(((aa)a^{'})b) = f((ba^{'})(aa)) = f((aa)(a^{'}b)) \geq f(a) \wedge \frac{1-k}{2}$$

and

$$f(ab) = f(a((bb)b^{'})) = f((bb)(ab^{'})) \geq f(b) \wedge \frac{1-k}{2}.$$

Thus f is an $(\in, \in \vee q_k)$-fuzzy $(1,2)$ ideal of S. ■

Note that $(\in, \in \vee q_k)$-fuzzy two-sided ideals, $(\in, \in \vee q_k)$-fuzzy bi-ideals, $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals, $(\in, \in \vee q_k)$-fuzzy $(1,2)$ ideals, $(\in, \in \vee q_k)$-fuzzy interior ideals and $(\in, \in \vee q_k)$-fuzzy quasi-ideals coincide in a right regular AG-groupoid with left identity.

Lemma 49 *Let S be an AG-groupoid with left identity, then the following conditions are equivalent.*

(i) S is right regular.

(ii) $f \circ_k f = f_k$ for every $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let S be an AG-groupoid with left identity and let (i) holds. Let $a \in S$, then since S is right regular so by using (1), $a = (aa)x = (xa)a$. Let f be an $(\in, \in \vee q_k)$-fuzzy left ideal of S, then clearly $f \circ_k f \leq f_k$ and also we have

$$\begin{aligned}
(f \circ_k f)(a) &= \bigvee_{a=(xa)a} \{f(xa) \wedge f(a) \wedge \frac{1-k}{2}\} \\
&\geq f(a) \wedge f(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2} \\
&= f_k(a).
\end{aligned}$$

Thus $f \circ_k f = f_k$.

$(ii) \Longrightarrow (i)$: Assume that $f \circ_k f = f_k$ for $(\in, \in \vee q_k)$-fuzzy left ideal of S.. Since Sa is a left ideal of S, therefore, $(C_{Sa})_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Since $a \in Sa$ therefore $(C_{Sa})_k(a) = \frac{1-k}{2}$. Now by using the given assumption and we get

$$(C_{Sa})_k \circ_k (C_{Sa})_k = (C_{Sa})_k \text{ and } (C_{Sa})_k \circ_k (C_{Sa})_k = \left(C_{(Sa)^2}\right)_k$$

Thus we have $\left(C_{(Sa)^2}\right)_k(a) = (C_{Sa})_k(a) = \frac{1-k}{2}$, which implies that $a \in (Sa)^2$. Now

$$a \in (Sa)^2 = (Sa)(Sa) = (aS)(aS) = a^2S.$$

This shows that S is right regular. ■

Note that if an AG-groupoid has a left identity then $S \circ S = S$.

Theorem 50 *For an AG-groupoid S with left identity, then the following conditions are equivalent.*

(i) S is right regular.

(ii) $f_k = (S \circ f) \circ_k (S \circ f)$, where f is any $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let S be a right regular AG-groupoid and let f be any $(\in, \in \vee q_k)$-fuzzy left ideal of S, then clearly $S \circ f$ is also an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Now

$$((S \circ f) \circ_k (S \circ f))(a) = (S \circ f)(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \leq f(a) \wedge \frac{1-k}{2} = f_k(a).$$

Now let $a \in S$, since S is right regular therefore there exists $x \in S$ such that $a = a^2x$ and we have

$$a = (aa)x = (xa)a = (xa)((aa)x) = (xa)((xa)a)$$

Therefore

$$\begin{aligned}
((S \circ f) \circ_k (S \circ f))(a) &= \bigvee_{a=(xa)((xa)a)} \{(S \circ f)(xa) \wedge (S \circ f)((xa)a) \wedge \frac{1-k}{2}\} \\
&\geq (S \circ f)(xa) \wedge (S \circ f)((xa)a) \wedge \frac{1-k}{2} \\
&= \bigvee_{xa=xa} \{S(x) \wedge f(a)\} \wedge \bigvee_{(xa)a=(xa)a} \{S(xa) \wedge f(a)\} \wedge \frac{1-k}{2} \\
&\geq S(x) \wedge f(a) \wedge S(xa) \wedge f(a) \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2} \\
&= f_k(a).
\end{aligned}$$

Thus we get the required $f_k = (S \circ f) \circ_k (S \circ f)$.

$(ii) \Longrightarrow (i)$: Let $f_k = (S \circ f) \circ_k (S \circ f)$ holds for any $(\in, \in \vee q_k)$-fuzzy left ideal f of S, then by given assumption, we have

$$\begin{aligned}
f_k(a) &= ((S \circ f) \wedge_k (S \circ f))(a) \leq (f \circ_k f)(a) \\
&\leq (S \circ_k f)(a) \leq f_k(a).
\end{aligned}$$

Thus S is right regular. ∎

Lemma 51 *In a right regular LA-semigroup S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy quasi ideal of S.

(ii) $(f \circ S) \wedge_k (S \circ f) = f_k$.

Proof. $(i) \Longrightarrow (ii)$ is easy.

$(ii) \Longrightarrow (i)$ is obvious. ∎

Theorem 52 *Let S be a right regular AG-groupoid with left identity, then the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S.
(ii) f is an $(\in, \in \vee q_k)$-fuzzy right ideal of S.
(iii) f is an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S.
(iv) f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.
(v) f is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S.
(vi) f is an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$ ideal of S.
(vii) f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S.
$(viii)$ f is an $(\in, \in \vee q_k)$-fuzzy quasi ideal of S.
(ix) $f \circ_k S = f_k$ and $S \circ_k f = f_k$.

Proof. $(i) \Longrightarrow (ix)$: Let f be an $(\in, \in \vee q_k)$-fuzzy left ideal of a right regular AG-groupoid S. Let $a \in S$ then there exists $a' \in S$ such that $a = a^2a'$. Now

$$a = (aa)a' = (a'a)a \text{ and } a = (aa)a' = (aa)(ea') = (a'e)(aa).$$

Therefore

$$\begin{aligned}(f \circ_k S)(a) &= \bigvee_{a=(a'a)a} \left\{ f(a'a) \wedge S(a) \wedge \frac{1-k}{2} \right\} \geq \left\{ f(a'a) \wedge 1 \wedge \frac{1-k}{2} \right\} \\ &\geq f(a) \wedge \frac{1-k}{2} = f_k(a)\end{aligned}$$

and

$$\begin{aligned}(S \circ_k f)(a) &= \bigvee_{a=(a'e)(aa)} \left\{ S(a'e) \wedge f(aa) \wedge \frac{1-k}{2} \right\} \geq \left\{ 1 \wedge f(aa) \wedge \frac{1-k}{2} \right\} \\ &\geq f(a) \wedge \frac{1-k}{2} = f_k(a).\end{aligned}$$

Now we get the required $f \circ_k S = f_k$ and $S \circ_k f = f_k$.

$(ix) \Longrightarrow (viii)$ is obvious.

$(viii) \Longrightarrow (vii)$: Let f be an $(\in, \in \vee q_k)$-fuzzy quasi ideal of a right regular AG-groupoid S. Now for $a \in S$ there exists $a' \in S$ such that $a = a^2a'$ and therefore by using (3) and (4), we have

$$(xa)y = (xa)(ey) = (ye)(ax) = a((ye)x)$$

and

$$(xa)y = (x((aa)a'))y = ((aa)(xa'))y = ((a'x)(aa))y = (a((a'x)a))y = (y((a'x)a))a.$$

Since f is a fuzzy quasi ideal of S, therefore we have

$$f_k((xa)y) = ((f \circ S) \wedge_k (S \circ f))((xa)y) = (f \circ S)((xa)y) \wedge (S \circ f)((xa)y) \wedge \frac{1-k}{2}.$$

Now

$$(f \circ S)((xa)y) = \bigvee_{(xa)y=a((ye)x)} \{f(a) \wedge S((ye)x)\} \geq f(a)$$

and

$$(S \circ f)((xa)y) = \bigvee_{(xa)y=(y((a'x)a))a} \left\{S(y((a'x)a)) \wedge f(a)\right\} \geq f(a).$$

Which implies that $f_k((xa)y) \geq f(a) \wedge \frac{1-k}{2} \Longrightarrow f((xa)y) \geq f(a) \wedge \frac{1-k}{2}$. Thus f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S.

$(vii) \Longrightarrow (vi)$ is followed by Theorem 48.
$(vi) \Longrightarrow (v)$ is followed by Theorem 47.
$(v) \Longrightarrow (iv)$ is followed by Lemma 44.
$(iv) \Longrightarrow (iii)$ is followed by Lemma 42.
$(iii) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (i)$ are an easy consequences of Lemma 34. ■

Theorem 53 *In a right regular AG-groupoid S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy bi-(generalized bi-) ideal of S.
(ii) $(f \circ_k S) \circ_k f = f_k$ and $f \circ_k f = f_k$.

Proof. $(i) \Longrightarrow (ii)$: Let f be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of a right regular AG-groupoid S with left identity and let $a \in S$ then there exists $x \in S$ such that $a = a^2x$. Now by using (1), (4) and (3), we have

$$\begin{aligned} a &= (aa)x = (xa)a = (x((aa)x))a = ((aa)(xx))a = ((xx)(aa))a = (((aa)x)x)a \\ &= (((xa)a)x)a = (((x((aa)x))a)x)a = ((((aa)(xx))a)x)a = ((((xx)(aa))a)x)a \\ &= (((a(x^2a))a)x)a \end{aligned}$$

Therefore

$$\begin{aligned} ((f \circ_k S) \circ_k f)(a) &= \bigvee_{a=(((a(x^2a))a)x)a} \left\{(f \circ_k S)(((a(x^2a))a)x) \wedge f(a) \wedge \frac{1-k}{2}\right\} \\ &\geq (f \circ_k S)(((a(x^2a))a)x) \wedge f(a) \wedge \frac{1-k}{2} \\ &= \bigvee_{((a(x^2a))a)x=((a(x^2a))a)x} \left[\left\{f((a(x^2a))a) \wedge S(x) \wedge \frac{1-k}{2}\right\} \wedge f(a) \wedge \frac{1-k}{2}\right] \\ &\geq f((a(x^2a))a) \wedge 1 \wedge f(a) \wedge \frac{1-k}{2} \geq \left\{f(a) \wedge f(a) \wedge \frac{1-k}{2}\right\} \wedge f(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge \frac{1-k}{2} = f_k(a). \end{aligned}$$

Now

$$a = (aa)x = (xa)a = (x((aa)x))a = ((aa)(xx))a = ((xx)(aa))a = (a(x^2a))a.$$

Therefore

$$\begin{aligned}
((f \circ_k S) \circ_k f)(a) &= \bigvee_{a=(a(x^2a))a} \left\{ (f \circ_k S)((a(x^2a))) \wedge f(a) \wedge \frac{1-k}{2} \right\} \\
&= \bigvee_{a=(a(x^2a))a} \left(\bigvee_{a(x^2a)=a(x^2a)} \left\{ f(a) \wedge S(x^2a) \wedge \frac{1-k}{2} \right\} \wedge f(a) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(a(x^2a))a} \left(\bigvee_{a(x^2a)=a(x^2a)} \{ f(a) \wedge 1 \} \right) \wedge f(a) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=(a(x^2a))a} \left(\bigvee_{a(x^2a)=a(x^2a)} f(a) \right) \wedge f(a) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=(a(x^2a))a} \left\{ f(a) \wedge f(a) \wedge \frac{1-k}{2} \right\} \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=(a(x^2a))a} \left\{ f((a(x^2a))a) \right\} \wedge \frac{1-k}{2} \\
&= f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}$$

Thus $(f \circ_k S) \circ_k f = f_k$.
Now

$$\begin{aligned}
a &= (aa)x = (xa)a = (x((aa)x))a = ((aa)(xx))a = (((xx)a)a)a \\
&= (((xx)((aa)x))a)a = (((xx)((xa)a))a)a = (((xx)((ae)(ax)))a)a \\
&= (((xx)(a((ae)x)))a)a = ((a((xx)((ae)x)))a)a
\end{aligned}$$

Therefore

$$\begin{aligned}
(f \circ_k f)(a) &= \bigvee_{a=((a((xx)((ae)x)))a)a} \left\{ f((a((xx)((ae)x)))a) \wedge f(a) \wedge \frac{1-k}{2} \right\} \\
&\geq \left\{ f((a((xx)((ae)x)))a) \wedge f(a) \wedge \frac{1-k}{2} \right\} \\
&\geq f(a) \wedge f(a) \wedge f(a) \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}$$

Now by using Lemma 24, $f \circ_k f = f_k$.
$(ii) \Longrightarrow (i)$: Let f be a fuzzy subset of a right regular AG-groupoid S,

then

$$\begin{aligned} f((xy)z) &= ((f \circ_k S) \circ_k f)((xy)z) = \bigvee_{(xy)z=(xy)z} \left\{ (f \circ_k S)(xy) \wedge f(z) \wedge \frac{1-k}{2} \right\} \\ &\geq \bigvee_{xy=xy} \left\{ f(x) \wedge S(y) \wedge \frac{1-k}{2} \right\} \wedge f(z) \wedge \frac{1-k}{2} \\ &\geq f(x) \wedge 1 \wedge f(z) \wedge \frac{1-k}{2} = f(x) \wedge f(z) \wedge \frac{1-k}{2}. \end{aligned}$$

Since $f \circ_k f = f_k$ therefore by Lemma 24, f is a $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S. This shows that f is an $(\in, \in \vee q_k)$-fuzzy bi ideal of S. ■

Theorem 54 *In a right regular AG-groupoid S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S.
(ii) $(S \circ_k f) \circ_k S = f_k$.
Proof. It is simple. ■

Theorem 55 *In a right regular AG-groupoid S with left identity, the following statements are equivalent.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$ ideal of S.
(ii) $(f \circ_k S) \circ_k (f \circ_k f) = f_k$ and $f \circ_k f = f_k$.
Proof. $(i) \Longrightarrow (ii)$: Let f be an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$ ideal of a right regular AG-groupoid S with left identity and let $a \in S$ then there exists $x \in S$ such that $a = a^2 x$. Now

$$\begin{aligned} a &= (aa)x = (xa)a = (xa)((aa)x) = (aa)((xa)x) = (a((aa)x))((xa)x) \\ &= ((aa)(ax))((xa)x) = (((xa)x)(ax))(aa) = (a(((xa)x)x))(aa). \end{aligned}$$

Therefore

$$\begin{aligned} ((f \circ_k S) \circ_k (f \circ_k f))(a) &= \bigvee_{a=(a(((xa)x)x))(aa)} \left\{ \begin{array}{c} (f \circ_k S)(a(((xa)x)x)) \wedge (f \circ_k f)(aa) \\ \wedge \frac{1-k}{2} \end{array} \right\} \\ &\geq \left\{ (f \circ_k S)(a(((xa)x)x)) \wedge (f \circ_k f)(aa) \wedge \frac{1-k}{2} \right\}. \end{aligned}$$

Now

$$\begin{aligned} (f \circ_k S)(a(((xa)x)x)) &= \left\{ (f \circ S)(a(((xa)x)x)) \wedge \frac{1-k}{2} \right\} \\ &= \bigvee_{a(((xa)x)x)=a(((xa)x)x)} \left\{ \{ f(a) \wedge S(((xa)x)x) \} \wedge \frac{1-k}{2} \right\} \\ &\geq f(a) \wedge S(((xa)x)x) \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2} \\ &= f_k(a) \end{aligned}$$

and

$$\begin{aligned}(f\circ_k f)(aa) &= \left\{(f\circ f)(aa)\wedge\frac{1-k}{2}\right\}=\left\{\bigvee_{aa=aa}\{f(a)\wedge f(a)\}\wedge\frac{1-k}{2}\right\}\\ &\geq f(a)\wedge\frac{1-k}{2}=f_k(a).\end{aligned}$$

Thus we get

$$((f\circ_k S)\circ_k(f\circ_k f))(a)\geq f_k(a).$$

Now

$$\begin{aligned}a &= (aa)x=(((aa)x)((aa)x))x=((aa)(((aa)x)x))x=((aa)((xx)(aa)))x\\ &= ((aa)(x^2(aa)))x=(x(x^2(aa)))(aa)=(x(a(x^2a)))(aa)\\ &= (a(x(x^2a)))(aa)=(a(x(x^2((aa)x))))(aa)=(a(x((aa)x^3)))(aa).\end{aligned}$$

Therefore

$$((f\circ_k S)\circ_k(f\circ_k f))(a)=\bigvee_{a=(a(x((aa)x^3)))(aa)}\left\{\begin{array}{c}(f\circ_k S)(a(x((aa)x^3)))\wedge\\ (f\circ_k f)(aa)\wedge\frac{1-k}{2}\end{array}\right\}.$$

Now

$$\begin{aligned}(f\circ_k S)(a(x((aa)x^3))) &= \bigvee_{a(x((aa)x^3))=a(x((aa)x^3))}\left\{f(a)\wedge S(x((aa)x^3))\wedge\frac{1-k}{2}\right\}\\ &= \bigvee_{a(x((aa)x^3))=a(x((aa)x^3))}\left\{f(a)\wedge\frac{1-k}{2}\right\}\end{aligned}$$

and

$$(f\circ_k f)(aa)=\bigvee_{aa=aa}\left\{f(a)\wedge f(a)\wedge\frac{1-k}{2}\right\}=\bigvee_{aa=aa}\left\{f(a)\wedge\frac{1-k}{2}\right\}.$$

Therefore

$$\begin{aligned}(f\circ_k S)(a(x((aa)x^3)))\wedge(f\circ_k f)(aa) &= \left\{\begin{array}{c}\bigvee_{a(x((aa)x^3))=a(x((aa)x^3))}\left\{f(a)\wedge\frac{1-k}{2}\right\}\wedge\\ \bigvee_{aa=aa}\left\{f(a)\wedge\frac{1-k}{2}\right\}\end{array}\right\}\\ &= \bigvee_{a(x((aa)x^3))=a(x((aa)x^3))}\left\{f(a)\wedge f(a)\wedge\frac{1-k}{2}\right\}.\end{aligned}$$

Thus from above, we get

$$
\begin{aligned}
((f \circ_k S) \circ_k (f \circ_k f))(a) &= \bigvee_{a=(a(x((aa)x^3)))(aa)} \left(\bigvee_{\substack{a(x((aa)x^3))=a(x((aa)x^3)) \\ \wedge \frac{1-k}{2}}} \left\{ \begin{array}{c} f(a) \wedge f(a) \\ \wedge \frac{1-k}{2} \end{array} \right\} \right) \\
&= \bigvee_{a=(a(x((aa)x^3)))(aa)} \left\{ f(a) \wedge f(a) \wedge \frac{1-k}{2} \right\} \\
&\leq \bigvee_{a=(a(x((aa)x^3)))(aa)} \left\{ f((a(x((aa)x^3)))(aa)) \wedge \frac{1-k}{2} \right\} \\
&= f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}
$$

Thus $(f \circ_k S) \circ_k (f \circ_k f) = f_k$.
Now by using (1) and (4), we have

$$
\begin{aligned}
a &= (aa)x = (xa)a = (x((aa)x))a = ((aa)(xx))a = ((a((aa)x))x^2)a \\
&= (((aa)(ax))x^2)a = ((x^2(ax))(aa))a = ((ax^3)(aa))a.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(f \circ_k f)(a) &= (f \circ f)(a) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=((ax^3)(aa))a} \left\{ f(((ax^3)(aa))) \wedge f(a) \right\} \wedge \frac{1-k}{2} \\
&\geq \left\{ f(((ax^3)(aa))) \wedge f(a) \right\} \wedge \frac{1-k}{2} \\
&\geq \left\{ f(a) \wedge \frac{1-k}{2} \right\} \wedge f(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}
$$

Now by using Lemma 24, $f \circ_k f = f_k$.
$(ii) \Longrightarrow (i)$: Let f be a fuzzy subset of a right regular AG-groupoid S. Now since $f \circ_k f = f_k$ therefore by Lemma 24, f is a $(\in, \in \vee q_k)$-fuzzy

AG-subgroupoid of S

$$\begin{aligned}
f((xa)(yz)) &= ((f \circ_k S) \circ_k (f \circ_k f))((xa)(yz)) \\
&= ((f \circ S) \circ (f \circ f))((xa)(yz)) \wedge \frac{1-k}{2} \\
&= ((f \circ S) \circ f)((xa)(yz)) \wedge \frac{1-k}{2} \\
&= \bigvee_{(xa)(yz)=(xa)(yz)} \{(f \circ S)(xa) \wedge f(yz)\} \wedge \frac{1-k}{2} \\
&\geq (f \circ S)(xa) \wedge f(yz) \wedge \frac{1-k}{2} \\
&= \bigvee_{(xa)=(xa)} \{f(x) \wedge S(a)\} \wedge f(yz) \wedge \frac{1-k}{2} \\
&\geq f(x) \wedge 1 \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2} \\
&= f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2}.
\end{aligned}$$

Thus we get $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2}$ and thus f is an $(\in, \in \vee q_k)$-fuzzy $(1,2)$ ideal of S. ■

A subset A of an AG-groupoid S is called semiprime if $a^2 \in A$ implies $a \in A$.

The subset $\{a, b\}$ of an AG-groupoid S in Example 30 is semiprime.

A fuzzy subset f of an AG-groupoid S is called a fuzzy semiprime if $f(a) \geq f(a^2)$ for all a in S.

Definition 56 *A fuzzy subset f is called an $(\in, \in \vee q_k)$-fuzzy semiprime if for all $x \in S,\ t \in (0,1]$ we have the following condition*

$$x_t^2 \in f \Longrightarrow x_t \in \vee q_k f.$$

Lemma 57 *Let f be a fuzzy subset of AG-groupoid S, then f is an $(\in, \in \vee q_k)$-fuzzy semiprime if and only if $f(x) \geq \min\{f(x^2), \frac{1-k}{2}\}$, for all $x \in S$.*

Proof. It is similar to the proof of Lemma 22. ■

Let us define a fuzzy subset f of an AG-groupoid S in Example 35 as follows: $f(a) = 0.2$, $f(b) = 0.5$, $f(c) = 0.6$, $f(d) = 0.1$ and $f(e) = 0.4$, then f is an $(\in, \in \vee q_k)$-fuzzy semiprime.

Lemma 58 *For a right regular AG-groupoid S, the following holds.*

(i) Every $(\in, \in \vee q_k)$-fuzzy right ideal of S is an $(\in, \in \vee q_k)$-fuzzy semiprime.

(ii) Every $(\in, \in \vee q_k)$-fuzzy left ideal of S is an $(\in, \in \vee q_k)$-fuzzy semiprime if S has a left identity.

Proof. (i) : It is simple.

(ii) : Let f be a $(\in, \in \vee q_k)$-fuzzy left ideal of a right regular AG-groupoid S and let $a \in S$ then there exists $x \in S$ such that $a = a^2x$. Now by using (3), we have

$$f(a) = f((aa)(ex)) = f((xe)a^2) \geq f(a^2) \wedge \frac{1-k}{2}.$$

Which shows that f is an $(\in, \in \vee q_k)$-fuzzy semiprime. ■

Lemma 59 *A right(resp: left and two-sided) ideal R of an AG-groupoid S is semiprime if and only if $(C_R)_k$ are $(\in, \in \vee q_k)$-fuzzy semiprime.*

Proof. Let R be any right ideal of an AG-groupoid S, then by Lemma 25, $(C_R)_k$ is a $(\in, \in \vee q_k)$-fuzzy right ideal of S. Now let $a \in S$ then by given assumption $(C_R)_k(a) \geq (C_R)_k(a^2)$. Let $a^2 \in R$, then $(C_R)_k(a^2) = \frac{1-k}{2} \Longrightarrow$ $(C_R)_k(a) = \frac{1-k}{2}$ which implies that $a \in R$. Thus every right ideal of S is semiprime. The converse is simple.

Similarly every left and two-sided ideal of an AG-groupoid S is semiprime if and only if their characteristic functions are $(\in, \in \vee q_k)$-fuzzy semiprime. ■

Lemma 60 *Let S be an AG-groupoid, then every right (left, two-sided) ideal of S is semiprime if every fuzzy right (left, two-sided) ideal of S is an $(\in, \in \vee q_k)$-fuzzy semiprime.*

Proof. The direct part can be easily followed by Lemma 59. ■

The converse is not true in general. For this, let us consider an AG-groupoid S in Example 35. It is easy to observe that the only left ideals of S are $\{a, b, e\}$, $\{a, c, e\}$, $\{a, b, c, e\}$ and $\{a, e\}$ which are semiprime. Clearly the right and two sided ideals of S are $\{a, b, c, e\}$ and $\{a, e\}$ which are also semiprime. Now on the other hand, if we define a fuzzy subset f of S as follows: $f(a) = f(b) = f(c) = 0.2$, $f(d) = 0.1$ and $f(e) = 0.3$, then f is a fuzzy right (left, two-sided) ideal of S but f is not an $(\in, \in \vee q_k)$-fuzzy semiprime because $f(c) \ngeq f(c^2) \wedge \frac{1-k}{2}$.

Lemma 61 *Let S be an AG-groupoid with left identity, then the following statements are equivalent.*

(i) S is right regular.

(ii) Every $(\in, \in \vee q_k)$-fuzzy right (left, two-sided) ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime.

Proof. $(i) \Longrightarrow (ii)$ is followed by Lemma 58.

$(ii) \Longrightarrow (i)$: Let S be an AG-groupoid with left identity and let every fuzzy right (left, two-sided) ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime. Since a^2S is a right and also a left ideal of S, therefore by using Lemma 60, $(C_{a^2S})_k$ is $(\in, \in \vee q_k)$-semiprime. Now clearly $a^2 \in a^2S$, therefore $a \in a^2S$, which shows that S is right regular. ■

Theorem 62 *For an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) S is right regular.

(ii) Every $(\in, \in \vee q_k)$-fuzzy right ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime.

(iii) Every $(\in, \in \vee q_k)$-fuzzy left ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime.

Proof. $(i) \Longrightarrow (iii)$ and $(ii) \Longrightarrow (i)$ are followed by Lemma 61.

$(iii) \Longrightarrow (ii)$: Let S be an AG-groupoid and let f be a $(\in, \in \vee q_k)$-fuzzy right ideal of S, then by using Lemma 45, f is a $(\in, \in \vee q_k)$-fuzzy left ideal of S and therefore by given assumption f is a $(\in, \in \vee q_k)$-fuzzy semiprime. ■

Theorem 63 *For an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) S is right regular.

(ii) $R \cap L = RL$, R is any right ideal and L is any left ideal of S where R is semiprime.

(iii) $f \wedge_k g = f \circ_k g$, where f is any $(\in, \in \vee q_k)$-fuzzy right ideal and g is any $(\in, \in \vee q_k)$-fuzzy left ideal of S where f is a $(\in, \in \vee q_k)$-fuzzy semiprime.

Proof. $(i) \Longrightarrow (iii)$: Let S be a right regular AG-groupoid and let f is any $(\in, \in \vee q_k)$-fuzzy right ideal and g is any $(\in, \in \vee q_k)$-fuzzy left ideal of S. Now for $a \in S$ there exists $x \in S$ such that $a = a^2x$. Now by using (1), we have

$$a = (aa)x = (xa)a = ((ex)a)a = ((ax)e)a.$$

Therefore

$$\begin{aligned}(f \circ_k g)(a) &= \bigvee_{a=((ax)e)a} \left\{ f((ax)e) \wedge g(a) \wedge \frac{1-k}{2} \right\} \geq f(a) \wedge g(a) \wedge \frac{1-k}{2} \\ &= (f \wedge_k g)(a).\end{aligned}$$

Which implies that $f \circ_k g \geq f \wedge_k g$ and obviously $f \circ_k g \leq f \wedge_k g$. Thus $f \circ_k g = f \wedge_k g$ and by Lemma 58, f is a $(\in, \in \vee q_k)$-fuzzy semiprime .

$(iii) \Longrightarrow (ii)$: Let R be any right ideal and L be any left ideal of an AG-groupoid S, then by Lemma 25, $(C_R)_k$ and $(C_L)_k$ are $(\in, \in \vee q_k)$-fuzzy right and $(\in, \in \vee q_k)$-fuzzy left ideals of S respectively. As $RL \subseteq R \cap L$ is obvious therefore let $a \in R \cap L$, then $a \in R$ and $a \in L$. Now by using Lemma 26 and given assumption, we have

$$\begin{aligned}(C_{RL})_k(a) &= (C_R \circ_k C_L)(a) = (C_R \wedge_k C_L)(a) \\ &= C_R(a) \wedge C_L(a) \wedge \frac{1-k}{2} = \frac{1-k}{2}.\end{aligned}$$

Which implies that $a \in RL$ and therefore $R \cap L = RL$. Now by using Lemma 59, R is semiprime.

$(ii) \Longrightarrow (i)$: Let S be an AG-groupoid, then clearly Sa is a left ideal of S such that $a \in Sa$ and a^2S is a right ideal of S such that $a^2 \in a^2S$. Since by assumption, a^2S is semiprime therefore $a \in a^2S$. Now by using $(3), (1)$ and (4), we have

$$\begin{aligned} a &\in a^2S \cap Sa = (a^2S)(Sa) = (aS)(Sa^2) = ((Sa^2)S)a = ((Sa^2)(SS))a \\ &= ((SS)(a^2S))a = (a^2((SS)S))a \subseteq (a^2S)S = (SS)(aa) = a^2S. \end{aligned}$$

Which shows that S is right regular. $\blacksquare$

2

Generalized Fuzzy Ideals of Abel Grassmann Groupoids

In this chapter, we investigate some characterizations of regular and intra-regular Abel-Grassmann's groupoids in terms of $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy quasi-ideals.

An element a of an AG-groupoid S is called **regular** if there exist $x \in S$ such that $a = (ax)a$ and S is called **regular**, if every element of S is regular. An element a of an AG-groupoid S is called **intra-regular** if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called **intra-regular**, if every element of S is intra-regular.

The following definitions for AG-groupoids are same as for semigroups in [36].

Definition 64 *(1) A fuzzy subset δ of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, r \in (0, 1]$, it satisfies,*

$x_t \in \delta$, $y_r \in \delta$ implies that $(xy)_{\min\{t,r\}} \in \vee q_k \delta$.

(2) A fuzzy subset δ of S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if for all $x, y \in S$ and $t, r \in (0, 1]$, it satisfies,

$x_t \in \delta$ implies $(yx)_t \in \vee q_k \delta$ ($x_t \in \delta$ implies $(xy)_t \in \vee q_k \delta$).

(3) A fuzzy AG-subgroupoid f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if for all $x, y, z \in S$ and $t, r \in (0, 1]$ the following condition holds.

$y_t \in f$ implies $((xy)z)_t \in \vee q_k f$.

(4) A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S if for all $x \in S$ it satisfies, $f(x) \geq \min(f \circ C_S(x), C_S \circ f(x), \frac{1-k}{2})$, where C_S is the fuzzy subset of S mapping every element of S on 1.

(5) A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if $x_t \in f$ and $z_r \in S$ implies $((xy)z)_{\min\{t,r\}} \in \vee q_k f$, for all $x, y, z \in S$ and $t, r \in (0, 1]$.

(6) A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if for all $x, y, z \in S$ and $t, r \in (0, 1]$ the following conditions hold

(i) If $x_t \in f$ and $y_r \in S$ implies $(xy)_{\min\{t,r\}} \in \vee q_k f$,

(ii) If $x_t \in f$ and $z_r \in f$ implies $((xy)z)_{\min\{t,r\}} \in \vee q_k f$.

Theorem 65 *[36] (1) Let δ be a fuzzy subset of S. Then δ is an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S if $\delta(xy) \geq \min\{\delta(x), \delta(y), \frac{1-k}{2}\}$.*

(2) A fuzzy subset δ of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if
$\delta(xy) \geq \min\{\delta(y), \frac{1-k}{2}\}$ $\left(\delta(xy) \geq \min\{\delta(x), \frac{1-k}{2}\}\right)$.

(3) A fuzzy subset f of an AG-groupoid S is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if and only if it satisfies the following conditions.
(i) $f(xy) \geq \min\left\{f(x), f(y), \frac{1-k}{2}\right\}$ *for all* $x, y \in S$ *and* $k \in [0,1)$.
(ii) $f((xy)z) \geq \min\left\{f(y), \frac{1-k}{2}\right\}$ *for all* $x, y, z \in S$ *and* $k \in [0,1)$.

(4) Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if
(i) $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$ *for all* $x, y \in S$ *and* $k \in [0,1)$,
(ii) $f((xy)z) \geq \min\{f(x), f(z), \frac{1-k}{2}\}$ *for all* $x, y, z \in S$ *and* $k \in [0,1)$.

Here we begin with examples of an AG-groupoid.

Example 66 *Let us consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$\cdot$	1	2	3
1	2	2	2
2	3	3	3
3	3	3	3

Note that S has no left identity. Define a fuzzy subset $F : S \longrightarrow [0,1]$ as follows:

$$F(x) = \begin{cases} 0.9 \text{ for } x = 1 \\ 0.5 \text{ for } x = 2 \\ 0.6 \text{ for } x = 3 \end{cases}$$

Then clearly F is an $(\in, \in \vee q_k)$-fuzzy ideal of S.

Example 67 *Let us consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$*$	1	2	3
1	3	1	2
2	2	3	1
3	1	2	3

Obviously 3 is the left identity in S. Define a fuzzy subset $G : S \longrightarrow [0,1]$ as follows:

$$G(x) = \begin{cases} 0.8 \text{ for } x = 1 \\ 0.6 \text{ for } x = 2 \\ 0.5 \text{ for } x = 3 \end{cases}$$

Then clearly G is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

Lemma 68 *Intersection of two ideals of an AG-groupoid is an ideal.*

Proof. It is easy. ■

Lemma 69 *Let S be an AG-groupoid. If $a = a(ax)$, for some x in S. Then $a = a^2y$, for some y in S.*

Proof. Using medial law, we get $a = a(ax) = [a(ax)](ax) = (aa)((ax)x) = a^2y$, where $y = (ax)x$. ■

Lemma 70 *Let S be an AG-groupoid with left identity. If $a = a^2x$, for some x in S. Then $a = (ay)a$, for some y in S.*

Proof. Using medial law, left invertive law, (1), paramedial law and medial law, we get

$$\begin{aligned} a &= a^2x = (aa)x = ((a^2x)(a^2x))x = ((a^2a^2)(xx))x = (xx^2)(a^2a^2) \\ &= a^2((xx^2)a^2)) = ((xx^2)a^2)a)a = ((aa^2)(xx^2))a = ((x^2x)(a^2a))a \\ &= [a^2\{(x^2x)a\}]a = [\{a(x^2x)\}(aa)]a = [a(\{a(x^2x)\}a)]a \\ &= (ay)a, \text{ where } y = \{a(x^2x)\}a. \end{aligned}$$

■

Lemma 71 *Let S be an AG-groupoid with left identity. Then the following holds.*

(i) $(aS)a^2 = (aS)a, \quad (aS)((aS)a) = (aS)a, \quad S((aS)a) = (aS)a.$

(ii) $(Sa)(aS) = a(aS), \quad (aS)(Sa) = (aS)a., \quad [a(aS)]S = (aS)a.$

(iii) $[(Sa)S](Sa) = (aS)(Sa), \quad (Sa)S = (aS), \quad S(Sa) = Sa, \quad Sa^2 = a^2S.$

Proof. Straightforward. ■

Lemma 72 *Any $(\in, \in \vee q_k)$-fuzzy left ideal of an intra regular AG-groupoid is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal.*

Proof. We get

$$\begin{aligned} S \circ_k f(a) &= \bigvee_{a=pq} \left\{ S(p) \wedge f(q) \wedge \frac{1-k}{2} \right\} \\ &\geq \left\{ S((y \cdot xa)) \wedge f(a) \wedge \frac{1-k}{2} \right\} \\ &= \left\{ 1 \wedge f(a) \wedge \frac{1-k}{2} \right\} \\ &= f(a) \wedge \frac{1-k}{2} \geq f_k(a). \end{aligned}$$

Also

$$
\begin{aligned}
S \circ_k f(a) &= \bigvee_{a=pq} S(p) \wedge f(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} 1 \wedge f(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} f(q) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \\
&\leq f(pq) \wedge \frac{1-k}{2} \\
&= f_k(a).
\end{aligned}
$$

Thus $S \circ_k f(a) = f_k(a) \leq f(a)$.

$$
\begin{aligned}
f(a) &\geq S \circ_k f(a) \wedge f \circ_k S(a) \\
&= S \circ f(a) \wedge \frac{1-k}{2} \wedge f \circ S(a) \wedge \frac{1-k}{2} \\
&= \min\{S \circ f(a), f \circ S(a), \frac{1-k}{2}\}
\end{aligned}
$$

■

Lemma 73 *Any $(\in, \in \vee q_k)$-fuzzy right ideal of an intra regular AG-groupoid is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal.*

Proof. We see that

$$
\begin{aligned}
f \circ_k S(a) &= \bigvee_{a=pq} \left\{ f(p) \wedge S(q) \wedge \frac{1-k}{2} \right\} \\
&\geq \left\{ f(a) \wedge S([(x \cdot y_2 y_1)a]) \wedge \frac{1-k}{2} \right\} \\
&= \left\{ f(a) \wedge 1 \wedge \frac{1-k}{2} \right\} \\
&= f_k(a).
\end{aligned}
$$

Also

$$\begin{aligned}
f \circ_k S(a) &= \bigvee_{a=pq} f(p) \wedge S(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} f(p) \wedge 1 \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} f(p) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=pq} f(pq) \wedge \frac{1-k}{2} \\
&= f_k(a).
\end{aligned}$$

Thus $f \circ_k S(a) = f_k(a) \leq f(a)$.

$$\begin{aligned}
f(a) &\geq S \circ_k f(a) \wedge f \circ_k S(a) \\
&= S \circ f(a) \wedge \frac{1-k}{2} \wedge f \circ S(a) \wedge \frac{1-k}{2} \\
&= \min\{S \circ f(a), f \circ S(a), \frac{1-k}{2}\}.
\end{aligned}$$

■

2.1 Some Characterizations of AG-groupoids by $(\in, \in \vee q_k)$-fuzzy Ideals

Theorem 74 *For an AG-groupoid with left identity, the following are equivalent.*
(i) S is intra-regular
(ii) $I[a] \cap J[a] \subseteq I[a]J[a]$, for all a in S.
(iii) $I \cap J \subseteq IJ$, for any left ideal I and quasi-ideal J of S.
(iv) $f \wedge_k g \leq f \circ_k g$, for any $(\in, \in \vee q_k)$-fuzzy left ideal f and $(\in, \in \vee q_k)$-fuzzy quasi-ideal g of S.

Proof. $(i) \Longrightarrow (iv)$
Let f and g be $(\in, \in \vee q)$-fuzzy left and quasi-ideals of an intra-regular AG-groupoid S with left identity. For each a in S there exists x, y in S such

that $a=(xa^2)y$. Then we get

$$\begin{aligned}
(f\circ_k g)(a) &= \bigvee_{a=pq}\left\{f(p)\wedge g(q)\wedge\frac{1-k}{2}\right\}\\
&\geq f(sa)\wedge g(a)\wedge\frac{1-k}{2}\\
&=\left\{f(sa)\wedge\frac{1-k}{2}\wedge g(a)\right\}\\
&= f_k(sa)\wedge g(a)\\
&= S\circ_k f(sa)\wedge g(a)\\
&=\bigvee_{sa} S(s)\wedge f(a)\wedge g(a)\wedge\frac{1-k}{2}\\
&= f\wedge_k g(a).
\end{aligned}$$

Therefore $f\circ_k g\geq f\wedge_k g$.

$(iv)\Longrightarrow(iii)$

Let I and J be left and quasi-ideals of an AG-groupoid S with left identity and let $a\in I\cap J$. Then we get

$$\begin{aligned}
(C_{IJ})_k(a) &= (C_I\circ_k C_J)(a)=(C_I\wedge_k C_J)(a)\\
&= (C_{I\cap J})_k(a)\geq\wedge\frac{1-k}{2}.
\end{aligned}$$

Thus $I\cap J\subseteq IJ$.

$(iii)\Longrightarrow(ii)$ It is obvious.

$(ii)\Longrightarrow(i)$

Since $a\cup Sa$ is a principal left and $a\cup Sa\cap aS$ is a principal quasi-ideal of an AG-groupoid S with left identity containing a. Using by (ii), medial law, left invertive law and paramedial law, we get

$$\begin{aligned}
(a\cup Sa)\cap[a\cup(Sa\cap aS)] &\subseteq (a\cup Sa)[a\cup(Sa\cap aS)]\\
&\subseteq (a\cup Sa)\cap(a\cup Sa)\\
&= a^2\cup a(Sa)\cup(Sa)a\cup(Sa)(Sa)\\
&= a^2\cup Sa^2\cup a^2S\cup Sa^2\\
&= a^2\cup Sa^2\\
&= Sa^2=Sa^2\cdot S
\end{aligned}$$

Hence S is intra-regular. ■

Similarly we can prove the following theorem.

Theorem 75 *For an AG-groupoid with left identity, the following are equivalent.*

(i) *S is intra-regular*

(ii) $I[a] \cap J[a] \subseteq I[a]J[a]$, *for all* a *in* S.
(iii) $I \cap J \subseteq IJ$, *for any quasi-ideal* I *and left-ideal* J *of* S.
(iv) $f \wedge_k g \leq f \circ_k g$, *for any* $(\in, \in \vee q_k)$*-fuzzy quasi-ideal* f *and* $(\in, \in \vee q_k)$*-fuzzy left ideal* g *of* S.

Theorem 76 *For an AG-groupoid with left identity* e*, the following are equivalent.*
(i) S *is intra-regular,*
(ii) $Q[a] \cap L[a] \subseteq Q[a]L[a]$, *for all* a *in* S.
(iii) $A \cap B \subseteq BA$, *for any quasi-ideal* A *and left ideal* B *of* S.
(iv) $f \wedge_k g \leq g \circ_k f$, *where* f *is any* $(\in, \in \vee q_k)$*-fuzzy-quasi-ideal and* g *is an* $(\in, \in \vee q_k)$*-fuzzy left ideal.*

Proof. $(i) \Longrightarrow (iv)$
Let f and g be $(\in, \in \vee q_k)$-fuzzy quasi and left ideals of an intra-regular AG-groupoid S with left identity. Since S is intra-regular so for each a in S there exists x, y in S such that $a = xa^2 \cdot y$. The we get

$$\begin{aligned}
(g \circ_k f)(a) &= \bigvee_{a=pq} \left\{ g(p) \wedge f(q) \wedge \frac{1-k}{2} \right\} \\
&\geq g(sa) \wedge f(a) \wedge \frac{1-k}{2} \\
&= (S \circ_k g(sa)) \wedge f(a) \\
&= \bigvee_{sa=cd} S(c) \wedge g(d) \wedge f(a) \wedge \frac{1-k}{2} \\
&\geq S(s) \wedge g(a) \wedge f(a) \wedge \frac{1-k}{2}) \\
&= g(a) \wedge f(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge g(a) \wedge \frac{1-k}{2} = f \wedge_k g(a).
\end{aligned}$$

Thus $g \circ_k f \geq f \wedge_k g$.
$(iv) \Longrightarrow (iii)$ Let A and B be quasi and left ideals of S and $a \in A \cap B$, then we get

$$\begin{aligned}
(C_{AB})_k(a) &= (C_A \circ_k C_B)(a) \geq (C_B \wedge_k C_A)(a) \\
&= (C_{B \cap A})_k(a) \geq \frac{1-k}{2}.
\end{aligned}$$

Therefore $A \cap B \subseteq BA$.
$(iii) \Longrightarrow (ii)$ is obvious
$(ii) \Longrightarrow (i)$
Since $a \cup Sa$ is a principal left and $a \cup Sa \cap aS$ is a principal quasi-ideal

of an AG-groupoid S with left identity containing a. Using by (ii), we get

$$\begin{aligned}(a \cup Sa) \cap [a \cup (Sa \cap aS)] &\subseteq [a \cup (Sa \cap aS)](a \cup Sa) \\ &\subseteq (a \cup Sa) \cap (a \cup Sa) \\ &= a^2 \cup a(Sa) \cup (Sa)a \cup (Sa)(Sa) \\ &= a^2 \cup Sa^2 \cup a^2 S \cup Sa^2 \\ &= a^2 \cup Sa^2 \\ &= Sa^2 = Sa^2 \cdot S\end{aligned}$$

Hence S is intra regular. ■

Similarly we can prove the following theorem.

Theorem 77 *For an AG-groupoid with left identity e, the following are equivalent.*

(i) *S is intra-regular,*

(ii) *$Q[a] \cap L[a] \subseteq Q[a]L[a]$, for all a in S.*

(iii) *$A \cap B \subseteq BA$, for any left ideal A and quasi-ideal B of S.*

(iv) *$f \wedge_k g \leq g \circ_k f$, where f is any $(\in, \in \vee q_k)$-fuzzy left ideal and g is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal.*

Lemma 78 *If I is an ideal of an intra-regular AG-groupoid S with left identity, then $I = I^2$.*

Proof. It is easy. ■

Theorem 79 *Let S be an AG-groupoid with left identity. Then the following are equivalent*

(i) *S is intra-regular.*

(ii) *$L[a] \cap B \cap Q[a] \subseteq L[a]B \cdot Q[a]$, for all a in S and B is any subset of S.*

(iii) *$A \cap B \cap C \subseteq AB \cdot C$, for any left ideal A, subset B and every quasi-ideal C of S.*

(iv) *$f \wedge_k g \wedge_k h \leq (f \circ_k g) \circ_k h$, for any $(\in, \in \vee q_k)$-fuzzy left ideal f, $(\in, \in \vee q_k)$-fuzzy subset g and $(\in, \in \vee q_k)$-fuzzy quasi-ideal h of S.*

Proof. $(i) \Longrightarrow (iv)$

Let f, g and h be $(\in, \in \vee q_k)$-fuzzy left ideal, fuzzy subset and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of an intra-regular AG-groupoid S with left identity. Then we get

$$\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} \left\{ f \circ_k g(p) \wedge h(a) \wedge \frac{1-k}{2} \right\} \\
&\geq \left\{ f \circ_k g(ua^2 \cdot a) \wedge h(a) \wedge \frac{1-k}{2} \right\} \\
&= \bigvee_{ua^2 \cdot a = cd} f(c) \wedge g(d) \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq f(ua^2) \wedge \frac{1-k}{2} \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} \\
&= f_k(ua^2) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} \\
&= S \circ_k f(ua^2) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} \\
&= \bigvee_{ua^2 = rs} S(r) \wedge f(r) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2}] \\
&\geq 1 \wedge f(a^2) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge_k g(a) \wedge_k h(a).
\end{aligned}$$

Thus $f \wedge_k g \wedge_k h \leq (f \circ_k g) \circ_k h$.

$(iv) \Longrightarrow (iii)$ We get

$$\begin{aligned}
(C_{AB \cdot C})_k (a) &= (C_A \circ_k C_B) \circ_k C_C (a) = (C_A \wedge_k C_B \wedge_k C_C)(a) \\
&= (C_{A \cap B \cap C})_k (a) \geq \frac{1-k}{2}.
\end{aligned}$$

Therefore $A \cap B \cap C \subseteq AB \cdot C$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$

$$\begin{aligned}
(a \cup Sa) \cap Sa \cap [a \cup (Sa \cap aS)] &\subseteq [\{(a \cup Sa)\}Sa][a \cup (Sa \cap aS)] \\
&\subseteq [\{(a \cup Sa)\}Sa][a \cup Sa] \\
&= [Sa \cdot Sa] \cdot Sa \\
&\subseteq Sa^2 \cdot S.
\end{aligned}$$

Hence S is intra-regular. ■

Similarly we can prove the following theorems.

Theorem 80 *Let S be an AG-groupoid with left identity. Then the following are equivalent* *(i) S is intra-regular.*

(ii) $L[a] \cap Q[a] \cap B \subseteq L[a]Q[a] \cdot B$, for all a in S and B is any subset of S.

(iii) $A \cap B \cap C \subseteq AB \cdot C$, for any left ideal A, subset C and every quasi-ideal B of S.

(iv) $f \wedge_k g \wedge_k h \leq (f \circ_k g) \circ_k h$, for any $(\in, \in \vee q_k)$-fuzzy left ideal f, $(\in, \in \vee q_k)$-fuzzy subset f and $(\in, \in \vee q_k)$-fuzzy quasi-ideal g of S.

Theorem 81 *Let S be an AG-groupoid with left identity. Then the following are equivalent*

(i) S is intra-regular.

(ii) $Q[a] \cap L[a] \cap A \subseteq Q[a]L[a] \cdot A$, for all a in S and for any subset A of S.

(iii) $Q \cap L \cap A \subseteq QL \cdot A$, for any quasi-ideal Q, subset A and every left ideal L of S.

(iv) $f \wedge_k g \wedge_k h \leq (f \circ_k g) \circ_k h$, for any $(\in, \in \vee q_k)$-fuzzy left ideal g, $(\in, \in \vee q_k)$-fuzzy subset h and $(\in, \in \vee q_k)$-fuzzy quasi-ideal f of S.

Theorem 82 *For an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$, for any $(\in, \in \vee q_k)$-fuzzy quasi-ideal f and for any $(\in, \in \vee q_k)$-fuzzy left ideal g and $(\in, \in \vee q_k)$-fuzzy subset h of S.

Proof. $(i) \Longrightarrow (ii)$ It is same as $(i) \Longrightarrow (iii)$ of theorem 82.

$(ii) \Longrightarrow (i)$

Let f and g be an $(\in, \in \vee q_k)$-fuzzy quasi, left ideals and $(\in, \in \vee q_k)$-fuzzy subset of an AG-groupoid S with left identity. Then

$$\begin{aligned} ((f \wedge_k g) \wedge_k S)(a) &= (f \wedge_k g)(a) \wedge S(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge g(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \\ &= f(a) \wedge g(a) \wedge \frac{1-k}{2} = f \wedge_k g(a). \end{aligned}$$

Therefore $((f \wedge_k g) \wedge_k S) = f \wedge_k g$. Also

$(f \circ_k g) \circ_k S = (S \circ_k g) \circ_k f$. Now

$$\begin{aligned} S \circ_k g(a) &= \bigvee_{a=pq} S(p) \wedge g(q) \wedge \frac{1-k}{2} \\ &= \bigvee_{a=pq} g(q) \wedge \frac{1-k}{2} \\ &\leq f(pq) \wedge \frac{1-k}{2} \\ &= f(a) \wedge \frac{1-k}{2} \leq f(a). \end{aligned}$$

Thus $S \circ_k g \leq g$. Now using (ii), we get

$$\begin{aligned} (f \wedge_k g)(a) &= ((f \wedge_k g) \wedge_k S)(a) \leq ((f \circ_k g) \circ_k S)(a) \\ &= ((S \circ_k g) \circ_k f)(a) \leq g \circ_k f(a). \end{aligned}$$

Therefore by theorem 76, S is intra-regular. ■

Similarly we can prove the following theorems.

Theorem 83 *For an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$, *for any $(\in, \in \vee q_k)$-fuzzy left ideal f and for any $(\in, \in \vee q_k)$-fuzzy quasi-ideal g and $(\in, \in \vee q_k)$-fuzzy subset h of S.*

Theorem 84 *For an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$, *for any $(\in, \in \vee q_k)$-fuzzy subset f and for any $(\in, \in \vee q_k)$-fuzzy left ideal g and $(\in, \in \vee q_k)$-fuzzy quasi-ideal h of S.*

Theorem 85 *For an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$, *for any $(\in, \in \vee q_k)$-fuzzy left ideal f and for any $(\in, \in \vee q_k)$-fuzzy subset ideal g and $(\in, \in \vee q_k)$-fuzzy quasi-ideal h of S.*

2.2 Medial and Para-medial Laws in Fuzzy AG-groupoids

Lemma 86 *Let S be an AG-groupoid with left identity. Then the following holds.*

(i) $(f\circ_k g)\circ_k(h\circ_k\gamma)=(f\circ_k h)\circ_k(g\circ_k\gamma)$.

(ii) $(f\circ_k g)\circ_k(h\circ_k\gamma)=(\gamma\circ_k g)\circ_k(h\circ_k f)$.

(iii) $f\circ_k(g\circ_k h)=g\circ_k(f\circ_k h)$.

Proof. (i) Using medial law we have,

$$\begin{aligned}
(f\circ_k g)\circ_k(h\circ_k\gamma)(a) &= \bigvee_{a=mn}(f\circ_k g)(m)\wedge(h\circ_k\gamma)(n)\wedge\frac{1-k}{2}\\
&= \bigvee_{a=mn}\left\{\begin{array}{c}\left(\bigvee_{m=op} f(o)\wedge g(p)\wedge\frac{1-k}{2}\right)\\ \wedge\left(\bigvee_{n=qr} h(q)\wedge\gamma(r)\wedge\frac{1-k}{2}\right)\end{array}\right\}\wedge\frac{1-k}{2}\\
&= \bigvee_{a=mn=(op)(qr)}\left(f(o)\wedge g(p)\wedge h(q)\wedge\gamma(r)\wedge\frac{1-k}{2}\right)\\
&= \bigvee_{a=mn=(oq)(pr)}\left(f(o)\wedge h(q)\wedge g(p)\wedge\gamma(r)\wedge\frac{1-k}{2}\right)\\
&= \bigvee_{a=m'n'}\left\{\begin{array}{c}\left(\bigvee_{m'=oq} f(o)\wedge h(q)\wedge\frac{1-k}{2}\right)\\ \wedge\left(\bigvee_{n'=pr} g(p)\wedge\gamma(r)\wedge\frac{1-k}{2}\right)\end{array}\right\}\wedge\frac{1-k}{2}\\
&= \bigvee_{a=m'n'}\left\{(f\circ_k g)\left(m'\right)\wedge(h\circ_k\gamma)\left(n'\right)\wedge\frac{1-k}{2}\right\}\\
&= (f\circ_k g)\circ_k(h\circ_k\gamma)(a).
\end{aligned}$$

(ii) Using paramedial law we get,

$$\begin{aligned}
(f \circ_k g) \circ_k (h \circ_k \gamma)(a) &= \bigvee_{a=mn} (f \circ_k g)(m) \wedge (h \circ_k \gamma)(n) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=mn} \left\{ \begin{array}{c} \left(\bigvee_{m=op} f(o) \wedge g(p) \wedge \frac{1-k}{2} \right) \\ \wedge \left(\bigvee_{n=qr} h(q) \wedge \gamma(r) \wedge \frac{1-k}{2} \right) \end{array} \right\} \wedge \frac{1-k}{2} \\
&= \bigvee_{a=mn=(op)(qr)} \left(f(o) \wedge g(p) \wedge h(q) \wedge \gamma(r) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=mn=(rp)(qo)} \left(\gamma(r) \wedge g(p) \wedge h(q) \wedge f(o) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=m'n'} \left\{ \begin{array}{c} \left(\bigvee_{m'=rp} \gamma(r) \wedge g(p) \wedge \frac{1-k}{2} \right) \\ \wedge \left(\bigvee_{n'=qo} h(q) \wedge f(o) \wedge \frac{1-k}{2} \right) \end{array} \right\} \wedge \frac{1-k}{2}
\end{aligned}$$

$$\begin{aligned}
&= \bigvee_{a=m'n'} \left\{ (\gamma \circ_k g) \left(m' \right) \wedge (h \circ_k f) \left(n' \right) \right\} \wedge \frac{1-k}{2} \\
&= (\gamma \circ_k g) \circ_k (h \circ_k f)(a).
\end{aligned}$$

(iii) Using (1) we get,

$$\begin{aligned}
((f \circ_k (g \circ_k h))(a) &= \bigvee_{a=mn} f(m) \wedge (g \circ_k h)(n) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=mn} \left(f(m) \wedge \left(\bigvee_{n=op} g(o) \wedge h(p) \wedge \frac{1-k}{2} \right) \right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=mn=m(op)} (f(m) \wedge \{g(o) \wedge h(p)\}) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=mn=o(mp)} \left(g(o) \wedge (f(m) \wedge h(p)) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=m'n'} \left\{ g\left(m'\right) \circ_k \left(\bigvee_{n'=mp} f(m) \wedge h(p) \wedge \frac{1-k}{2} \right) \right\} \wedge \frac{1-k}{2} \\
&= \bigvee_{a=m'n'} \left(g\left(m'\right) \circ_k (f \circ_k h)\left(n'\right) \right) = g \circ_k (f \circ_k h).
\end{aligned}$$

■

2.3 Certain Characterizations of Regular AG-groupoids

Theorem 87 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For left ideals $L_1,\ L_2$ and ideal I of S, $L_1 \cap I \cap L_2 \subseteq (L_1 I) L_2$.

(iii) $L[a] \cap I[a] \cap L[a] \subseteq (L[a]\, I[a])\, L[a]$,

for some $a \in S$.

Proof. $(i) \Rightarrow (ii)$

Assume that L_1, L_2 are left ideal and I is an ideal of a regular AG-groupoid S. Let $a \in L_1 \cap I \cap L_2$. This implies that $a \in L_1$, $a \in I$ and $a \in L_2$. Now since S is regular so for $a \in S$, there exist $x \in S$, such that $a = (ax)a$. Therefore using left invertive law we get

$$a = [\{(ax)a\}x]a = [(xa)(ax)]a \in [(SL_1)(IS)]L_2 \subseteq (L_1 I) L_2.$$

Hence $L_1 \cap I \cap L_2 \subseteq (L_1 I) L_2$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$

Since $a \cup Sa \cup aS$ and $a \cup Sa$ are principle ideal and left ideal of S generated by a respectively. Thus by (iii) and paramedial law, we have

$$\begin{aligned}(a \cup Sa) \cap (a \cup Sa \cup aS) \cap (a \cup Sa) &\subseteq ((a \cup Sa)(a \cup Sa \cup aS))(a \cup Sa) \\ &\subseteq ((a \cup Sa) S)(a \cup Sa) \\ &= \{aS \cup (Sa) S\} (a \cup Sa) = (aS)(a \cup Sa) \\ &= (aS) a \cup (Sa)(Sa) = (aS) a.\end{aligned}$$

Hence S is regular. ■

Theorem 88 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular

(ii) For $(\in, \in \vee q_k)$-fuzzy left ideals f, h and $(\in, \in \vee q_k)$-fuzzy ideal g of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

(iiii) For $(\in, \in \vee q_k)$-fuzzy quasi-ideals f, h and $(\in, \in \vee q_k)$-fuzzy ideal g of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. $(i) \Rightarrow (iii)$ Assume that f, g are $(\in, \in \vee q_k)$-fuzzy quasi-ideals and g is an $(\in, \in \vee q_k)$-fuzzy ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$, there exist $x \in S$, such that $a = (ax) a$. Therefore using left invertive law and (1), we get

$$a = [\{(ax) a\} x] a = [(xa)(ax)] a = [a\{(xa)(x)\}] a.$$

Thus

$$
\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&\geq \{f(a) \wedge g((xa)\, x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= \{f\,(a) \wedge g\,(a) \wedge \frac{1-k}{2}\} \wedge h\,(a) \wedge \frac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)\,(a)\,.
\end{aligned}
$$

Thus $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$ Assume that L_1, L_2 left ideals and I is an ideal of S. Then $(C_{L_1})_k$, $(C_I)_k$ and $(C_{L_2})_k$ are $(\in, \in \vee q_k)$-fuzzy left ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy left ideal of S respectively. Therefore we have,

$$(C_{L_1 \cap I \cap L_2})_k = (C_{L_1} \wedge_k C_I) \wedge_k C_{L_2} \leq (C_{L_1} \circ_k C_I) \circ_k C_{L_2} = (C_{(L_1 I) L_2})_k.$$

Thus $L_1 \cap I \cap L_2 \subseteq (L_1 I)\, L_2$. Hence S is regular. ■

Theorem 89 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For ideal I and quasi-ideal Q of S, $I \cap Q \subseteq IQ$.

(iii) $I\,[a] \cap Q\,[a] \subseteq I\,[a]\, Q\,[a]\,,$

for some $a \in S$.

Proof. $(i) \Rightarrow (ii)$ Assume that I and Q are ideal and quasi-ideal of a regular AG-groupoid S respectively. Let $a \in I \cap Q$. This implies that $a \in I$ and $a \in Q$. Since S is regular so for $a \in S$ there exist $x \in S$ such that $a = (ax)\, a \in (IS)\, Q \subseteq IQ$. Thus $I \cap Q \subseteq IQ$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$

Since $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (Sa \cap aS)$ are principle ideal and principle quasi-ideal of S generated by a respectively. Thus by (ii), (1), Left invertive law, paramedial law we have,

$$\begin{aligned}
(a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) &\subseteq (a \cup Sa \cup aS)(a \cup (Sa \cap aS)) \\
&\subseteq (a \cup Sa \cup aS)(a \cup Sa) \\
&= aa \cup a(Sa) \cup (Sa)a \cup (Sa)(Sa) \\
&\quad \cup (aS)a \cup (aS)(Sa) \\
&= a^2 \cup a^2S \cup a^2S \cup a^2S \cup (aS)a \cup (aS)a \\
&= a^2 \cup (aS)a \cup a^2S.
\end{aligned}$$

If $a = a^2$, then $a = a^2a$. If $a = a^2x$, for some x in S, then S is regular. ■

Theorem 90 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For $(\in, \in \vee q_k)$-fuzzy ideal f, and $(\in, \in \vee q_k)$-fuzzy quasi-ideal g of S, $f \wedge_k g \leq f \circ_k g$.

Proof. $(i) \Rightarrow (ii)$ Assume that f and g are $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of a regular AG-groupoid S respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that $a = (ax)a$. Thus,

$$\begin{aligned}
(f \circ_k g)(a) &= \bigvee_{a=pq} f(p) \wedge g(q) \wedge \frac{1-k}{2} = \bigvee_{a=pq=(ax)a} f(p) \wedge g(q) \wedge \frac{1-k}{2} \\
&\geq f(ax) \wedge g(a) \wedge \frac{1-k}{2} \geq \left(f(a) \wedge \frac{1-k}{2}\right) \wedge g(a) \wedge \frac{1-k}{2} \\
&= (f \wedge_k g)(a).
\end{aligned}$$

Hence $(f \wedge_k g) \leq (f \circ_k g)$.

$(ii) \Longrightarrow (i)$

Assume that I and Q are ideal and quasi-ideal of S respectively. Then $(C_I)_k$, and $(C_Q)_k$ are $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S. Therefore we have, $(C_{I \cap Q})_k = (C_I \wedge_k C_Q) \leq (C_I \circ_k C_Q) = (C_{IQ})_k$. Therefore $I \cap Q \subseteq IQ$. Hence S is regular. ■

Theorem 91 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For bi-ideals B_1, B_2 and ideal I of S, $B_1 \cap I \cap B_2 \subseteq (B_1 I) B_2$.

(iii) $B[a] \cap I[a] \cap B[a] \subseteq (B[a] I[a]) B[a]$, for some $a \in S$.

Proof. $(i) \Rightarrow (ii)$

Assume that B_1, B_2 are bi-ideal and I is an ideal of regular AG-groupoid S. Let $a \in B_1 \cap I \cap B_2$. This implies that $a \in B_1$, $a \in I$ and $a \in B_2$. Now since S is regular so for $a \in S$, there exist $x \in S$, such that $a = (ax)\,a$. Therefore $a = (a\,((xa)\,x))\,a \in (B_1\,((SI)\,S))\,B_2 \subseteq (B_1 I)\,B_2$. Thus $B_1 \cap I \cap B_2 \subseteq (B_1 I)\,B_2$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$

Since $B\,[a] = a \cup a^2 \cup (aS)\,a$ and $I\,[a] = a \cup Sa \cup aS$ are principle bi-ideal and principle ideal of S generated by a respectively. Thus by (iii), (1), and left invertive law, medial law and paramedial law, we have

$$\begin{aligned}
& \left(a \cup a^2 \cup (aS)\,a\right) \cap (a \cup Sa \cup aS) \cap \left(a \cup a^2 \cup (aS)\,a\right) \\
\subseteq\; & \{[\left(a \cup a^2 \cup (aS)\,a\right)][(a \cup Sa \cup aS)]\}\left(a \cup a^2 \cup (aS)\,a\right) \\
\subseteq\; & \{[\left(a \cup a^2 \cup (aS)\,a\right)][(a \cup Sa \cup aS)]\}S \\
=\; & [a^2 \cup a\,(Sa) \cup a\,(aS) \cup a^2 a \cup a^2\,(Sa) \cup a^2\,(aS) \cup \\
& ((aS)\,a)\,a \cup ((aS)\,a)\,(Sa) \cup ((aS)\,a)\,(aS)]S \\
\subseteq\; & [a^2 \cup a^2 S \cup a\,(aS) \cup (Sa)(aS)]S \subseteq [a^2 \cup a^2 S \cup a\,(aS)]S \\
=\; & a^2 S \cup \left(a^2 S\right) S \cup (a\,(aS))\,S = a^2 S \cup a^2 S \cup (aS)\,a \\
=\; & a^2 S \cup (aS)\,a.
\end{aligned}$$

Therefore $a = a^2 u$ or $a = (ax)a$, for some u and x in S. Hence S is regular. ■

Theorem 92 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For $(\in, \in \vee q_k)$-fuzzy bi-ideals f, h and $(\in, \in \vee q_k)$-fuzzy ideal g of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

(iii) For $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals f, h and $(\in, \in \vee q_k)$-fuzzy ideal g of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. $(i) \Rightarrow (iii)$

Assume that f, g and h are $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a regular AG-groupoid S respectively. Now since S is regular so for $a \in S$, there exist

$x \in S$, such that $a = (ax)\, a$. Therefore $a = (a\,((xa)\,x))\, a \in$ Thus,

$$\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&\geq \{f(a) \wedge g\,((xa)\,x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= \{f\,(a) \wedge g\,(a) \wedge \frac{1-k}{2}\} \wedge h\,(a) \wedge \frac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)\,(a)\,.
\end{aligned}$$

Therefore $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$

Assume that $B_1,\ B_2$ are bi-ideals and I is an ideal of S respectively. Then $(C_{B_1})_k$, $(C_I)_k$ and $(C_{B_2})_k$ are $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy bi-ideal of S respectively. Therefore we have, $(C_{B_1 \cap I \cap B_2})_k = (C_{B_1} \wedge_k C_I) \wedge_k C_{B_2} \leq (C_{B_1} \circ_k C_I) \circ_k C_{B_2} = (C_{(B_1 I) B_2})_k$. Therefore $B_1 \cap I \cap B_2 \subseteq (B_1 I)\, B_2$. Hence S is regular. ■

Theorem 93 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For ideals $I_1,\ I_2$ and quasi-ideal Q of S, $I_1 \cap I_2 \cap Q \subseteq (I_1 I_2)\, Q$.

(iii) $I\,[a] \cap I\,[a] \cap Q\,[a] \subseteq (I\,[a]\, I\,[a])\, Q\,[a]\,,$

for some $a \in S$.

Proof. $(i) \Rightarrow (ii)$

Assume that $I_1,\ I_2$ are ideals and Q is quasi-ideal of a regular AG-groupoid S, respectively. Let $a \in I_1 \cap I_2 \cap Q$. This implies that $a \in I_1$, $a \in I_1$ and $a \in Q$. Now since S is regular so for $a \in S$, there exist $x \in S$, such that $a = (ax)\, a$. Therefore $a = (a\,((xa)\,x))\, a \in (I_1\,((SI_2)\,S))\, Q \subseteq (I_1 I_2)\, Q$. Thus $I_1 \cap I_2 \cap Q \subseteq (I_1 I_2)\, Q$.

$(ii) \Rightarrow (iii)$ is obvious.
$(iii) \Rightarrow (i)$
Since $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (Sa \cap aS)$ are principle ideal and principle quasi-ideal of S generated by a respectively. Thus by left invertive law and medial law we have,

$$\begin{aligned}
(a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) &\subseteq (a \cup Sa \cup aS)(a \cup Sa \cup aS) \\
&\quad (a \cup (Sa \cap aS)) \\
&\subseteq ((a \cup Sa \cup aS) S)(a \cup aS) \\
&= \{aS \cup (Sa) S \cup (aS) S\}(a \cup aS) \\
&= \{aS \cup aS \cup Sa\}(a \cup aS) \\
&= (aS \cup Sa)((a \cup aS)) \\
&= (aS) a \cup (aS)(aS) \cup (Sa) a \cup (Sa)(aS) \\
&= (aS) a \cup a^2 S \cup a (aS).
\end{aligned}$$

Hence S is regular. ■

Theorem 94 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For $(\in, \in \vee q_k)$-fuzzy ideals f, g and $(\in, \in \vee q_k)$-fuzzy quasi-ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. $(i) \Rightarrow (ii)$
Assume that f, g are $(\in, \in \vee q_k)$-fuzzy ideals and h is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$, there exist $x \in S$, such that $a = (ax) a$. Therefore

$a = (a((xa)x))a$. Thus,

$$\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&\geq \{f(a) \wedge g((xa)x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= \{f(a) \wedge g(a) \wedge \frac{1-k}{2}\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)(a).
\end{aligned}$$

Therefore $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

$(ii) \Longrightarrow (i)$

Assume that I_1, I_2 are ideals and Q is a quasi-ideal of S respectively. Then $(C_{I_1})_k$, $(C_{I_2})_k$ and $(C_Q)_k$ are $(\in, \in \vee q_k)$-fuzzy ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S respectively. Therefore we have,

$$(C_{I_1 \cap I_2 \cap Q})_k = (C_{I_1} \wedge_k C_{I_2}) \wedge_k C_Q \leq (C_{I_1} \circ_k C_{I_2}) \circ_k C_Q = (C_{(I_1 I_2)Q})_k.$$

Thus $I_1 \cap I_2 \cap Q \subseteq (I_1 I_2) Q$. Hence S is regular. ■

Theorem 95 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) $I[a] \cap J[a] = I[a] J[a]$ for some a in S.

(iii) For ideals I, J of S, $I \cap J = IJ$ $(I \cap J = JI)$.

(iv) For bi-ideal B of S, $B = (BS)B$.

Proof. $(i) \Rightarrow (iv)$

Assume that B is a bi-ideal of a regular AG-groupoid S. Clearly $(BS)B \subseteq B$. Let $b \in B$. Since S is regular so for $b \in S$ there exist $x \in S$ such that $b = (bx)b \in (BS)B$. Thus $B = (BS)B$.

$(iv) \Rightarrow (iii)$

Assume that I and J are ideals of regular AG-groupoid S. Now,

$$((I \cap J) S)(I \cap J) \subseteq (SS)(I \cap J) = S(I \cap J) = SI \cap SJ \subseteq I \cap J$$

and

$$I \cap J = ((I \cap J) S)(I \cap J) \subseteq (IS) J \subseteq IJ.$$

Moreover $IJ \subseteq SJ \subseteq J$, also $IJ \subseteq IS \subseteq I$. Therefore $IJ \subseteq I \cap J$. Thus $I \cap J = IJ$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$

Since $I[a] = a \cup Sa \cup aS$ is a principle ideal of S generated by a. Thus by (ii), (1), left invertive law and paramedial law, we have,

$$\begin{aligned}(a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) &= (a \cup Sa \cup aS)(a \cup Sa \cup aS) \\ &= a^2 \cup a(Sa) \cup a(aS) \cup (Sa) a \\ &\quad \cup (Sa)(Sa) \cup (Sa)(aS) \cup (aS) a \\ &\quad \cup (aS)(Sa) \cup (aS)(aS) \\ &= a^2 \cup a(aS) \cup (aS) a \cup a^2 S.\end{aligned}$$

Hence S is regular. ■

Theorem 96 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For $(\in, \in \vee q_k)$-fuzzy ideals f, g of S, $(f \wedge_k g) \leq (f \circ_k g)$.

(iii) For $(\in, \in \vee q_k)$-fuzzy right ideals f, g of S, $(f \wedge_k g) \leq (f \circ_k g)$.

Proof. $(i) \Rightarrow (iii)$

Assume that f and g are $(\in, \in \vee q_k)$-fuzzy right ideals of a regular AG-groupoid S. Now since S is regular so for $a \in S$ there exist $x \in S$ such that $a = (ax) a$. Thus,

$$\begin{aligned}(f \circ_k g)(a) &= \bigvee_{a=pq} f(p) \wedge g(q) \wedge \frac{1-k}{2} = \bigvee_{a=pq=(ax)a} f(p) \wedge g(q) \wedge \frac{1-k}{2} \\ &\geq f(ax) \wedge g(a) \wedge \frac{1-k}{2} \geq \left(f(a) \wedge \frac{1-k}{2}\right) \wedge g(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).\end{aligned}$$

Hence $(f \wedge_k g) \leq (f \circ_k g)$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$

Assume that I and J are ideals S. Then $(C_I)_k$, and $(C_J)_k$ are $(\in, \in \vee q_k)$-fuzzy ideals of S. Therefore we have, $(C_{I \cap J})_k = (C_I \wedge_k C_J) \leq (C_I \circ_k C_J) = (C_{IJ})_k$. Thus $I \cap J \subseteq IJ$. Hence S is regular. ■

Theorem 97 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For $(\in, \in \vee q_k)$-fuzzy ideals f, g of S, $(f \wedge_k g) \leq (g \circ_k f)$.

(iii) For $(\in, \in \vee q_k)$-fuzzy right ideals f, g of S, $(f \wedge_k g) \leq (g \circ_k f)$.

Proof. It is easy. ∎

Theorem 98 *Let S be an AG-groupoid with left identity. Then the following are equivalent.*

(i) S is regular.

(ii) For $(\in, \in \vee q_k)$-fuzzy ideals f, g and h of S, $f \wedge_k g \wedge_k h \leq (f \circ_k g) \circ_k h$.

(iii) For $(\in, \in \vee q_k)$-fuzzy right ideals f, g and h of S, $f \wedge_k g \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. $(i) \Rightarrow (iii)$

Assume that f, g and h are $(\in, \in \vee q_k)$-fuzzy right ideals of a regular AG-groupoid S. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using paramedial law and medial law we have, $a = (ax)\,a = (ax)\,((ax)\,a) = (a\,(ax))\,(xa)$. Thus,

$$\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(a(ax))(xa)=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&\geq \{f(a) \wedge g\,(ax)\} \wedge h(xa) \wedge \frac{1-k}{2} \\
&\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= \{f\,(a) \wedge g\,(a) \wedge \frac{1-k}{2}\} \wedge h\,(a) \wedge \frac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)\,(a)\,.
\end{aligned}$$

Therefore $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$

Assume that f, g and h are $(\in, \in \vee q_k)$-fuzzy ideals of S. Now by using left invertive law, we have, $(f \wedge_k g) \leq (f \wedge_k g) \wedge_k S \leq (f \circ_k g) \circ_k S = (S \circ_k g) \circ_k f \leq g \circ_k f$. Thus $(f \wedge_k g) \leq g \circ_k f$. Hence S is regular. ■

3

Generalized Fuzzy Left Ideals in AG-groupoids

In this chapter, we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals in an AG-groupoid. We characterize intra-regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals.

3.1 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals of AG-groupoids

Let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For any $B \subseteq A$, let $X_{\gamma B}^{\delta}$ be a fuzzy subset of X such that $X_{\gamma B}^{\delta}(x) \geq \delta$ for all $x \in B$ and $X_{\gamma B}^{\delta}(x) \leq \gamma$ otherwise. Clearly, $X_{\gamma B}^{\delta}$ is the characteristic function of B if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and a fuzzy subset f of X, we say that

(1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.

(2) $x_r q_\delta f$ if $f(x) + r > 2\delta$.

(3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

Now we introduce a new relation on $\mathcal{F}(X)$, denoted by "$\subseteq \vee q_{(\gamma,\delta)}$", as follows:

For any $f, g \in \mathcal{F}(X)$, by $f \subseteq \vee q_{(\gamma,\delta)} g$ we mean that $x_r \in_\gamma f$ implies $x_r \in_\gamma \vee q_\delta g$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover f and g are said to be (γ,δ)-equal, denoted by $f =_{(\gamma,\delta)} g$, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$.

The above definitions can be found in [41].

Lemma 99 *[41] Let f and g be fuzzy subsets of $\mathcal{F}(X)$. Then $f \subseteq \vee q_{(\gamma,\delta)} g$ if and only if* $\max\{g(x), \gamma\} \geq \min\{f(x), \delta\}$ *for all $x \in X$.*

Lemma 100 *[41] Let f, g and $h \in \mathcal{F}(X)$. If $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} h$, then $f \subseteq \vee q_{(\gamma,\delta)} h$.*

The relation "$=_{(\gamma,\delta)}$" is equivalence relation on $\mathcal{F}(X)$, see [41]. Moreover, $f =_{(\gamma,\delta)} g$ if and only if $\max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$ for all $x \in X$.

Lemma 101 *Let A, B be any non-empty subsets of an AG -groupoid S with a left identity. Then we have*

(1) $A \subseteq B$ if and only if $X_{\gamma A}^{\delta} \subseteq \vee q_{(\gamma,\delta)} X_{\gamma B}^{\delta}$, where $r \in (\gamma, 1]$ and $\gamma, \delta \in [0,1]$.

(2) $X_{\gamma A}^{\delta} \cap X_{\gamma B}^{\delta} =_{(\gamma,\delta)} X_{\gamma(A \cap B)}^{\delta}$.

(3) $X_{\gamma A}^{\delta} \circ X_{\gamma B}^{\delta} =_{(\gamma,\delta)} X_{\gamma(AB)}^{\delta}$.

3.2 Some Basic Results

Lemma 102 *If S is an AG-groupoid with a left identity then $(ab)^2 = a^2b^2 = b^2a^2$ for all a and b in S.*

Proof. It follows by medial and paramedial laws. ■

Definition 103 *A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$, such that $x_t \in_\gamma f$, $y_s \in_\gamma f$ we have $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.*

Theorem 104 *Let f be a fuzzy subset of an AG groupoid S. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG subgroupoid of S if and only if $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$, where $\gamma, \delta \in [0, 1]$.*

Proof. Let f be a fuzzy subset of an AG-groupoid S which is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subgroupoid of S. Assume that there exists $x, y \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}.$$

Then $\max\{f(xy), \gamma\} < t$, this implies that $f(xy) < t \leq \gamma$, which further implies that $(xy)_{\min\{t,s\}}\overline{\in_\gamma \vee q_\delta} f$ and $\min\{f(x), f(y), \delta\} \geq t$, therefore $\min\{f(x), f(y)\} \geq t$ this implies that $f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$, implies that $x_t \in_\gamma f$, $y_s \in_\gamma f$ but $(xy)_{\min\{t,s\}}\overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \text{ for all } x, y \in S.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$, $y_s \in_\gamma f$ by definition we write $f(x) \geq t > \gamma$, $f(y) \geq s > \gamma$, then $\max\{f(xy), \delta\} \geq \min\{f(x), f(y), \delta\}$ this implies that $f(xy) \geq \min\{t, s, \delta\}$. Here arises two cases,

Case(a): If $\{t, s\} \leq \delta$ then $f(xy) \geq \min\{t, s\} > \gamma$ this implies that $(xy)_{\min\{t,s\}} \in_\gamma f$.

Case(b): If $\{t, s\} > \delta$ then $f(xy) + \min\{t, s\} > 2\delta$ this implies that $(xy)_{\min\{t,s\}} q_\delta f$.

From both cases we write $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y in S. ■

Definition 105 *A fuzzy subset f of an AG-groupoid S with a left identity is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (respt-right) ideal of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $y_t \in_\gamma f$ we have $(xy)_t \in_\gamma \vee q_\delta f$ (resp $x_t \in_\gamma f$ implies that $(xy)_t \in_\gamma \vee q_\delta f$).*

Theorem 106 *A fuzzy subset f of an AG-groupoid S with left identity is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (respt right) ideal of S. if and only if*

$$\max\{f(xy), \gamma\} \geq \min\{f(y), \delta\} \text{ (respt } \max\{f(xy), \gamma\} \geq \min\{f(x), \delta\}).$$

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Let there exists $x, y \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(y), \delta\}.$$

Then $\max\{f(xy), \gamma\} < t \leq \gamma$ this implies that $(xy)_t \bar{\in}_\gamma f$ which further implies that $(xy)_t \overline{\in_\gamma \vee q_\delta} f$. As $\min\{f(y), \delta\} \geq t > \gamma$ which implies that $f(y) \geq t > \gamma$, this implies that $y_t \in_\gamma f$. But $(xy)_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Thus

$$\max\{f(xy), \gamma\} \geq \min\{f(y), \delta\}.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $y_s \in_\gamma f$ but $(xy)_t \overline{\in_\gamma \vee q_\delta} f$, then $f(y) \geq t > \gamma$, $f(xy) < \min\{f(y), \delta\}$ and $f(xy) + t \leq 2\delta$. It follows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(y), \delta\}$ which is a contradiction. Hence $y_t \in_\gamma f$ this implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ (respt $x_t \in_\gamma f$ implies $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$) for all x, y in S. ■

Definition 107 *A fuzzy subset f of an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if for all x, y and $z \in S$ and $t, s \in (\gamma, 1]$, the following conditions hold.*
(1) *if $x_t \in_\gamma f$ and $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.*
(2) *if $x_t \in_\gamma f$ and $z_s \in_\gamma f$ implies that $((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.*

Theorem 108 *A fuzzy subset f of an AG-groupoid S with left identity is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if*
(I) $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.
(II) $\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}$.

Proof. $(1) \Leftrightarrow (I)$ is the same as theorem 104.
$(2) \Rightarrow (II)$ Assume that $x, y \in S$ and $t, s \in (\gamma, 1]$ such that

$$\max\{f((xy)z), \gamma\} < t \leq \min\{f(x), f(z), \delta\}.$$

Then $\max\{f((xy)z), \gamma\} < t$ which implies that $f((xy)z) < t$ this implies that $((xy)z)_t \bar{\in}_\gamma f$ which further implies that $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$. Also $\min\{f(x), f(z), \delta\} \geq t > \gamma$, this implies that $f(x) \geq t > \gamma$, $f(z) \geq t > \gamma$ implies that $x_t \in_\gamma f$, $z_t \in_\gamma f$. But $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$, a contradiction. Hence

$$\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}.$$

$(II) \Rightarrow (2)$ Assume that x, y in S and $t, s \in (\gamma, 1]$, such that $x_t \in_\gamma f$, $z_s \in_\gamma f$ but $((xy)z)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$, then $f(x) \geq t > \gamma$, $f(z) \geq s > \gamma$, $f((xy)z) < \min\{f(x), f(y), \delta\}$ and $f((xy)z) + \min\{t, s\} \leq 2\delta$. It follows that $f((xy)z) < \delta$ and so $\max\{f((xy)z), \gamma\} < \min\{f(x), f(y), \delta\}$ a contradiction. Hence $x_t \in_\gamma f$, $z_s \in_\gamma f$ implies that $((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y in S. ■

Example 109 *Consider an AG-groupoid* $S = \{1, 2, 3\}$ *in the following multiplication table.*

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Define a fuzzy subset f *on* S *as follows:*

$$f(x) = \begin{cases} 0.41 \text{ if } x = 1, \\ 0.44 \text{ if } x = 2, \\ 0.42 \text{ if } x = 3. \end{cases}$$

Then, we have

- f *is an* $(\in_{0.1}, \in_{0.1} \vee q_{0.11})$*-fuzzy left ideal,*
- f *is not an* $(\in, \in \vee q_{0.11})$*-fuzzy left ideal,*
- f *is not a fuzzy left ideal.*

Example 110 *Let* $S = \{1, 2, 3\}$ *and the binary operation* $\circ$ *be defined on* S *as follows:*

$\circ$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then clearly $(S, \circ)$ *is an AG-groupoid. Defined a fuzzy subset* f *on* S *as follows:*

$$f(x) = \begin{cases} 0.44 \text{ if } x = 1, \\ 0.6 \text{ if } x = 2, \\ 0.7 \text{ if } x = 3. \end{cases}$$

Then, we have

- f *is an* $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$*-fuzzy left ideal of* S*.*
- f *is not an* $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$*-fuzzy right ideal of* S*.*

Example 111 *Let* $S = \{1, 2, 3\}$*, then binary operation* $\cdot$ *defined on* S *as follows:*

$\cdot$	1	2	3
1	1	1	1
2	3	3	3
3	1	1	1

Clearly $(S, \cdot)$ *is an AG-groupoid. Let us defined a fuzzy subset* f *on* S *as follows:*

$$f(x) = \begin{cases} 0.6 \text{ if } x = 1 \\ 0.4 \text{ if } x = 2 \\ 0.3 \text{ if } x = 3 \end{cases}$$

Clearly f *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left ideal of* S*.*

Lemma 112 *Let f be a fuzzy subset of an AG-groupoid S. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if* $\max\{f(a),\gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$.

Proof. Assume that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of an AG-groupoid S. If $a \in S$, then there exist c, d, p and q in S such that $a = pq$ and $p = cd$. Since f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S, we have $\max\{f((cd)q), \gamma\} \geq \min\{f(c), f(q), \delta\}$. Therefore,

$$\begin{aligned}
\min\{((f \circ S) \circ f)(a), \delta\} &= \min\left\{\bigvee_{a=pq} \{(f \circ S)(p) \wedge f(q)\}, \delta\right\} \\
&= \min\left\{\bigvee_{a=pq} \{\bigvee_{p=cd} \{f(c) \wedge S(d)\} \wedge f(q)\}, \delta\right\} \\
&= \min\left\{\bigvee_{a=(cd)q} \{\{f(c) \wedge 1 \wedge f(q)\}, \delta\}\right\} \\
&= \min\left\{\bigvee_{a=(cd)q} \{f(c) \wedge f(q)\}, \delta\right\} \\
&= \bigvee_{a=(cd)q} \{\min\{f(p), f(q), \delta\}\} \\
&\leq \bigvee_{a=(cd)q} \{\max\{f((cd)q), \gamma\}\} \\
&= \max\{f(a), \gamma\}.
\end{aligned}$$

Hence, $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$.

Conversely, assume that $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$. Let a in S, there exist c, d and q in S such that $a = (cd)q$. Then we have

$$\begin{aligned}
\max\{f((cd)q), \gamma\} &= \max\{f(a), \gamma\} \\
&\geq \min\{((f \circ S) \circ f)(a), \delta\} \\
&= \min\left\{\bigvee_{a=bc} \{(f \circ S)(b) \wedge f(c), \delta\}\right\} \\
&\geq \min\{\{(f \circ S)(cd) \wedge f(q)\}, \delta\} \\
&= \min\left\{\bigvee_{cd=st} \{f(s) \wedge S(t)\} \wedge f(q)\}, \delta\right\} \\
&\geq \min\{\min(f(c), f(q), \delta\} \\
&= \min\{(f(c), f(q), \delta\}.
\end{aligned}$$

Hence $\max\{f((cd)q), \gamma\} \geq \min\{f(c), f(q), \delta\}$. ■

Lemma 113 *Every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left ideal of an AG-groupoid* S *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal of* S*.*

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of an AG-groupoid S. For any a in S, there exist p, q and for p there exists s, t in S, such that $a = pq$ and $p = st$. Then

$$\begin{aligned}
\min\{((f \circ S) \circ f)(a), \delta\} &= \min\{\bigvee_{a=pq} \{(f \circ S)(p) \wedge f(q)\}, \delta\} \\
&= \min\{\bigvee_{a=pq} \{\bigvee_{p=st} \{f(s) \wedge S(t)\} \wedge f(q)\}, \delta\} \\
&= \min\{\bigvee_{a=(st)q} \{f(s) \wedge f(q)\}, \delta\} \\
&= \min\{\bigvee_{a=(st)q} [\min\{f(s), f(q)\}], \delta\} \\
&= \bigvee_{a=(st)q} \min[\min\{f(s), \delta\}, \min\{f(q), \delta\}] \\
&\leq \bigvee_{a=(st)q=(qt)s} \min\{\max\{f(qt)s, \gamma\}, \max\{f(st)q, \gamma\}\} \\
&= \min\{\max\{f(a), \gamma\}, \max\{f(a), \gamma\}\} \\
&= \max\{f(a), \gamma\}.
\end{aligned}$$

Hence $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$. Hence f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. ■

Lemma 114 *Let* f *and* g *be* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left ideals of an AG-groupoid* S *with left identity. Then* $(f \circ g)$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left ideal of* S*.*

Proof. Let f be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of an AG-groupoid S with left identity. So for any y in S there exists a and b in S such that $y = ab$. Therefore

$$xy = x(ab) = a(xb).$$

then

$$\begin{aligned}
\min\{f\circ g(y),\delta\} &= \min\left\{\bigvee_{y=ab}\{f(a)\wedge g(b)\},\delta\right\} \\
&= \bigvee_{y=ab}\{\min\{\min\{f(a),\delta\},\min\{g(b),\delta\}\}\} \\
&\leq \bigvee_{xy=a(xb)}\{\min\{\max\{f(a),\gamma\},\max\{g(xb),\gamma\}\}\} \\
&= \bigvee_{xy=a(xb)}\{\max\{\min\{f(a),g(xb)\},\gamma\}\} \\
&\leq \bigvee_{xy=ac}\{\max\{\min\{f(a),g(c)\},\gamma\}\} \\
&= \max\{(f\circ g)(xy),\gamma\}.
\end{aligned}$$

■

Lemma 115 *If L is a left ideal of an AG-groupoid S if and only if $X_{\gamma L}^{\delta}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S.*

Proof. (i) Let $x, y \in L$ which implies that $xy \in L$. Then by definition we get $X_{\gamma L}^{\delta}(xy) \geq \delta$, $X_{\gamma L}^{\delta}(x) \geq \delta$ and $X_{\gamma L}^{\delta}(y) \geq \delta$ but $\delta \geq \gamma$. Thus

$$\max\{X_{\gamma L}^{\delta}(xy),\gamma\} = X_{\gamma L}^{\delta}(xy) \text{ and } \min\{X_{\gamma L}^{\delta}(y),\delta\} = \delta.$$

Hence $\max\{X_{\gamma L}^{\delta}(xy),\gamma\} \geq \min\{X_{\gamma L}^{\delta}(y),\delta\}$.

(ii) Let $x \in L$ and $y \notin L$, which implies that $xy \notin L$. Then by definition $X_{\gamma L}^{\delta}(x) \geq \delta$, $X_{\gamma L}^{\delta}(y) \leq \gamma$ and $X_{\gamma L}^{\delta}(xy) \leq \gamma$. Therefore

$$\max\{X_{\gamma L}^{\delta}(xy),\gamma\} = \gamma \text{ and } \max\{X_{\gamma L}^{\delta}(y),\delta\} = X_{\gamma L}^{\delta}(y).$$

Hence $\max\{X_{\gamma L}^{\delta}(xy),\gamma\} \geq \min\{X_{\gamma L}^{\delta}(y),\delta\}$.

(iii) Let $x \notin L, y \in L$ which implies that $xy \in L$. Then by definition, we get $X_{\gamma L}^{\delta}(xy) \geq \delta$, $X_{\gamma L}^{\delta}(y) \geq \delta$ and $X_{\gamma L}^{\delta}(x) \leq \gamma$. Thus

$$\max\{X_{\gamma L}^{\delta}(xy),\gamma\} = X_{\gamma L}^{\delta}(xy) \text{ and } \min\{X_{\gamma L}^{\delta}(y),\delta\} = \delta.$$

Hence $\max\{X_{\gamma L}^{\delta}(xy),\gamma\} \geq \min\{X_{\gamma L}^{\delta}(y),\delta\}$.

(iv) Let $x, y \notin L$ which implies that $xy \notin L$. Then by definition we get such that $X_{\gamma L}^{\delta}(xy) < \gamma$, $X_{\gamma L}^{\delta}(y) < \gamma$ and $X_{\gamma L}^{\delta}(x) < \gamma$. Thus

$$\max\{X_{\gamma L}^{\delta}(xy),\gamma\} = \gamma \text{ and } \min\{X_{\gamma L}^{\delta}(y),\delta\} = X_{\gamma L}^{\delta}(y).$$

Hence $\max\{X_{\gamma L}^{\delta}(xy),\gamma\} \geq \min\{X_{\gamma L}^{\delta}(y),\delta\}$.

Converse, let $sl \in SL$, where $l \in L$ and $s \in S$. Now by hypothesis $\max\{X^{\delta}_{\gamma L}((sl), \gamma\} \geq \min\{X^{\delta}_{\gamma L}(l), \delta\}$. Since $l \in L$, therefore $X^{\delta}_{\gamma L}(l) \geq \delta$ which implies that $\min\{X^{\delta}_{\gamma L}(l), \delta\} = \delta$. Thus

$$\max\{X^{\delta}_{\gamma L}(sl), \gamma\} \geq \delta.$$

This clearly implies that $\max X^{\delta}_{\gamma L}(sl) \geq \delta$. Therefore $sl \in L$. Hence L is a left ideal of S. ■

Similarly we can prove the following lemma.

Lemma 116 *If B is a bi-ideal of an AG-groupoid S if and only if $X^{\delta}_{\gamma B}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal ideal of S.*

3.3 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals of Intra Regular AG-groupoids

An element a of an AG-groupoid S is called **intra-regular** if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called **intra-regular**, if every element of S is intra-regular.

Theorem 117 *Let S be an AG- groupoid with left identity then the following conditions are equivalent.*

(i) S is intra regular.

(ii) $L[a] \cap L[a] \subseteq L[a]L[a]$, for all a in S.

(iii) $L_1 \cap L_2 \subseteq L_1L_2$, for all left ideals L_1, L_2 of S.

(iv) $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$, for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals f and g of S.

Proof. $(i) \Rightarrow (iv)$ Since S is intra regular therefore for any a in S there exist x, y in S such that $a = (xa^2)y$. Then

$$\begin{aligned} a &= (xa^2)y = (xa^2)(y_1y_2) = (y_2y_1)(a^2x) = a^2[(y_2y_1)x] \\ &= [a(y_2y_1)](ax) = (xa)[(y_2y_1)a]. \end{aligned}$$

Let for any a in S there exist p and q in S such that $a = pq$, then

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{\bigvee_{a=pq} \{\{f(p) \wedge g(q)\}, \gamma\}\right\} \\ &\geq \max\{\min\{f(xa)), g((y_2y_1)a)\}, \gamma\} \\ &= \min\{\max\{f(a), \gamma\}, \max\{g(a), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\ &= \min\{f(a), g(a), \delta\}. \end{aligned}$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$.

$(iv) \Rightarrow (iii)$ If B is a bi- ideal of S. Then by (iii), we get

$$X^{\delta}_{\gamma L_1 \cap L_2} =_{(\gamma,\delta)} X^{\delta}_{\gamma L_1} \cap X^{\delta}_{\gamma L_2} \subseteq \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma L_1} \circ X^{\delta}_{\gamma L_2} =_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma L_1 L_2}.$$

Hence $L_1 \cap L_2 \subseteq L_1 L_2$.
$(iii) \Rightarrow (ii)$ It is obvious.
$(ii) \Rightarrow (i)$

$$\begin{aligned}(a \cup Sa) \cap (a \cup Sa) &\subseteq (a \cup Sa)(a \cup Sa) \\ &= a^2 \cup a(Sa) \cup (Sa)a \cup (Sa)(Sa) \\ &= a^2 \cup S(aa) \cup (aa)S \cup (SS)(aa) \\ &= a^2 \cup Sa^2 \cup a^2 S \cup Sa^2 \\ &= a^2 \cup Sa^2\end{aligned}$$

Thus $a = a^2$ or $a \in Sa^2$. Hence S is intra regular. ■

Corollary 118 *Let S be an AG- groupoid with left identity then the following conditions are equivalent.*
(i) *S is intra regular.*
(ii) *$L[a] \subseteq L[a]L[a]$, for all a in S.*
(iii) *$L \subseteq L^2$, for all left ideals L of S.*
(iv) *$f \subseteq \vee q_{(\gamma,\delta)} f \circ f$, for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals f of S.*

Theorem 119 *Let S be an AG- groupoid with left identity then the following conditions are equivalent.*
(i) *S is intra regular.*
(ii) *$L[a] \cap L[a] \cap L[a] \subseteq (L[a]L[a])L[a]$, for all a in S.*
(ii) *$A \cap B \cap C \subseteq (AB)C$, for all left ideals A, B and C of S.*
(iv) *$f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)} (f \circ g) \circ h$, for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals f, g and h of S, .*

Proof. $(i) \Rightarrow (iv)$ For every a in S, we have $a = (xa^2)y$, this by (1) and left invertive law implies that $a = (y(xa))a$. Now using (1) , medial, paramedial laws, we get

$$\begin{aligned}y(xa) &= y[x\{(xa^2)y\}] = y[(xa^2)(xy)] = (xa^2)[y(xy)] = (xa^2)(xy^2) \\ &= (xx)(a^2y^2) = a^2(x^2y^2) = (ax^2)(ay^2) = (y^2a)(x^2a).\end{aligned}$$

Let for any a in S there exist p and q in S such that $a = pq$. Then

$$\begin{aligned}
\max\{(f\circ g)\circ h(a),\gamma\} &= \max[\bigvee_{a=pq}\{\{(f\circ g)(p)\wedge h(q)\},\gamma\}\}] \\
&= \max\{\min\{(f\circ g)(y(xa)),h(a)\},\gamma\}\} \\
&= \max\left\{\min\bigvee_{y(xa)=rs}\{f(r)\wedge g(s),h(a)\},\gamma\}\right\} \\
&\geq \min\left\{\max\{\{f(y^2a),g(x^2a)\},h(a)\},\gamma\right\} \\
&= \min\{\max\{f(y^2a),\gamma\},\max\{g(x^2a),\gamma\},\max\{h(a),\gamma\}\} \\
&\geq \min\{\min\{f(a),\delta\},\min\{g(a),\delta\},\min\{h(a),\delta\}\} \\
&= \min\{(f\cap g\cap h)(a),\delta\}.
\end{aligned}$$

Thus $f\cap g\cap h \subseteq \vee q_{(\gamma,\delta)}(f\circ g)\circ h$.
$(iii)\Rightarrow(ii)$ If A, B and C are left ideals of S. Then by (iii), we get

$$\begin{aligned}
X^{\delta}_{\gamma(A\cap B\cap C)} &= {}_{(\gamma,\delta)}X^{\delta}_{\gamma A\cap B}\cap X^{\delta}_{\gamma C}\subseteq \vee q_{(\gamma,\delta)}(X^{\delta}_{\gamma A}\circ X^{\delta}_{\gamma B})\circ X^{\delta}_{\gamma C} \\
&= {}_{(\gamma,\delta)}\vee q_{(\gamma,\delta)}X^{\delta}_{\gamma AB}\circ X^{\delta}_{\gamma C} ={}_{(\gamma,\delta)}\vee q_{(\gamma,\delta)}X^{\delta}_{\gamma(AB)C}.
\end{aligned}$$

Hence we get $A\cap B\cap C\subseteq (AB)C$.
$(iii)\Rightarrow(ii)$ It is obvious.
$(ii)\Rightarrow(i)$

$$\begin{aligned}
& (a\cup Sa)\cap(a\cup Sa)\cap(a\cup Sa) \\
\subseteq\ & [(a\cup Sa)(a\cup Sa)](a\cup Sa) \\
=\ & [a^2\cup Sa^2](a\cup Sa)\subseteq (Sa^2)S.
\end{aligned}$$

Hence S is intra regular. ■

Theorem 120 *Let S be an AG- groupoid with left identity then the following conditions are equivalent.*
(i) S is intra regular.
(ii) $L[a]\cap L[a]\subseteq (L[a]L[a])L[a]$,for all a in S.
(iii) $A\cap B\subseteq (AB)A$, for all left ideals A and B of S.
(iv) $f\cap g\subseteq \vee q_{(\gamma,\delta)}(f\circ g)\circ f$, for all $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideals f and g of S.

Proof. $(i) \Rightarrow (iv)$ Let for any a in S there exist p and q in S such that $a = pq$. Then

$$\begin{aligned}
\max\{(f \circ g) \circ f(a), \gamma\} &= \max[\bigvee_{a=pq} \{\{(f \circ g)(p) \wedge f(q)\}, \gamma\}\}] \\
&\geq \max[\{\{(f \circ g)((y^2a)(x^2a)) \wedge f(a)\}, \gamma\}\}] \\
&= \max\left\{\min\{\bigvee_{(y^2a)(x^2a)=rs} f(r) \wedge g(s), f(a)\}, \gamma\}\right\} \\
&\geq \max\left\{\min\{f(y^2a), g(x^2a), f(a)\}, \gamma\}\right\} \\
&= \min\{\max\{f(y^2a), \gamma\}, \max\{g(x^2a), \gamma\}, \max\{f(a), \gamma\}\} \\
&\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{f(a), \delta\}\} \\
&= \min\{(f \cap g \cap f)(a), \delta\}.
\end{aligned}$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ f$.

$(iv) \Rightarrow (iii)$ It is obvious.

$(vi) \Rightarrow (ii)$ If A, B are any left ideals of S. Then by (iii), we get

$$\begin{aligned}
X^{\delta}_{\gamma(A\cap B)} &= X^{\delta}_{\gamma(A\cap B\cap A)} =_{(\gamma,\delta)} X^{\delta}_{\gamma A\cap B} \cap X^{\delta}_{\gamma A} \subseteq \vee q_{(\gamma,\delta)}(X^{\delta}_{\gamma A} \circ X^{\delta}_{\gamma B}) \circ X^{\delta}_{\gamma A} \\
&= {}_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma AB} \circ X^{\delta}_{\gamma A} =_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma(AB)A}.
\end{aligned}$$

Hencewe get $A \cap B \subseteq (AB)A$.

$(iii) \Rightarrow (ii)$ It is obvious.

$(ii) \Rightarrow (i)$ It is same as $(ii) \Rightarrow (i)$ of theorem 119. ■

Definition 121 *A fuzzy subset f of an AG-groupoid S is called an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy semiprime ideal if $x^2_t \in_{\gamma} f$ implies that $x_t \in_{\gamma} \vee q_{\delta}$ for all $x \in S$ and $t \in (\gamma, 1]$.*

Example 122 *Consider an AG-groupoid $S = \{1, 2, 3, 4, 5\}$ with the following multiplication table*

.	1	2	3	4	5
1	4	5	1	2	3
2	3	4	5	1	2
3	2	3	4	5	1
4	1	2	3	4	5
5	5	1	2	3	4

Clearly $(S, .)$ is intra-regular because $1 = (3.1^2).2$, $2 = (1.2^2).5$, $3 = (5.3^2).2$, $4 = (2.4^2).1$, $5 = (3.5^2).1$. *Define a fuzzy subset f on S as given:*

$$f(x) = \begin{cases} 0.7 \; if \; x = 1, \\ 0.6 \; if \; x = 2, \\ 0.68 \; if \; x = 3, \\ 0.63 \; if \; x = 4, \\ 0.52 \; if \; x = 5. \end{cases}$$

Then it is easy to see that f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.5})$-fuzzy semiprime ideal of S.

Theorem 123 *A fuzzy subset f of an AG-groupoid S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal if and only if* $\max\{f(x), \gamma\} \geq \min\{f(x^2), \delta\}$.

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S. Let there exists $x, y \in S$ and $t \in (\gamma, 1]$ such that $\max\{f(x), \gamma\} < t \leq \min\{f(x^2), \delta\}$. Then $\max\{f(x), \gamma\} < t$ implies that $x_t \bar{\in}_\gamma f$ implies that $x_t \overline{\in_\gamma \vee q_\delta} f$. As $\min\{f(x^2), \delta\} \geq t > \gamma$ this implies that $f(x^2) \geq t > \gamma$ implies that $x_t^2 \in_\gamma f$. But $x_t \overline{\in_\gamma \vee q_\delta} f$, a contradiction to the definition of semiprime ideals. Thus $\max\{f(x), \gamma\} \geq \min\{f(x^2), \delta\}$.

Conversely, assume that there exist x, y in S and $t \in (\gamma, 1]$ such that $x_t^2 \in_\gamma f$ but $x_t \overline{\in_\gamma \vee q_\delta} f$, then $f(x^2) \geq t > \gamma$, $f(x) < \min\{f(x^2), \delta\}$ and $f(x) + t \leq 2\delta$. It follows that $f(x) < \delta$ and so $\max\{f(x), \gamma\} < \min\{f(x^2), \delta\}$ which is a contradiction to the definition of semiprime ideals. Hence $x_t^2 \in_\gamma f$ implies that $(x^2)_t \in_\gamma \vee q_\delta f$ for all x, y in S. ■

Theorem 124 *For a non empty subset I of an AG-groupoid S with left identity, the following conditions are equivalent.*
(i) I is semiprime.
(ii) $X_{\gamma I}^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (ii)$ Let I be semiprime of an AG-groupoid S. Let a be any element of S such that $a \in I$, then I is an ideal so $a^2 \in I$. Hence $X_{\gamma I}^\delta(a), X_{\gamma I}^\delta(a^2) \geq \delta$ which implies that $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$.

Now let $a \notin I$, since I is semiprime, thus $a^2 \notin I$. This implies that $X_{\gamma I}^\delta(a) \leq \gamma$ and $X_{\gamma I}^\delta(a^2) \leq \gamma$. Therefore $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$. Hence, we have $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$ for all $a \in S$.

$(ii) \Rightarrow (i)$ Let $X_{\gamma I}^\delta$ is fuzzy semiprime. Let $a^2 \in I$, for some a in S, this implies that $X_{\gamma I}^\delta(a^2) \geq \delta$. Now since $X_{\gamma I}^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. Thus $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$. Therefore $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \delta$. But $\delta > \gamma$, so $X_{\gamma I}^\delta(a) \geq \delta$. Thus $a \in I$. Hence I is semiprime. ■

Theorem 125 *Let S be an AG-groupoid with left identity then the following conditions are equivalent.*
(i) S is intra-regular.
(ii) For every ideal of S is semiprime.
(iii) For every left ideal of S is semiprime.
(iv) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S is fuzzy semiprime.

Proof. $(i) \Rightarrow (iv)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of an intra regular AG-groupoid S. Now since S is intra-regular so for each $a \in S$ there exists x, y in S such that $a = (xa^2)y$. Now using medial law, paramedial law and left invertive law, we get

$$a = (xa^2)y = [(ex)(aa)]y = [(aa)(xe)]y = [y(xe)]a^2.$$

Thus

$$\begin{aligned}\max\{f(a),\gamma\} &= \max\{f([y(xe)]a^2),\gamma\}\\ &\geq \min\{f(a^2),\delta\}.\end{aligned}$$

$(iv) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$ are obvious.

$(ii) \Rightarrow (i)$ Assume that every ideal is semiprime and since Sa^2 is an ideal containing a^2. Thus

$$a \in Sa^2 = (SS)a^2 = (a^2S)S = (Sa^2)S.$$

Hence S is an intra-regular AG-groupoid. ■

4

Generalized Fuzzy Prime and Semiprime Ideals of Abel Grassmann Groupoids

In this chapter we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime (semiprime) ideals in AG-groupoids. We characterize intra regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideals.

Lemma 126 *If A is an ideal of an AG-groupoid S if and only if $X^\delta_{\gamma A}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S.*

Proof. (i) Let $x, y \in A$ which implies that $xy \in A$. Then by definition we get $X^\delta_{\gamma A}(xy) \geq \delta$, $X^\delta_{\gamma A}(x) \geq \delta$ and $X^\delta_{\gamma A}(y) \geq \delta$ but $\delta > \gamma$. Thus

$$\begin{aligned} \max\{X^\delta_{\gamma A}(xy), \gamma\} &= X^\delta_{\gamma A}(xy) \text{ and} \\ \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\} &= \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y)\} = \delta. \end{aligned}$$

Hence $\max\{X^\delta_{\gamma A}(xy), \gamma\} \geq \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\}$.

(ii) Let $x \notin A$ and $y \in A$, which implies that $xy \notin A$. Then by definition $X^\delta_{\gamma A}(x) \leq \gamma$, $X^\delta_{\gamma R}(y) \geq \delta$ and $X^\delta_{\gamma R}(xy) \leq \gamma$. Therefore

$$\begin{aligned} \max\{X^\delta_{\gamma A}(xy), \gamma\} &= \gamma \text{ and} \\ \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\} &= X^\delta_{\gamma A}(x). \end{aligned}$$

Hence $\max\{X^\delta_{\gamma A}(xy), \gamma\} \geq \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\}$.

(iii) Let $x \in A, y \notin A$ which implies that $xy \notin A$. Then by definition, we get $X^\delta_{\gamma A}(xy) \leq \gamma$, $X^\delta_{\gamma A}(y) \leq \gamma$ and $X^\delta_{\gamma A}(x) \geq \delta$. Thus

$$\begin{aligned} \max\{X^\delta_{\gamma A}(xy), \gamma\} &= \gamma \text{ and} \\ \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\} &= X^\delta_{\gamma A}(y). \end{aligned}$$

Hence $\max\{X^\delta_{\gamma A}(xy), \gamma\} \geq \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\}$.

(iv) Let $x, y \notin A$ which implies that $xy \notin A$. Then by definition we get such that $X^\delta_{\gamma A}(xy) \leq \gamma$, $X^\delta_{\gamma A}(y) \leq \gamma$ and $X^\delta_{\gamma A}(x) \leq \gamma$. Thus

$$\begin{aligned} \max\{X^\delta_{\gamma A}(xy), \gamma\} &= \gamma \text{ and} \\ \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\} &= \{X^\delta_{\gamma A}(x), X^\delta_{\gamma R}(y)\} = \gamma. \end{aligned}$$

Hence $\max\{X^{\delta}_{\gamma A}(xy),\gamma\}\geq\min\{X^{\delta}_{\gamma A}(x),X^{\delta}_{\gamma A}(y),\delta\}$.

Converse, let $(xy)\in AS$ where $x\in A$ and $y\in S$, and $(xy)\in SA$ where $y\in A$ and $x\in S$. Now by hypothesis $\max\{X^{\delta}_{\gamma A}(xy),\gamma\}\geq\min\{X^{\delta}_{\gamma B}(x),X^{\delta}_{\gamma B}(y),\delta\}$. Since $x\in A$, therefore $X^{\delta}_{\gamma A}(x)\geq\delta$, and $y\in A$ therefore $X^{\delta}_{\gamma A}(y)\geq\delta$ which implies that $\min\{X^{\delta}_{\gamma A}(x),X^{\delta}_{\gamma A}(y),\delta\}=\delta$. Thus

$$\max\{X^{\delta}_{\gamma A}(xy),\gamma\}\geq\delta.$$

This clearly implies that $X^{\delta}_{\gamma A}(xy)\geq\delta$. Therefore $xy\in A$. Hence A is an ideal of S. ■

Example 127 *Let $S=\{1,2,3\}$, and the binary operation "$\cdot$" be defined on S as follows.*

$\cdot$	1	2	3
1	1	1	1
2	1	1	1
3	1	2	1

Then $(S,\cdot)$ is an AG-groupoid. Define a fuzzy subset $f:S\rightarrow[0,1]$ as follows.

$$f(x)=\begin{cases}0.31 \text{ for } x=1\\ 0.32 \text{ for } x=2\\ 0.30 \text{ for } x=3\end{cases}$$

Then clearly

- *f is an $(\in_{0.2},\in_{0.2}\vee q_{0.3})$-fuzzy ideal of S,*
- *f is not an $(\in,\in\vee q_{0.3})$-fuzzy ideal of S, because $f(1\cdot 2)<f(2)\wedge\frac{1-0.3}{2}$,*
- *f is not a fuzzy ideal of S, because $f(1\cdot 2)<f(2)$.*

Definition 128 *A fuzzy subset f of an AG-groupoid S is called an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy bi-ideal of S if for all x,y and $z\in S$ and $t,s\in(\gamma,1]$, the following conditions holds.*

(1) if $x_t\in_{\gamma}f$ and $y_s\in_{\gamma}f$ implies that $(xy)_{\min\{t,s\}}\in_{\gamma}\vee q_{\delta}f$.

(2) if $x_t\in_{\gamma}f$ and $z_s\in_{\gamma}f$ implies that $((xy)z)_{\min\{t,s\}}\in_{\gamma}\vee q_{\delta}f$.

Theorem 129 *A fuzzy subset f of an AG-groupoid S is $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy bi-ideal of S if and only if*

(I) $\max\{f(xy),\gamma\}\geq\min\{f(x),f(y),\delta\}$.

(II) $\max\{f((xy)z),\gamma\}\geq\min\{f(x),f(z),\delta\}$.

Proof. $(1)\Leftrightarrow(I)$ is the same as theorem 104.

$(2)\Rightarrow(II)$. Assume that $x,y\in S$ and $t,s\in(\gamma,1]$ such that

$$\max\{f((xy)z),\gamma\}<t\leq\min\{f(x),f(z),\delta\}.$$

Then $\max\{f((xy)z),\gamma\}<t$ which implies that $f((xy)z)<t\leq\gamma$ this implies that $((xy)z)_t\overline{\in}_{\gamma}f$ which further implies that $((xy)z)_t\overline{\in_{\gamma}\vee q_{\delta}}f$. Also

$\min\{f(x), f(z), \delta\} \geq t > \gamma$, this implies that $f(x) \geq t > \gamma$, $f(z) \geq t > \gamma$ implies that $x_t \in_\gamma f$, $z_t \in_\gamma f$ but $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$, a contradiction. Hence

$$\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}.$$

$(II) \Rightarrow (2)$ Assume that x, y, z in S and $t, s \in (\gamma, 1]$, such that $x_t \in_\gamma f$, $z_s \in_\gamma f$ by definition we can write $f(x) \geq t > \gamma$, $f(z) \geq s > \gamma$, then $\max\{f((xy)z), \delta\} \geq \min\{f(x), f(y), \delta\}$ this implies that $f((xy)z) \geq \min\{t, s, \delta\}$. We consider two cases here,

Case(i): If $\{t, s\} \leq \delta$ then $f((xy)z) \geq \min\{t, s\} > \gamma$ this implies that $((xy)z)_{\min\{t,s\}} \in_\gamma f$.

Case(ii): If $\{t, s\} > \delta$ then $f((xy)z) + \{t, s\} > 2\delta$ this implies that $((xy)z)_{\min\{t,s\}} q_\delta f$.

From both cases we write $((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y, z in S. ■

Lemma 130 *A subset B of an AG-groupoid S is a bi-ideal if and only if $X^\delta_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. (i) Let B be a bi-ideal and assume that $x, y \in B$ then for any a in S we have $(xa)y \in B$, thus $X^\delta_{\gamma B}((xa)y) \geq \delta$. Now since $x, y \in B$ so $X^\delta_{\gamma B}(x) \geq \delta$, $X^\delta_{\gamma B}(y) \geq \delta$ which clearly implies that $\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \geq \delta$. Thus

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} = X^\delta_{\gamma B}((xa)y) \text{ and}$$
$$\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} = \delta.$$

Hence $\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}$.

(ii) Let $x \in B, y \notin B$, then $(xa)y \notin B$, for all a in S. This implies that $X^\delta_{\gamma B}((xa)y) \leq \gamma$, $X^\delta_{\gamma B}(x) \geq \delta$ and $X^\delta_{\gamma B}(y) < \gamma$. Therefore

$$\begin{aligned} \max\{X^\delta_{\gamma B}((xa)y), \gamma\} &= \gamma \text{ and} \\ \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} &= X^\delta_{\gamma B}(y). \end{aligned}$$

Hence $\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}$.

(iii) Let $x \notin B, y \in B$ implies that $(xa)y \notin B$, for all a in S. This implies that $X^\delta_{\gamma B}((xa)y) \leq \gamma$, $X^\delta_{\gamma B}(x) \leq \gamma$, $X^\delta_{\gamma B}(y) \geq \delta$ then

$$\begin{aligned} \max\{X^\delta_{\gamma B}((xa)y), \gamma\} &= \gamma, \text{ and} \\ \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} &= X^\delta_{\gamma B}(x) \end{aligned}$$

Therefore

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}.$$

(iv) Let $x, y \notin B$ which implies that $(xa)y \notin B$, for all a in S. This implies that $\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \leq \gamma$, $X^\delta_{\gamma B}((xa)y) \leq \gamma$. Thus

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} = \gamma \text{ and}$$
$$\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} = \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \leq \gamma.$$

Hence $\max\{X^{\delta}_{\gamma B}((xa)y),\gamma\} \geq \min\{X^{\delta}_{\gamma B}(x), X^{\delta}_{\gamma B}(y),\delta\}$.

If $(xa)y \in B$, then $\min\{X^{\delta}_{\gamma B}(x), X^{\delta}_{\gamma B}(y)\} \geq \delta$, $X^{\delta}_{\gamma B}((xa)y) \geq \delta$. Thus

$$\max\{X^{\delta}_{\gamma B}((xa)y),\gamma\} = X^{\delta}_{\gamma B}((xa)y) \text{ and}$$
$$\min\{X^{\delta}_{\gamma B}(x), X^{\delta}_{\gamma B}(y),\delta\} = \delta.$$

Hence $\max\{X^{\delta}_{\gamma B}((xa)y),\gamma\} \geq \min\{X^{\delta}_{\gamma B}(x), X^{\delta}_{\gamma B}(y),\delta\}$.

Converse, let $(b_1 s)b_2 \in (BS)B$, where $b_1, b_2 \in B$ and $s \in S$. Now by hypothesis $\max\{X^{\delta}_{\gamma B}((b_1 s)b_2),\gamma\} \geq \min\{X^{\delta}_{\gamma B}(b_1), X^{\delta}_{\gamma B}(b_2),\delta\}$. Since $b_1, b_2 \in B$, therefore $X^{\delta}_{\gamma B}(b_1) \geq \delta$ and $X^{\delta}_{\gamma B}(b_2) \geq \delta$ which implies that $\min\{X^{\delta}_{\gamma B}(b_1), X^{\delta}_{\gamma B}(b_2),\delta\} = \delta$. Thus

$$\max\{X^{\delta}_{\gamma B}((b_1 s)b_2),\gamma\} \geq \delta.$$

This clearly implies that $X^{\delta}_{\gamma B}((b_1 s)b_2) \geq \delta$. Therefore $(b_1 s)b_2 \in B$. Hence B is a bi-ideal of S. ■

Definition 131 *A fuzzy AG-subgroupoid f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if for all $x, y, z \in S$ and $t, r \in (\gamma, 1]$ the following conditions holds.*

(I) $x_t \in_\gamma f$, $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

(II) $y_t \in_\gamma f$ implies $((xy)z)_t \in_\gamma \vee q_\delta f$.

Lemma 132 *A fuzzy subset f of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of an AG-groupoid S if and only if it satisfies the following conditions.*

(III) $\max\{f(xy),\gamma\} \geq \min\{f(x), f(y),\delta\}$ for all $x, y \in S$ and $\gamma, \delta \in [0,1]$.

(IV) $\max\{f(xyz),\gamma\} \geq \min\{f(y),\delta\}$ for all $x, y, z \in S$ and $\gamma, \delta \in [0,1]$.

Proof. $(I) \Rightarrow (III)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S. Let (I) holds. Let us consider on contrary. If there exists $x, y \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f(xy),\gamma\} < t \leq \min\{f(x), f(y),\delta\}.$$

Then $\max\{f(xy),\gamma\} < t \leq \gamma$ this implies that $(xy)_t \bar{\in}_\gamma f$ again implies that $(xy)_t \overline{\in_\gamma \vee q_\delta} f$. As $\min\{f(x), f(y),\delta\} \geq t > \gamma$ this implies that $f(x) \geq t > \gamma$ and $f(y) \geq t > \gamma$ implies that $x_t \in_\gamma f$ and $y_t \in_\gamma f$.

But $(xy)_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction .Thus

$$\max\{f(xy),\gamma\} \geq \min\{f(x), f(y),\delta\}.$$

$(III) \Rightarrow (I)$ Assume that $x, y,$ in S and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$ and $y_s \in_\gamma f$. Then $f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$, $\max\{f(xy),\gamma\} \geq \min\{f(x), f(y),\delta\} \geq \min\{t, s,\delta\}$. We consider two cases here,

Case(1): If $\{t, s\} \leq \delta$ then $\max\{f(xy),\gamma\} \geq \min\{t, s\} > \gamma$ this implies that $(xy)_{\min\{t,s\}} \in_\gamma f$.

Case(2): If $\{t,s\} > \delta$ then $f(xy) + \min\{t,s\} > 2\delta$ this implies that $(xy)_{\min\{t,s\}} q_\delta f$.

Hence $x_t \in_\gamma f$, $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

$(II) \Rightarrow (IV)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S. Let (II) holds. Let us consider on contrary. If there exists $x, y \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f((xy)z), \gamma\} < t \leq \min\{f(y), \delta\}.$$

Then $\max\{f((xy)z), \gamma\} < t \leq \gamma$ this implies that $((xy)z)_t \bar{\in}_\gamma f$ further implies that $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$. As $\min\{f(y), \delta\} \geq t > \gamma$ this implies that $f(y) \geq t > \gamma$ implies that $y_t \in_\gamma f$. But $(xyz)_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction according to definition. Thus (IV) is valid

$$\max\{f((xy)z), \gamma\} \geq \min\{f(y), \delta\}$$

$(IV) \Rightarrow (II)$ Assume that x, y, z in S and $t, s \in (\gamma, 1]$ such that $y_t \in_\gamma f$. Then $f(y) \geq t > \gamma$, by (IV) we write $\max\{f((xy)z), \gamma\} \geq \min\{f(y), \delta\} \geq \min\{t, \delta\}$. We consider two cases here,

Case(i): If $t \leq \delta$ then $f((xy)z) \geq t > \gamma$ this implies that $((xy)z)_t \in_\gamma f$.

Case(ii): If $t > \delta$ then $f((xy)z) + t > 2\delta$ this implies that $((xy)z) q_\delta f$.

From both cases $((xy)z)_t \in_\gamma \vee q_\delta f$. Hence f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S. ■

Lemma 133 *If I is a interior ideal of an AG-groupoid S if and only if $X_{\gamma I}^{\delta}$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy interior ideal of S.*

Proof. (i) Let $x, a, y \in I$ which implies that $(xa)y \in I$. Then by definition we get $X_{\gamma I}^{\delta}((xa)y) \geq \delta$ and $X_{\gamma I}^{\delta}(a) \geq \delta$, but $\delta > \gamma$. Thus

$$\begin{aligned} \max\{X_{\gamma I}^{\delta}((xa)y), \gamma\} &= X_{\gamma I}^{\delta}((xa)y) \text{ and} \\ \min\{X_{\gamma I}^{\delta}(a), \delta\} &= \delta. \end{aligned}$$

Hence $\max\{ X_{\gamma I}^{\delta}((xa)y), \gamma\} \geq \min\{X_{\gamma I}^{\delta}(a), \delta\}$.

(ii) Let $x \notin I$, $y \notin I$ and $a \in I$, which implies that $(xa)y \in I$. Then by definition $X_{\gamma I}^{\delta}((xa)y) \geq \delta$ and $X_{\gamma I}^{\delta}(a) \geq \delta$. Therefore

$$\begin{aligned} \max\{X_{\gamma I}^{\delta}((xa)y), \gamma\} &= X_{\gamma I}^{\delta}((xa)y), \text{ and} \\ \min\{X_{\gamma I}^{\delta}(a), \delta\} &= \delta. \end{aligned}$$

Hence $\max\{X_{\gamma I}^{\delta}((xa)y), \gamma\} \geq \min\{X_{\gamma I}^{\delta}(a), \delta\}$.

(iii) Let $x \in I, y \in I$ and $a \notin I$ which implies that $(xa)y \notin I$. Then by definition, we get $X_{\gamma I}^{\delta}((xa)y) \leq \gamma$, $X_{\gamma I}^{\delta}(a) \leq \gamma$. Thus

$$\begin{aligned} \max\{X_{\gamma I}^{\delta}((xa)y), \gamma\} &= \gamma \text{ and} \\ \min\{X_{\gamma I}^{\delta}(a), \delta\} &= X_{\gamma I}^{\delta}(a). \end{aligned}$$

Hence $\max\{X_{\gamma I}^{\delta}((xa)y), \gamma\} \geq \min\{X_{\gamma I}^{\delta}(a), \delta\}$.

(iv) Let $x, a, y \notin I$ which implies that $(xa)y \notin I$. Then by definition we get such that $X_{\gamma I}^{\delta}((xa)y) \leq \gamma$, $X_{\gamma I}^{\delta}(a) \leq \gamma$. Thus

$$\begin{aligned} \max\{X_{\gamma I}^{\delta}((xa)y), \gamma\} &= \gamma \text{ and} \\ \min\{X_{\gamma I}^{\delta}(a), \delta\} &= X_{\gamma I}^{\delta}(a). \end{aligned}$$

Hence $\max\{X_{\gamma I}^{\delta}((xa)y), \gamma\} \geq \min\{X_{\gamma I}^{\delta}(a), \delta\}$.

Conversely, let $(xa)y \in (SI)S$, where $a \in I$ and $x, y \in S$. Now by hypothesis $\max\{X_{\gamma I}^{\delta}((xa)y), \gamma\} \geq \min\{X_{\gamma I}^{\delta}(a), \delta\}$. Since $a \in I$, therefore $X_{\gamma I}^{\delta}(a) \geq \delta$ which implies that $\min\{X_{\gamma I}^{\delta}(a), \delta\} = \delta$. Thus

$$\max\{X_{\gamma I}^{\delta}((xa)y), \gamma\} \geq \delta.$$

This clearly implies that $X_{\gamma I}^{\delta}((xa)y) \geq \delta$. Therefore $(xa)y \in I$. Hence I is an interior ideal of S. ■

Example 134 *Consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.41 \text{ if } x = 1, \\ 0.44 \text{ if } x = 2, \\ 0.42 \text{ if } x = 3. \end{cases}$$

Then, we have

- *f is an $(\in_{0.1}, \in_{0.1} \vee q_{0.11})$-fuzzy quasi-ideal,*
- *f is not an $(\in, \in \vee q_{0.11})$-fuzzy quasi-ideal.*

4.1 $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy Prime Ideals of AG-groupoids

Definition 135 *An $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy subset f of an AG-groupoid S is said to be prime if for all a, b in S and $t \in (\gamma, 1]$. It satisfies,*

(1)$(ab)_t \in_{\gamma} f$ implies that $(a)_t \in_{\gamma} \vee q_{\delta} f$ or $(b)_t \in_{\gamma} \vee q_{\delta} f$.

Theorem 136 *An $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy prime ideal f of an AG-groupoid S if for all a, b in S, and $t \in (\gamma, 1]$.It satisfies*

(2) $\max\{f(a), f(b), \gamma\} \geq \min\{f(ab), \delta\}$.

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime ideal of an AG-groupoid S. If there exists a, b in S and $t \in (\gamma, 1]$, such that $\max\{f(a), f(b), \gamma\} < t \leq \min\{f(ab), \delta\}$ then $\min\{f(ab), \delta\} \geq t$ implies that $f(ab) \geq t > \gamma$ and $\min\{f(a), f(b), \gamma\} < t$ this implies that $f(a) < t \leq \gamma$ or $f(b) < t \leq \gamma$ again implies that $(a)_t \bar{\in}_\gamma f$ or $(b)_t \bar{\in}_\gamma f$ i.e. $(ab)_t \in_\gamma f$ but $(a)_t \overline{\in_\gamma \vee q_\delta} f$ or $(b)_t \overline{\in_\gamma \vee q_\delta} f$, which is a contradiction. Hence (2) is valid.

Conversely, assume that (2) is holds. Let $(ab)_t \in_\gamma f$. Then $f(ab) \geq t > \gamma$ and by (2) we have $\max\{f(a), f(b), \gamma\} \geq \min\{f(ab), \delta\} \geq \min\{t, \delta\}$. We consider two cases here,

Case(a): If $t \leq \delta$, then $f(a) \geq t > \gamma$ or $f(b) \geq t > \gamma$ this implies that $(a)_t \in_\gamma f$ or $(b)_t \in_\gamma f$.

Case(b): If $t > \delta$, then $f(a) + t > 2\delta$ or $f(b) + t > 2\delta$ this implies that $(a)_t q_\delta f$ or $(b)_t q_\delta f$. Hence f is prime. ■

Theorem 137 *Let I be an non empty subset of an AG-groupoid S with left identity. Then*

(i) I is a prime ideal.

(ii) $\chi^\delta_{\gamma I}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime ideal of S.

Proof. $(i) \Rightarrow (ii)$. Let I be a prime ideal of an AG-groupoid S. Let $(ab) \in I$ then $\chi^\delta_{\gamma I}(ab) \geq \delta$, this implies that so $ab \in I$ and I is prime, so $a \in I$ or $b \in I$, by definition we can get $\chi^\delta_{\gamma I}(a) \geq \delta$ or $\chi^\delta_{\gamma I}(b) \geq \delta$, therefore

$$\begin{aligned} \min\{\chi^\delta_{\gamma I}(ab), \delta\} &= \delta \text{ and} \\ \max\{\chi^\delta_{\gamma I}(a), \chi^\delta_{\gamma I}(b), \gamma\} &= \max\{\chi^\delta_{\gamma I}(a), \chi^\delta_{\gamma I}(b)\} \geq \delta. \end{aligned}$$

which implies that $\max\{\chi^\delta_{\gamma I}(a), \chi^\delta_{\gamma I}(b), \gamma\} \geq \min\{\chi^\delta_{\gamma I}(ab), \delta\}$. Hence $\chi^\delta_{\gamma I}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime ideal of S.

$(ii) \Rightarrow (i)$. Assume that $\chi^\delta_{\gamma I}$ is a prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S, then I is prime. Let $(ab) \in I$ by definition we can write $\chi^\delta_{\gamma I}(ab) \geq \delta$, therefore,by given condition we have $\max\{\chi^\delta_{\gamma I}(a), \chi^\delta_{\gamma I}(b), \gamma\} \geq \min\{\chi^\delta_{\gamma I}(ab), \delta\} = \delta$. this implies that $\chi^\delta_{\gamma I}(a) \geq \delta$ or $\chi^\delta_{\gamma I}(b) \geq \delta$ this implies that $a \in I$ or $b \in I$. Hence I is prime. ■

Example 138 *Let $S = \{1, 2, 3\}$, and the binary operation "$\cdot$" be defined on S as follows.*

$\cdot$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Then $(S, \cdot)$ is an intra-regular AG-groupoid with left identity 1. Define a fuzzy subset $f : S \to [0, 1]$ as follows.

$$f(x) = \begin{cases} 0.34 \text{ for } x = 1 \\ 0.36 \text{ for } x = 2 \\ 0.35 \text{ for } x = 3 \end{cases}$$

Then clearly

- f *is an* $(\in_{0.2}, \in_{0.2} \vee q_{0.22})$*-fuzzy prime ideal,*
- f *is not an* $(\in, \in \vee q_{0.22})$*-fuzzy prime ideal,*
- f *is not fuzzy prime ideal.*

Theorem 139 *An* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy subset* f *of an AG-groupoid* S *is prime if and only if* $U(f,t)$ *is prime in AG-groupoid* S*, for all* $0 < t \leq \delta$.

Proof. Let us consider an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset f of an AG-groupoid S is prime and $0 < t \leq \delta$. Let $(ab) \in_\gamma U(f,t)$ this implies that $f(ab) \geq t > \gamma$. Then by theorem 136 $\max\{f(a), f(b), \gamma\} \geq \min\{f(ab), \delta\} \geq \min\{t, \delta\} = t$, so $f(a) \geq t > \gamma$ or $f(b) \geq t > \gamma$, which implies that $a \in_\gamma U(f,t)$ or $b \in_\gamma U(f,t)$. Therefore $U(f,t)$ is prime in AG-groupoid S, for all $0 < t \leq \delta$.

Conversely, assume that $U(f,t)$ is prime in AG-groupoid S, for all $0 < t \leq \delta$. Let $(ab)_t \in_\gamma f$ implies that $ab \in_\gamma U(f,t)$, and $U(f,t)$ is prime, so $a \in_\gamma U(f,t)$ or $b \in_\gamma U(f,t)$, that is $a_t \in_\gamma f$ or $b_t \in_\gamma f$. Thus $a_t \in_\gamma \vee q_\delta f$ or $b_t \in_\gamma \vee q_\delta f$. Therefore f must be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime in AG-groupoid S. ■

Definition 140 *A fuzzy subset* f *of an AG-groupoid* S *is said to be* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy semiprime for all* $s, t \in (\gamma, 1]$ *and* $a \in S$*. it satisfies*

(1) $a_t^2 \in_\gamma f$ implies that $a_t \in_\gamma \vee q_\delta f$.

Theorem 141 *A fuzzy subset* f *of an AG-groupoid* S *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy semiprime if and only if it satisfies*

(2) $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}$ for all $a \in S$.

Proof. (1) $\Rightarrow$ (2) Let f be a fuzzy subset of an AG-groupoid S which is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime of S. Assume that there exists $a \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(a), \gamma\} < t \leq \min\{f(a^2), \delta\}.$$

Then $\max\{f(a), \gamma\} < t$ this implies that $f(a) < t \leq \gamma$, implies that , $f(a) + t < 2t \leq 2\delta$ this implies that $a_t \overline{\in_\gamma \vee q_\delta} f$ and $\min\{f(a^2), \delta\} \geq t$ this implies that $f(a^2) \geq t > \gamma$, further implies that $a_t^2 \in_\gamma f$ but $a_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Hence (2) is valid,

$$\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}, \text{ for all } a \in S.$$

(2) $\Rightarrow$ (1). Assume that there exist $a \in S$ and $t \in (\gamma, 1]$ such that $a_t^2 \in_\gamma f$, then $f(a^2) \geq t > \gamma$, thus by (2), we have $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\} \geq \min\{t, \delta\}$. We consider two cases here,

Case(i): if $t \leq \delta$, then $f(a) \geq t > \gamma$, this implies that $a_t \in_\gamma f$.

Case(ii) : if $t > \delta$, then $f(a) + t > 2\delta$, that is $a_t q_\delta f$. From (i) and (ii) we write $a_t \in_\gamma \vee q_\delta f$. Hence f is semiprime for all $a \in S$. ■

Theorem 142 *For a non empty subset I of an AG-groupoid S with left identity the following conditions are equivalent.*
(i) I is semiprime.
(ii) $\chi_{\gamma I}^{\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (ii)$ Let I is semiprime of an AG-groupoid S.
Case(a): Let a be any element of S such that $a^2 \in I$. Then I is semiprime, so $a \in I$. Hence $\chi_{\gamma I}^{\delta}(a^2) \geq \delta$ and $\chi_{\gamma I}^{\delta}(a) \geq \delta$. Therefore

$$\begin{aligned} \max\{\chi_{\gamma I}^{\delta}(a), \gamma\} &= \chi_{\gamma I}^{\delta}(a) \text{ and} \\ \min\{\chi_{\gamma I}^{\delta}(a^2), \delta\} &= \delta. \end{aligned}$$

which implies that $\max\{\chi_{\gamma I}^{\delta}(a), \gamma\} \geq \min\{\chi_{\gamma I}^{\delta}(a^2), \delta\}$.
Case(b): Let $a \notin I$, since I is semiprime therefore $a^2 \notin I$. This implies that $\chi_{\gamma I}^{\delta}(a) \leq \gamma$ and $\chi_{\gamma I}^{\delta}(a^2) \leq \gamma$, such that

$$\begin{aligned} \max\{\chi_{\gamma I}^{\delta}(a), \gamma\} &= \gamma \text{ and} \\ \min\{\chi_{\gamma I}^{\delta}(a^2), \delta\} &= \chi_{\gamma I}^{\delta}(a^2). \end{aligned}$$

Therefore $\max\{\chi_{\gamma I}^{\delta}(a), \gamma\} \geq \min\{\chi_{\gamma I}^{\delta}(a^2), \delta\}$. Hence in both cases

$$\max\{\chi_{\gamma I}^{\delta}(a), \gamma\} \geq \min\{\chi_{\gamma I}^{\delta}(a^2), \delta\} forallainS.$$

$(ii) \Rightarrow (i)$ Let $\chi_{\gamma I}^{\delta}$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. Let $a^2 \in I$ for some a in S. Then $\chi_{\gamma I}^{\delta}(a^2) \geq \delta$. Therefore $\max\{\chi_{\gamma I}^{\delta}(a), \gamma\} \geq \min\{\chi_{\gamma I}^{\delta}(a^2), \delta\} = \delta$ this implies that $\chi_{\gamma I}^{\delta}(a) \geq \delta$ again this implies that $a \in I$. Hence I is semiprime. ■

Example 143 *Let $S = \{1, 2, 3\}$, and the binary operation "$\cdot$" be defined on S as follows.*

$\cdot$	1	2	3
1	3	2	3
2	3	3	3
3	3	3	3

Then $(S, \cdot)$ is an AG-groupoid. Define a fuzzy subset $f : S \to [0, 1]$ as follows.

$$f(x) = \begin{cases} 0.41 \text{ for } x = 1 \\ 0.39 \text{ for } x = 2 \\ 0.42 \text{ for } x = 3 \end{cases}$$

Then clearly

- *f is $(\in_{0.1}, \in_{0.1} \vee q_{0.2})$-fuzzy semiprime,*
- *f is not $(\in, \in \vee q_{0.2})$-fuzzy semiprime,*
- *f is not fuzzy semiprime.*

4.2 $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Semiprime Ideals of Intra-regular AG-groupoids

Lemma 144 *If f is a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an intra-regular AG-groupoid S, then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime in S.*

Proof. Let S be a intra regular AG-groupoid. Then for any $a \in S$ there exists some $x, y \in S$ such that $a = (xa^2)y$. Now

$$\max\{f(a), \gamma\} = \max\{f(xa^2)y, \gamma\} \geq \min\{f(a^2), \delta\}.$$

Hence f is a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime in S. ■

Theorem 145 *Let S be an AG- groupoid then the following conditions are equivalent.*
(i) S is intra regular.
(ii) For every ideal A of S, $A \subseteq A^2$ and A is semiprime.
(iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy ideal f of S, $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$, and f is fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an intra regular AG-groupoid S with left identity. Now since S is intra regular therefore for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using paramedial law, medial law and left invertive law, we get

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a.$$

Let for any a in S there exist p and q in S such that $a = pq$, then

$$\begin{aligned}
\max\{(f \circ f)(a), \gamma\} &= \max\left\{\bigvee_{a=pq} \{\{f(p) \wedge f(q)\}, \gamma\}\right\} \\
&\geq \max\{\min\{f(y(xa)), f(a)\}, \gamma\} \\
&\geq \max\{\min\{f(y(xa)), f(a)\}, \gamma\} \\
&= \min\{\max\{f(y(xa)), \gamma\}, \max\{f(a), \gamma\}\} \\
&\geq \min\{\min\{f(a), \delta\}, \min\{f(a), \delta\}\} \\
&= \min\{f(a), \delta\}.
\end{aligned}$$

Thus $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$.

Now we show that f is a fuzzy semiprime ideal of intra-regular AG-groupoid S, Since S is intra-regular therefore for any a in S there exist x, y in S such that $a = (xa^2)y$. Then

$$\begin{aligned}
\max\{f(a), \gamma\} &= \max\{f((xa^2)y), \gamma\} \\
&\geq \min\{f(a^2), \delta\}.
\end{aligned}$$

$(iii) \Rightarrow (ii)$. Suppose A be any ideal of S. Then by (iii), we get

$$\chi^{\delta}_{\gamma A} = \chi^{\delta}_{\gamma A\cap A} = \chi^{\delta}_{\gamma A} \cap \chi^{\delta}_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma A} =_{(\gamma,\delta)} X^{\delta}_{\gamma A^2}.$$

Hence we get $A \subseteq A^2$. Now we show that A is semiprime. Let A is an ideal then $(\chi^{\delta}_{\gamma A})$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S. Let $a^2 \in A$, then since $\chi^{\delta}_{\gamma A}$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an AG-groupoid S, hence by (iii), $\max\{\chi^{\delta}_{\gamma A}(a), \gamma\} \geq \min\{\chi^{\delta}_{\gamma A}(a^2), \delta\} = \delta$ this implies that $\chi^{\delta}_{\gamma A}(a) \geq \delta$. Thus $a \in A$. This implies that A is semiprime.

$(ii) \Rightarrow (i)$. Assume that every ideal is semiprime of S. Since Sa^2 is a ideal of an AG-groupoid S generated by a^2. Therefore

$$a \in (Sa^2) \subseteq (SS)a^2 \subseteq (a^2S)S = ((aa)(SS))S = ((SS)(aa))S = (Sa^2)S.$$

Hence S is intra regular. ■

Lemma 146 *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an AG-groupoid S, is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.*

Proof. Let S be an AG-groupoid then for any $a, x, y \in S$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal. Now

$$\begin{aligned} \max\{f((xa)y), \gamma\} &\geq \max\{f(xa), \gamma\} \\ &\geq \min\{f(a), \delta\}. \end{aligned}$$

Hence f is a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S. ■

Theorem 147 *For an AG-groupoid S with left identity the following are equivalent.*

(i) *S is intra regular.*

(ii) *Every two sided ideal is semiprime.*

(iii) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal f of S is fuzzy semiprime.*

(iv) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal f of S is fuzzy semiprime.*

(v) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized interior ideal f of S is semiprime.*

Proof. $(i) \Rightarrow (v)$ Let S be an intra-regular and f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized interior ideal of an AG-groupoid S. Then for all $a \in S$ there exists x, y in S such that $a = (xa^2)y$. We have

$$\begin{aligned} \max\{f(a), \gamma\} &= \max\{f((xa^2)y), \gamma\} \\ &\geq \min\{f(a^2), \delta\}. \end{aligned}$$

$(v) \Rightarrow (iv$ is obvious.

$(iv) \Rightarrow (iii)$ it is obvious by lemma 146.

$(iii) \Rightarrow (ii)$. Let A be a two sided ideal of an AG-groupoid S, then $(\chi^{\delta}_{\gamma A})$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal of S. Let $a^2 \in A$, then since $\chi^{\delta}_{\gamma A}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal therefore $\chi^\delta_{\gamma A}(a^2) \geq \delta$, thus by (iii) $\max\{\chi^\delta_{\gamma A}(a), \gamma\} \geq \min\{\chi^\delta_{\gamma A}(a^2), \delta\} = \delta$ this implies that $\chi^\delta_{\gamma A}(a) \geq \delta$. Thus $a \in A$. Hence A is semiprime.

$(ii) \Rightarrow (i)$. Assume that every two sided ideal is semiprime and since Sa^2 is a two sided ideal contain a^2. Thus

$$a \in (Sa^2) \subseteq (SS)a^2 \subseteq (a^2S)S = ((aa)(SS))S = ((SS)(aa))S = (Sa^2)S.$$

Hence S is an intra-regular. ■

Theorem 148 *Let S be an AG-groupoid with left identity, then the following conditions equivalent*

(i) *S is intra-regular.*

(ii) *Every ideal of S is semiprime.*

(iii) *Every bi-ideal of S is semiprime.*

(iv) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f of S is semiprime.*

(v) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f of S is semiprime.*

Proof. $(i) \Rightarrow (v)$. Let S be an intra-regular and f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-generalized bi-ideal of S. Then for all $a \in S$ there exists x, y in S such that $a = (xa^2)y$.

$$\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\
&= \{y(x((xa^2)y)))\}a = \{x(y((xa^2)y))\}a = \{x((xa^2)y^2))\}a \\
&= \{(xa^2)(xy^2)\}a = \{x^2(a^2y^2)\}a = \{a^2(x^2y^2)\}a = \{a(x^2y^2)\}a^2 \\
&= \{((xa^2)y)(x^2y^2)\}a^2 = \{(y^2y)(x^2(xa^2))\}a^2 = \{(y^2x^2)(y(xa^2))\}a^2 \\
&= \{(y^2x^2)((y_1y_2)(xa^2))\}a^2 = \{(y^2x^2)((a^2y_2)(xy_1))\}a^2 = \{(y^2x^2)((a^2x)(y_2y_1))\}a^2 \\
&= \{(y^2x^2)(((y_2y_1)x)(aa))\}a^2 = \{(y^2x^2)(a^2(x(y_2y_1)))\}a^2 = \{(aa)(x((x^2y^2))(y_2y_1))\}a^2 \\
&= \{a^2\{(x(x^2y^2)(y_2y_1))\}\}a^2 = (a^2t)a^2, \text{where } t = (x(x^2y^2)(y_2y_1)).
\end{aligned}$$

we have
$\max\{f(a), \gamma\} = \max\{f(a^2t)a^2, \gamma\} \geq \max\{\min\{f(a^2), f(a^2)\}, \delta\} = \min\{f(a^2), \delta\}$.
Therefore $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}$.

$(v) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (iii)$. Let B is a bi-ideal of S, then $\chi^\delta_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of an AG-groupoid S. let $a^2 \in B$ then since $\chi^\delta_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal therefore $\chi^\delta_{\gamma B}(a^2) \geq \delta$, thus by (iv), $\max\{\chi^\delta_{\gamma B}(a), \gamma\} \geq \min\{\chi^\delta_{\gamma B}(a^2), \delta\} = \delta$ this implies that $\chi^\delta_{\gamma B}(a) \geq \delta$. Thus $a \in B$. Hence B is semiprime.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Assume that every ideal of S is semiprime and since Sa^2 is an ideal containing a. Thus

$$a \in (Sa^2) \subseteq (SS)a^2 \subseteq (a^2S)S = ((aa)(SS))S = ((SS)(aa))S = (Sa^2)S.$$

Hence S is an intra-regular. ■

Theorem 149 *Let S be an AG-groupoid with left identity, then the following conditions equivalent*
(i) S is intra-regular.
(ii) Every ideal of S is semiprime.
(iii) Every quasi-ideal of S is semiprime.
(iv) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f of S is semiprime.

Proof. $(i) \Rightarrow (iv)$. Let S be an intra-regular AG-groupoid with left identity and f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal of S. Then for all $a \in S$ there exists x, y in S such that $a = (xa^2)y$. Now using left invertive law and medial law, then

$$a = (xa^2)(y_1y_2) = (y_2y_1)(a^2x) = a^2((y_2y_1)x) = a^2t, \text{ where } t = (y_2y_1)x.$$

we have
$\max\{f(a), \gamma\} = \max\{f(a^2t), \gamma\} \geq \min\{f(a^2), \delta\}$.
Therefore $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}$.
$(iv) \Rightarrow (iii)$. let Q be an quasi ideal of S, then $\chi^\delta_{\gamma Q}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal of an AG-groupoid S. let $a^2 \in Q$ then since $\chi^\delta_{\gamma Q}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal as then $\chi^\delta_{\gamma Q}(a^2) \geq \delta$ therefore by (iv), $\max\{\chi^\delta_{\gamma Q}(a), \gamma\} \geq \min\{\chi^\delta_{\gamma Q}(a^2), \delta\} = \delta$ this implies that $\chi^\delta_{\gamma Q}(a) \geq \delta$. Thus $a \in Q$. Hence Q is semiprime.
$(iii) \Rightarrow (ii)$ is obvious.
$(ii) \Rightarrow (i)$ Assume that every ideal of S is semiprime and since Sa^2 is an ideal containing a^2. Thus

$$a \in (Sa^2) \subseteq (Sa)(Sa) = (SS)(aa) = (a^2S)S = (Sa^2)S.$$

Hence S is an intra-regular. ∎

5

Fuzzy Soft Abel Grassmann Groupoids

In this chapter we introduce generalized fuzzy soft ideals in a non-associative algebraic structure namely Abel Grassmann groupoid. We discuss some basic properties concerning these new types of generalized fuzzy ideals in Abel-Grassmann groupoids. Moreover we characterize a regular Abel Grassmann groupoid in terms of its classical and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideals.

5.1 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Soft Ideals of AG-groupoids

Let U be an initial universe set and E the set of all possible parameters under consideration with respect to U. Then

A pair $\langle F, A\rangle$ is called a fuzzy soft set over U, where $A \subseteq E$ and F is a mapping given by $F : A \longrightarrow \mathcal{F}(U)$, where $\mathcal{F}(U)$ is the set of all fuzzy subsets of U. In general, for every $\varepsilon \in A$, $F(\varepsilon)$ is a fuzzy set of U and it is called fuzzy value set of parameter ε [26].

The extended intersection of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is a fuzzy soft set denoted by $\langle H, C\rangle$, where $C = A \cup B$ and defined as

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cap G(\varepsilon) & \text{if } \varepsilon \in A \cap B. \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle H, C\rangle = \langle F, A\rangle \,\tilde{\cap}\, \langle G, B\rangle$.

A new relation is defined on $\mathcal{F}(S)$ denoted as " $\subseteq \vee q_{(\gamma,\delta)}$",as follows.

For any $f, g \in \mathcal{F}(S)$, by $f \subseteq \vee q_{(\gamma,\delta)} g$, we mean that $x_r \in_\gamma f$ implies $x_r \in_\gamma \vee q_\delta g$ for all $x \in S$ and $r \in (\gamma, 1]$.

The following definition is available in [35].

Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two fuzzy soft sets over U. We say that $\langle F, A\rangle$ is a fuzzy soft subset of $\langle G, B\rangle$ and write $\langle F, A\rangle \subset \langle G, B\rangle$ if

(i) $A \subseteq B$;

(ii) For any $\varepsilon \in A$, $F(\varepsilon) \subseteq G(\varepsilon)$.

$\langle F, A\rangle$ and $\langle G, B\rangle$ are said to be fuzzy soft equal and write $\langle F, A\rangle = \langle G, B\rangle$ if $\langle F, A\rangle \subset \langle G, B\rangle$ and $\langle G, B\rangle \subset \langle F, A\rangle$.

Let $V \subseteq U$. A fuzzy soft set $\langle F, A\rangle$ over V is said to be a relative whole $(\gamma; \delta)$-fuzzy soft set (with respect to universe set V and parameter set A), denoted by $\Sigma(V; A)$, if $F(\varepsilon) = \mathcal{X}_\gamma^\delta$ for all $\varepsilon \in A$.

The product of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over an AG-groupoid S is a fuzzy soft set over S, denoted by $\langle F \circ G, C\rangle$, where $C = A \cup B$ and

$$(F \circ G)(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \circ G(\varepsilon) & \text{if } \varepsilon \in A \cap B. \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle F \circ G, C\rangle = \langle F, A\rangle \odot \langle G, B\rangle$.

A fuzzy soft set $\langle F, A\rangle$ over an AG-groupoid S is called

- Fuzzy soft left (right) ideal over S if $\Sigma \langle S, A\rangle \odot \langle F, A\rangle \subset \langle F, A\rangle (\langle F, A\rangle \odot \Sigma \langle S, E\rangle \subset \langle F, A\rangle)$.
- Fuzzy soft bi-ideal over S if $\langle F, A\rangle \odot \langle F, A\rangle \subset \langle F, A\rangle$ and $[\langle F, A\rangle \odot \Sigma \langle S, A\rangle] \odot \langle F, A\rangle \subset \langle F, A\rangle$.
- Fuzzy soft quasi-ideal over S if $[\langle F, A \odot \Sigma(S, A)]\tilde{\cap}[\Sigma(S, A) \odot \langle F, A\rangle] \subset \langle F, A\rangle$.

$\langle F, A\rangle$ is an (γ, δ)-fuzzy soft subset of $\langle G, B\rangle$ and write $\langle F, A\rangle \subset_{(\gamma,\delta)} \langle G, B\rangle$ if (i) $A \subseteq B$, and (ii) For any $\varepsilon \in A, F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} G(\varepsilon)$.

A fuzzy soft set $\langle F, A\rangle$ over an AG-groupoid S is called

- An $(\in_\gamma, \in_\gamma \vee q\ _\delta)$-fuzzy soft left ideal over S if $\Sigma(S, A) \odot \langle F, A\rangle \subset_{(\gamma,\delta)} \langle F, A\rangle$.
- An $(\in_\gamma, \in_\gamma \vee q\ _\delta)$-fuzzy soft bi-ideal over S if (i) $\langle F, A\rangle \odot \langle F, A\rangle \subset_{(\gamma,\delta)} \langle F, A\rangle$, and (ii) $[\langle F, A\rangle \odot \Sigma(S, A)] \odot \langle F, A\rangle \subset_{(\gamma,\delta)} \langle F, A\rangle$.
- An $(\in_\gamma, \in_\gamma \vee q\ _\delta)$-fuzzy soft quasi-ideal over S if $[\langle F, A \odot \Sigma(S, A)]\tilde{\cap}[\Sigma(S, A) \odot \langle F, A\rangle] \subset_{(\gamma,\delta)} \langle F, A\rangle$.

Example 150 *Let $S = \{1, 2, 3\}$ and the binary operation "$\cdot$" defines on S as follows:*

$\cdot$	1	2	3
1	2	2	3
2	3	3	3
3	3	3	3

Then $(S, \cdot)$ is an AG-groupoid. Let $A = \{0.35, 0.4\}$ and define a fuzzy soft set $\langle F, A\rangle$ over S as follows:

$$F(\epsilon)(x) = \begin{cases} 2\epsilon & \textit{if } x \in \{1, 2\}, \\ \frac{2}{5} & \textit{otherwise.} \end{cases}$$

Then $\langle F, A\rangle$ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$- fuzzy soft left ideal of S.

Again let $E = \{0.7, 0.8\}$ and define a fuzzy soft set $\langle G, E\rangle$ over S as follows:

$$G(\epsilon)(x) = \begin{cases} \epsilon & \textit{if } x \in \{1, 2\}, \\ \frac{2}{5} & \textit{otherwise.} \end{cases}$$

Then $\langle F, E\rangle$ is an $(\in_{0.2}, \in_{0.2} \vee q_{0.4})$- fuzzy soft bi-ideal of S.

5.2 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Soft Ideals in Regular AG-groupoids

Theorem 151 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) S is regular.

(ii) For bi-ideal B, ideal I and left ideal L of S, $B \cap L \subseteq (BS)L$.

(iii) $\langle F, B\rangle \tilde{\cap} \langle G, L\rangle \subset_{(\gamma,\delta)} (\langle F, B\rangle \odot \Sigma\langle S, E\rangle) \odot \langle G, L\rangle$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle F, A\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle H, B\rangle$ of S.

(iv) $\langle F, B\rangle \tilde{\cap} \langle G, L\rangle \subset_{(\gamma,\delta)} (\langle F, B\rangle \odot \langle S, E\rangle \odot \langle G, L\rangle)$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft bi-ideal $\langle F, A\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft left ideal $\langle H, B\rangle$ of S.

Proof. $(i) \Rightarrow (iv)$

Let $\langle F, B\rangle$ and $\langle G, L\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft left ideal over S, respectively. Let a be any element of S, $\langle F, B\rangle \tilde{\cap} \langle G, L\rangle = \langle K_1, B \cup L\rangle$.For any $\varepsilon \in B \cup L$. We consider the following cases,

Case 1: $\varepsilon \in B - L$. Then $K_1(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $K_2(\varepsilon) = (F \circ G)(\varepsilon)$, so we have $K_1(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} K_2(\varepsilon)$

Case 2: $\varepsilon \in L - B$. Then $K_1(\varepsilon) = H(\varepsilon)$ and $K_1(\varepsilon) = H(\varepsilon) = K_2(\varepsilon)$

Case 3: $\varepsilon \in B \cap L$. Then $K_1(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $K_2(\varepsilon) = F(\varepsilon) \circ G(\varepsilon)$. Now we show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ G(\varepsilon))$. Now since S is regular AG-groupoid, so for $a \in S$ there exist $x \in S$ such that using left invertive law and also using law $a(bc) = b(ac)$, we have,

$$a = (ax)a = [\{(ax)a\}x]a \in (BS)L.$$

Thus we have,

$$\begin{aligned}
& \max\left\{((F(\varepsilon) \circ \mathcal{X}^\delta_{\gamma S}) \circ G(\varepsilon))(a), \gamma\right\} \\
= \quad & \max\left\{ \bigvee_{a=pq} (F(\varepsilon) \circ \mathcal{X}^\delta_{\gamma S})(p) \wedge G(\varepsilon)(q), \gamma \right\} \\
\geq \quad & \max\{(F(\varepsilon) \circ \mathcal{X}^\delta_{\gamma S})[\{(ax)a\}(x)] \wedge G(\varepsilon))(a), \gamma\} \\
= \quad & \max\{ \bigvee_{\{(ax)a\}(x)=uv} (F(\varepsilon)(u) \wedge \mathcal{X}^\delta_{\gamma S}(v)) \wedge G(\varepsilon)(a)\}, \gamma\} \\
\geq \quad & \max\left\{F(\varepsilon)\{(ax)a\} \wedge \mathcal{X}^\delta_{\gamma S}(x) \wedge G(\varepsilon)(a), \gamma\right\} \\
= \quad & \max\left\{F(\varepsilon)\{(ax)a\} \wedge 1 \wedge G(\varepsilon)(a), \gamma\right\} \\
= \quad & \min\left\{\max\{F(\varepsilon)\{(ax)a\}, \gamma\}, \max\{G(\varepsilon)(a), \gamma\}\right\} \\
\geq \quad & \min\{\min\{(F(\varepsilon)(a), \delta\}, \min\{G(\varepsilon)(a), \delta\}\} \\
= \quad & \min\left\{(F(\varepsilon) \wedge G(\varepsilon))(a), \delta\right\} \\
= \quad & \min\left\{(F(\varepsilon) \cap G(\varepsilon))(a), \delta\right\}
\end{aligned}$$

Thus $\min\{(F(\varepsilon)\cap G(\varepsilon))(a),\delta\}\leq\max\left\{((F(\varepsilon)\circ\mathcal{X}^{\delta}_{\gamma S})\circ G(\varepsilon))(a),\gamma\right\}$. This implies that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F(\varepsilon)\circ\mathcal{X}^{\delta}_{\gamma S})\circ G(\varepsilon)$.

Therefore in any case, we have $K_1(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}K_2(\varepsilon)$. Hence

$$\langle F,B\rangle\tilde{\cap}\langle G,L\rangle\subset_{(\gamma,\delta)}(\langle F,B\rangle\odot\langle S,E\rangle)\odot\langle G,L\rangle.$$

$(iv)\implies(iii)$ is obvious.

$(iii)\implies(ii)$

Assume that B and L are bi-ideal and left ideal of S, respectively, then $\Sigma(B,E)$ and $\Sigma(L,E)$ are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S,respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(B,E)\tilde{\cap}\Sigma(L,E)\subset_{(\gamma,\delta)}(\Sigma(B,E)\odot\Sigma(S,E))\odot\Sigma(L,E).$$

So,

$$\begin{aligned}\chi^{\delta}_{\gamma(B\cap L)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma B}\cap\chi^{\delta}_{\gamma L}\\ &\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma B}\circ\chi^{\delta}_{\gamma S})\circ\chi^{\delta}_{\gamma L}\\ &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma(BS)L}.\end{aligned}$$

Thus $B\cap L\subseteq(BS)L$.

$(ii)\Rightarrow(i)$

$B[a]=a\cup a^2\cup(aS)a$, and $L[a]=a\cup Sa$ are principle bi-ideal and principle left ideal of S generated by a respectively. Thus by (ii) left invertive law,medial law, paramedial law and using law $a(bc)=b(ac)$, we have,

$$\begin{aligned}(Sa)\cap(Sa) &\subseteq [Sa)S](Sa)=[S(aS)](Sa)\\ &= (aS)(Sa)\subseteq(aS)\,a.\end{aligned}$$

Hence S is regular. ■

Theorem 152 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) S is regular.

(ii) For an ideal I and left ideal L of S, $I\cap L\subseteq(IS)L$.

(iii) $\langle F,I\rangle\tilde{\cap}\langle G,L\rangle\subset_{(\gamma,\delta)}(\langle F,I\rangle\odot\Sigma\langle S,E\rangle\odot\langle G,L\rangle)$, for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal $\langle F,A\rangle$ and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal $\langle H,B\rangle$ of S.

(iv) $\langle F,I\rangle\tilde{\cap}\langle G,L\rangle\subset_{(\gamma,\delta)}(\langle F,I\rangle\odot\langle S,E\rangle)\odot\langle G,L\rangle$, for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized soft ideal $\langle F,A\rangle$ and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized soft left ideal $\langle H,B\rangle$ of S.

Proof. $(i)\Rightarrow(iv)$

Let $\langle F,I\rangle$ and $\langle G,L\rangle$ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized soft ideal and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized soft left ideal over S, respectively. Let a be any element of S, $\langle F,I\rangle\tilde{\cap}\langle G,L\rangle=\langle K_1,I\cup L\rangle$.For any $\varepsilon\in I\cup L$. We consider the following cases, Case 1: $\varepsilon \in I - L$. Then $K_1(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $K_2(\varepsilon) = (F \circ G)(\varepsilon)$, so we have $K_1(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} K_2(\varepsilon)$

Case 2: $\varepsilon \in L - I$. Then $K_1(\varepsilon) = H(\varepsilon)$ and $K_1(\varepsilon) = H(\varepsilon) = K_2(\varepsilon)$

Case 3: $\varepsilon \in I \cap L$. Then $K_1(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $K_2(\varepsilon) = F(\varepsilon) \circ G(\varepsilon)$. Now we show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ G(\varepsilon))$. Now since S is regular AG-groupoid, so for $a \in S$ there exist $x \in S$ such that using medial law, we have,

$$a = (ax)\,a = (ax)\{(ax)\,a\} = \{a(ax)\}(xa) \in (IS)L.$$

Thus we have,

$$\begin{aligned}
& \max\left\{((F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S}) \circ G(\varepsilon))(a), \gamma\right\} \\
= \; & \max\left\{\bigvee_{a=pq} (F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S})(p) \wedge G(\varepsilon)(q), \gamma\right\} \\
\geq \; & \max\{(F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S})\{a(ax)\} \wedge G(\varepsilon))(xa), \gamma\} \\
= \; & \max\{\bigvee_{\{a(ax)\}=uv} (F(\varepsilon)(u) \wedge \mathcal{X}^{\delta}_{\gamma S}(v)) \wedge G(\varepsilon)(xa)\}, \gamma\} \\
\geq \; & \max\left\{F(\varepsilon)(a) \wedge \mathcal{X}^{\delta}_{\gamma S}(ax) \wedge G(\varepsilon)(xa), \gamma\right\} \\
= \; & \max\{F(\varepsilon)(a) \wedge 1 \wedge G(\varepsilon)(xa), \gamma\} \\
= \; & \min\{\max\{F(\varepsilon)(a), \gamma\}, \max\{G(\varepsilon)(xa), \gamma\}\} \\
\geq \; & \min\{\min\{(F(\varepsilon)(a), \delta\}, \min\{G(\varepsilon)(xa), \delta\}\} \\
= \; & \min\{(F(\varepsilon) \wedge G(\varepsilon))(a), \delta\} \\
= \; & \min\{(F(\varepsilon) \cap G(\varepsilon))(a), \delta\}
\end{aligned}$$

Thus $\min\{(F(\varepsilon) \cap G(\varepsilon))(a), \delta\} \leq \max\left\{((F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S}) \circ G(\varepsilon))(a), \gamma\right\}$. This implies that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S}) \circ G(\varepsilon)$.

Therefore in any case, we have $K_1(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} K_2(\varepsilon)$. Hence

$$\langle F, I\rangle \tilde{\cap} \langle G, L\rangle \subset_{(\gamma,\delta)} (\langle F, I\rangle \odot \Sigma\langle S, E\rangle) \odot \langle G, L\rangle.$$

$(iv) \implies (iii)$ is obvious.

$(iii) \implies (ii)$ ■

Assume that I and L are ideal and left ideal of S, respectively, then $\Sigma(I, E)$ and $\Sigma(L, E)$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal over S,respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(I, E) \tilde{\cap} \Sigma(L, E) \subset_{(\gamma,\delta)} (\Sigma(I, E) \odot \Sigma(S, E)) \odot \Sigma(L, E).$$

So,

$$\begin{aligned}
\chi^{\delta}_{\gamma(I\cap L)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma I} \cap \chi^{\delta}_{\gamma L} \\
&\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma I} \circ \chi^{\delta}_{\gamma S}) \circ \chi^{\delta}_{\gamma L} \\
&= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma(IS)L}.
\end{aligned}$$

Thus $I \cap L \subseteq (IS)L$.
Proof. $(ii) \Rightarrow (i)$

$I[a] = aS \cup Sa$, and $L[a] = a \cup Sa$ are principle .ideal and principle left ideal of S generated by a respectively. Thus left invertive law, paramedial law and using law $a(bc) = b(ac)$, we have,

$$\begin{aligned}
\{(aS)\cup(Sa)\}\cap(Sa) &\subseteq [\{(aS)\cup(Sa)\}S](Sa) \\
&= [\{(aS)S\}\cup\{(Sa)S\}](Sa) \\
&= [\{S(Sa)\}\cup\{(Sa)S\}](Sa) \\
&= [Sa\cup\{(Sa)S\}](Sa) \\
&= [(Sa)(Sa)]\cup\{(Sa)S\}(Sa) \\
&\subseteq (Sa^2)S\cup(aS)\,a.
\end{aligned}$$

Hence S is regular. ■

Theorem 153 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For bi-ideal B of S, $B \subseteq B^2S$.*

(iii) $\langle F,B\rangle \subset_{(\gamma,\delta)} (\langle F,B\rangle \odot \langle F,B\rangle)\odot\Sigma\langle S,E\rangle$, *for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle F,A\rangle$ of S.*

(iv) $\langle F,B\rangle \subset_{(\gamma,\delta)} (\langle F,B\rangle \odot \langle F,B\rangle)\odot\Sigma\langle S,E\rangle$, *for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft bi-ideal $\langle F,A\rangle$ of S.*

Proof. $(i) \Rightarrow (iv)$

Let $\langle F,B\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft bi-ideal. Let a be any element of S, Now since S is regular AG-groupoid, so for $a \in S$ there exist $x \in S$ such that using left invertive law, medial law, paramedial law and also using law $a(bc) = b(ac)$, we have,

$$\begin{aligned}
a &= (ax)\,a = (ax)\{(ax)\,a\}\{a(ax)\}(xa) = x[\{a(ax)\}a] \\
&= x[\{(ea)(ax)\}a] = x[\{(xa)(ae)\}a] = (ex)[\{(xa)(ae)\}a] \\
&= [a\{(xa)(ae)\}](xe) = [a\{((ae)a)x\}](xe) = \{(xe)x\}[a\{(ae)a\}] \\
&= a[\{(xe)x\}\{(ae)a\}] = a[(ae)\{((xe)x)a\}] \\
&= (ea)[(ae)\{((xe)x)a\}] = [\{(ae)\{((xe)x)a\}\}a]e \\
&= [\{(ae)(ta)\}a]e = [\{(at)(ea)\}a]e = [\{(at)a\}a]e \in B^2S, \text{ where } t = \{(xe)x\}.
\end{aligned}$$

Thus we have,

$$\begin{aligned}
& \max\left\{((F(\varepsilon)\circ F(\varepsilon))\circ(\mathcal{X}^{\delta}_{\gamma S}))(a),\gamma\right\} \\
= & \max\left\{\bigvee_{a=pq}(F(\varepsilon)\circ F(\varepsilon))(p)\wedge(\mathcal{X}^{\delta}_{\gamma S})(q),\gamma\right\} \\
\geq & \max\{(F(\varepsilon)\circ F(\varepsilon))\{((at)a)a\}\wedge(\mathcal{X}^{\delta}_{\gamma S})(e),\gamma\} \\
= & \max\{\bigvee_{\{((at)a)a\}=uv}\{F(\varepsilon)(u)\wedge F(\varepsilon)(v)\wedge\mathcal{X}^{\delta}_{\gamma S}(e)\},\gamma\} \\
\geq & \max\left\{F(\varepsilon)\{(at)a\}\wedge F(\varepsilon)(a)\wedge\mathcal{X}^{\delta}_{\gamma S}(e),\gamma\right\} \\
= & \max\{F(\varepsilon)\{(at)a\}\wedge F(\varepsilon)(a)\wedge 1,\gamma\} \\
= & \min\{\max\{F(\varepsilon)\{(at)a\},\gamma\},\max\{F(\varepsilon)(a),\gamma\}\} \\
\geq & \min\min\{F(\varepsilon)(a),\delta\},\min\{F(\varepsilon)(a),\delta\}\} \\
= & \min\{(F(\varepsilon)(a),\delta\} \\
= & \min\{(F(\varepsilon)(a),\delta\}
\end{aligned}$$

Thus $\min\{(F(\varepsilon)(a),\delta\}\leq\max\left\{((F(\varepsilon)\circ F(\varepsilon))\circ(\mathcal{X}^{\delta}_{\gamma S}))(a),\gamma\right\}$. This implies that $F(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}((F(\varepsilon)\circ F(\varepsilon))\circ\mathcal{X}^{\delta}_{\gamma S})$.

Therefore in any case, we have $K_1(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}K_2(\varepsilon)$. Hence

$$\langle F,B\rangle\subset_{(\gamma,\delta)}(\langle F,B\rangle\odot\langle F,B\rangle)\odot\Sigma\langle S,E\rangle.$$

$(iv)\implies(iii)$ is obvious.

$(iii)\implies(ii)$

Assume that B is bi-ideal of S, then $\Sigma(B,E)$ is $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal over S,respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(B,E)\subset_{(\gamma,\delta)}(\Sigma(B,E)\odot\Sigma(B,E))\odot\Sigma(S,E).$$

So,

$$\begin{aligned}
\chi^{\delta}_{\gamma B} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma B^2}\cap\chi^{\delta}_{\gamma S} \\
&\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma B}\circ\chi^{\delta}_{\gamma B})\circ\chi^{\delta}_{\gamma S} \\
&= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma B^2S}.
\end{aligned}$$

Thus $B\subseteq B^2S$.

$(ii)\Rightarrow(i)$

$B[a]=a\cup a^2\cup(aS)a$, and $L[a]=a\cup Sa$ are principle bi-ideal and principle left ideal of S generated by a respectively. Thus by (ii), left invertive law, paramedial law and using law $a(bc)=b(ac)$, we have,

$$\begin{aligned}
Sa &\subseteq [(Sa)(Sa)]S=[S(Sa)](Sa) \\
&= (SS)[a(Sa)]=S[a(Sa)]\subseteq(aS)a.
\end{aligned}$$

Hence S is regular. ■

Theorem 154 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For bi-ideal B, ideal I and left ideal L of S, $L \cap B \subseteq (LS)\,B$.*

(iii) *$\langle F, L\rangle \tilde{\cap} \langle G, B\rangle \subset_{(\gamma,\delta)} (\langle F, L\rangle \odot \Sigma\langle S, E\rangle) \odot \langle G, B\rangle$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F, A\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle H, B\rangle$ of S.*

(iv) *$\langle F, L\rangle \tilde{\cap} \langle G, B\rangle \subset_{(\gamma,\delta)} (\langle F, L\rangle \odot \langle S, E\rangle) \odot \langle G, B\rangle$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft left ideal $\langle F, A\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft bi-ideal $\langle H, B\rangle$ of S.*

Proof. $(i) \Rightarrow (iv)$

Let $\langle F, L\rangle$ and $\langle G, B\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft left ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft bi-ideal over S, respectively. Let a be any element of S, $\langle F, L\rangle \tilde{\cap} \langle G, B\rangle = \langle K_1, L \cup B\rangle$.For any $\varepsilon \in L \cup B$. We consider the following cases,

Case 1: $\varepsilon \in L - B$. Then $K_1(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $K_2(\varepsilon) = (F \circ G)(\varepsilon)$, so we have $K_1(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} K_2(\varepsilon)$

Case 2: $\varepsilon \in B - L$. Then $K_1(\varepsilon) = H(\varepsilon)$ and $K_1(\varepsilon) = H(\varepsilon) = K_2(\varepsilon)$

Case 3: $\varepsilon \in L \cap B$. Then $K_1(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $K_2(\varepsilon) = F(\varepsilon) \circ G(\varepsilon)$. Now we show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ G(\varepsilon))$. Now since S is regular AG-groupoid, so for $a \in S$ there exist $x \in S$ such that using left invertive law, we have,

$$a = (ax)\,a = [\{(ax)\,a\}x]a = [(xa)(ax)]a \in (LS)B.$$

Thus we have,

$$\begin{aligned}
& \max\left\{((F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S}) \circ G(\varepsilon))(a), \gamma\right\} \\
= \;& \max\left\{\bigvee_{a=pq} (F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S})(p) \wedge G(\varepsilon)(q), \gamma\right\} \\
\geq \;& \max\{(F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S})\{(xa)(ax)\} \wedge G(\varepsilon))(a), \gamma\} \\
= \;& \max\{\bigvee_{\{(xa)(ax)\}=uv} (F(\varepsilon)(u) \wedge \mathcal{X}^{\delta}_{\gamma S}(v)) \wedge G(\varepsilon)(a)\}, \gamma\} \\
\geq \;& \max\left\{F(\varepsilon)(xa) \wedge \mathcal{X}^{\delta}_{\gamma S}(ax) \wedge G(\varepsilon)(a), \gamma\right\} \\
= \;& \max\left\{F(\varepsilon)(xa) \wedge 1 \wedge G(\varepsilon)(a), \gamma\right\} \\
= \;& \min\left\{\max\{F(\varepsilon)(a), \gamma\}, \max\{G(\varepsilon)(a), \gamma\}\right\} \\
\geq \;& \min\{\min\{(F(\varepsilon)(a), \delta\}, \min\{G(\varepsilon)(a), \delta\}\} \\
= \;& \min\left\{(F(\varepsilon) \wedge G(\varepsilon))(a), \delta\right\} \\
= \;& \min\left\{(F(\varepsilon) \cap G(\varepsilon))(a), \delta\right\}
\end{aligned}$$

Thus $\min\left\{(F(\varepsilon) \cap G(\varepsilon))(a), \delta\right\} \leq \max\left\{((F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S}) \circ G(\varepsilon))(a), \gamma\right\}$. This implies that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ \mathcal{X}^{\delta}_{\gamma S}) \circ G(\varepsilon)$.

Therefore in any case, we have $K_1(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} K_2(\varepsilon)$. Hence

$$\langle F, L\rangle \tilde{\cap} \langle G, B\rangle \subset_{(\gamma,\delta)} (\langle F, L\rangle \odot \langle S, E\rangle) \odot \langle G, B\rangle.$$

$(iv) \implies (iii)$ is obvious.
$(iii) \implies (ii)$
Assume that L and B are left ideal and bi-ideal of S, respectively, then $\Sigma(L, E)$ and $\Sigma(B, E)$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal over S,respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(L, E) \tilde{\cap} \Sigma(B, E) \subset_{(\gamma,\delta)} (\Sigma(L, E) \odot \Sigma(S, E)) \odot \Sigma(B, E).$$

So,

$$\begin{aligned} \chi^{\delta}_{\gamma(L\cap B)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma L} \cap \chi^{\delta}_{\gamma B} \\ &\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma L} \circ \chi^{\delta}_{\gamma S}) \circ \chi^{\delta}_{\gamma B} \\ &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma(LS)B}. \end{aligned}$$

Thus $L \cap B \subseteq (LS)B$.
$(ii) \Rightarrow (i)$
$L[a] = a \cup Sa$, and $B[a] = a \cup a^2 \cup (aS)\, a$ are principle left ideal and principle bi-ideal of S generated by a respectively. Thus by (ii), left invertive law, paramedial law and using law $a(bc) = b(ac)$, we have,

$$\begin{aligned} (Sa) \cap (Sa) &\subseteq [(Sa)S](Sa) = [S(aS)](Sa) \\ &= (aS)(Sa) \subseteq (aS)\, a. \end{aligned}$$

Hence S is regular. ■

Theorem 155 *For an AG-groupoid S, with left identity, the following are equivalent.*
(i) S is regular.
(ii) For left ideal L,quasi ideal Q and an ideal I of S, $L \cap Q \cap I \subseteq (LQ)\, I$.
(iii) $\langle F, L\rangle \tilde{\cap} \langle G, Q\rangle \tilde{\cap} \langle H, I\rangle \subset_{(\gamma,\delta)} (\langle F, L\rangle \odot \langle G, Q\rangle \odot \langle H, I\rangle)$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F, A\rangle$, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi ideal $\langle G, B\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal $\langle H, C\rangle$ of S.
(iv) $\langle F, L\rangle \tilde{\cap} \langle G, Q\rangle \tilde{\cap} \langle H, I\rangle \subset_{(\gamma,\delta)} (\langle F, L\rangle \odot \langle G, Q\rangle \odot \langle H, I\rangle)$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft left ideal $\langle F, A\rangle, (\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft quasi ideal $\langle G, B\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft ideal $\langle H, C\rangle$ of S.

Proof. $(i) \Rightarrow (iv)$
Let $\langle F, L\rangle, \langle G, Q\rangle$ and $\langle H, I\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft left ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft quasi ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft ideal over S, respectively. Let a be any element of S,

$\langle F,A\rangle\tilde{\cap}\langle G,A\rangle\tilde{\cap}\langle H,B\rangle = \langle K_1, A\cup B\rangle$.For any $\varepsilon \in A\cup B$. We consider the following cases,

Case 1: $\varepsilon \in A-B$. Then $K_1(\varepsilon) = F(\varepsilon)\cap G(\varepsilon)$ and $K_2(\varepsilon) = (F\circ G)(\varepsilon)$, so we have $K_1(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} K_2(\varepsilon)$

Case 2: $\varepsilon \in B-A$. Then $K_1(\varepsilon) = H(\varepsilon)$ and $K_1(\varepsilon) = H(\varepsilon) = K_2(\varepsilon)$

Case 3: $\varepsilon \in A\cap B$. Then $K_1(\varepsilon) = F(\varepsilon)\cap G(\varepsilon)$ and $K_2(\varepsilon) = F(\varepsilon)\circ G(\varepsilon)$. Now we show that $F(\varepsilon)\cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon)\circ G(\varepsilon))$. Now since S is regular AG-groupoid, so for $a\in S$ there exist $x\in S$ such that using medial law and left invertive law, we have,

$$\begin{aligned} a &= (ax)\,a = (ax)\{(ax)\,a\} = \{a(ax)\}(xa) \\ &= \{(xa)(ax)\}a \in (LQ)I. \end{aligned}$$

Thus we have,

$$\begin{aligned} & \max\{((F(\varepsilon)\circ G(\varepsilon))\circ(H(\varepsilon))(a),\gamma\} \\ = & \max\left\{\bigvee_{a=pq}(F(\varepsilon)\circ G(\varepsilon))(p)\wedge H(\varepsilon)(q),\gamma\right\} \\ \geq & \max\{(F(\varepsilon)\circ G(\varepsilon))\{(xa)(ax)\}\wedge H(\varepsilon))(a),\gamma\} \\ = & \max\{\bigvee_{\{(xa)(ax)\}=uv}(F(\varepsilon)(u)\wedge G(\varepsilon)(v))\wedge H(\varepsilon)(a)\},\gamma\} \\ \geq & \max\{F(\varepsilon)(xa)\wedge G(\varepsilon)(ax)\wedge H(\varepsilon)(a),\gamma\} \\ = & \min\{\max\{F(\varepsilon)(xa),\gamma\},\max\{G(\varepsilon)(ax),\gamma\},\max\{H(\varepsilon)(a),\gamma\}\} \\ \geq & \min\{\min\{(F(\varepsilon)(a),\delta\},\min\{G(\varepsilon)(a),\delta\},\min\{H(\varepsilon)(a),\delta\}\} \\ = & \min\{(F(\varepsilon)\wedge G(\varepsilon)\wedge H(\varepsilon))(a),\delta\} \\ = & \min\{(F(\varepsilon)\cap G(\varepsilon)\cap H(\varepsilon))(a),\delta\} \end{aligned}$$

Thus $\min\{(F(\varepsilon)\cap G(\varepsilon)\cap H(\varepsilon))(a),\delta\} \leq \max\{((F(\varepsilon)\circ G(\varepsilon)\circ H(\varepsilon))(a),\gamma\}$. This implies that $F(\varepsilon)\cap G(\varepsilon)\cap H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon)\circ G(\varepsilon))\circ H(\varepsilon)$.

Therefore in any case, we have $K_1(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} K_2(\varepsilon)$. Hence

$$\langle F,L\rangle\tilde{\cap}\langle G,Q\rangle\tilde{\cap}\langle H,I\rangle \subset_{(\gamma,\delta)} (\langle F,L\rangle\odot\langle G,Q\rangle)\odot\langle H,I\rangle.$$

$(iv) \Longrightarrow (iii)$ is obvious.

$(iii) \Longrightarrow (ii)$

Assume that L,Q and I are left ideal,quasi ideal and ideal of S, respectively, then $\Sigma(L,E)$, $\Sigma(Q,E)$ and $\Sigma(H,E)$ are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi ideal and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal over S,respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(L,E)\tilde{\cap}\Sigma(Q,E)\tilde{\cap}\Sigma(H,E) \subset_{(\gamma,\delta)} (\Sigma(L,E)\odot\Sigma(Q,E))\odot\Sigma(I,E).$$

So,

$$\begin{aligned} \chi^{\delta}_{\gamma(L\cap Q\cap I)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma L}\cap\chi^{\delta}_{\gamma Q}\cap\chi^{\delta}_{\gamma I} \\ &\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma L}\circ\chi^{\delta}_{\gamma Q})\circ\chi^{\delta}_{\gamma I} \\ &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma(LQ)I}. \end{aligned}$$

Thus $L\cap Q\cap I\subseteq (LQ)I$.

$(ii)\Rightarrow(i)$

$L[a]=a\cup Sa$, $Q[a]=a$ and $I[a]=aS\cup Sa$, are principle bi-ideal and principle left ideal of S generated by a respectively. Thus by (ii), left invertive law, paramedial law and using law $a(bc)=b(ac)$, we have,

$$\begin{aligned} \{(Sa)\cap(Sa)\}\cap(Sa) &\subseteq [\{(aS)\cup(Sa)\}S](Sa) \\ &= [\{(aS)S\}\cup\{(Sa)S\}](Sa) \\ &= [\{S(Sa)\}\cup\{(Sa)S\}](Sa) \\ &= [Sa\cup\{(Sa)S\}](Sa) \\ &= [(Sa)(Sa)]\cup\{(Sa)S\}(Sa) \\ &\subseteq (aS)\,a. \end{aligned}$$

Hence S is regular. ■

Theorem 156 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For an ideal I and bi-ideal B of S, $I\cap B\subseteq I\,(IB)$.*

(iii) $\langle F,I\rangle\tilde{\cap}\langle G,B\rangle\subset_{(\gamma,\delta)}\langle F,I\rangle\odot(\langle F,I\rangle\odot\langle G,B\rangle)$, *for any $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy soft ideal $\langle F,A\rangle$ and $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy soft bi-ideal $\langle H,B\rangle$ of S.*

(iv) $\langle F,I\rangle\tilde{\cap}\langle G,B\rangle\subset_{(\gamma,\delta)}\langle F,I\rangle\odot(\langle F,I\rangle\odot\langle G,B\rangle)$, *for any $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy generalized soft ideal $\langle F,A\rangle$ and $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy generalized soft bi-ideal $\langle H,B\rangle$ of S.*

Proof. $(i)\Rightarrow(iv)$

Let $\langle F,I\rangle$ and $\langle G,B\rangle$ be any $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy generalized soft ideal and $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy generalized soft bi-ideal over S, respectively. Let a be any element of S, $\langle F,I\rangle\tilde{\cap}\langle G,B\rangle=\langle K_1,I\cup B\rangle$.For any $\varepsilon\in I\cup B$. We consider the following cases,

Case 1: $\varepsilon\in I-B$. Then $K_1(\varepsilon)=F(\varepsilon)\cap G(\varepsilon)$ and $K_2(\varepsilon)=(F\circ G)(\varepsilon)$, so we have $K_1(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}K_2(\varepsilon)$

Case 2: $\varepsilon\in B-I$. Then $K_1(\varepsilon)=H(\varepsilon)$ and $K_1(\varepsilon)=H(\varepsilon)=K_2(\varepsilon)$

Case 3: $\varepsilon\in I\cap B$. Then $K_1(\varepsilon)=F(\varepsilon)\cap G(\varepsilon)$ and $K_2(\varepsilon)=F(\varepsilon)\circ G(\varepsilon)$. Now we show that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F(\varepsilon)\circ G(\varepsilon))$. Now since S is regular AG-groupoid, so for $a\in S$ there exist $x\in S$ such that using left invertive law, we have,

$$a=(ax)\,a=[\{(ax)\,a\}x]a=(ax)[(ax)a]\in I(IB).$$

Thus we have,

$$\begin{aligned}
& \max\{(F(\varepsilon)\circ(F(\varepsilon)\circ G(\varepsilon))(a),\gamma\} \\
= & \max\left[\left\{\bigvee_{a=pq} F(\varepsilon)(p)\wedge(F(\varepsilon)\circ G(\varepsilon))(q)\right\},\gamma\right] \\
\geq & \max\{(F(\varepsilon)(ax)\wedge(F(\varepsilon)\circ G(\varepsilon))\{(ax)a\}),\gamma\} \\
= & \max\{(F(\varepsilon)(ax) \bigvee_{\{(ax)a\}=uv} (F(\varepsilon)(u)\wedge G(\varepsilon)(v)\},\gamma\} \\
\geq & \max\{F(\varepsilon)(ax)\wedge(F(\varepsilon)(ax)\wedge G(\varepsilon)(a),\gamma\} \\
= & \max\{F(\varepsilon)(a)\wedge F(\varepsilon)(a)\wedge G(\varepsilon)(a),\gamma\} \\
= & \min\{\max\{F(\varepsilon)(a),\gamma\},\max\{G(\varepsilon)(a),\gamma\}\} \\
\geq & \min\{\min\{(F(\varepsilon)(a),\delta\},\min\{G(\varepsilon)(xa),\delta\}\} \\
= & \min\{(F(\varepsilon)\wedge G(\varepsilon))(a),\delta\} \\
= & \min\{(F(\varepsilon)\cap G(\varepsilon))(a),\delta\}
\end{aligned}$$

Thus $\min\{(F(\varepsilon)\cap G(\varepsilon))(a),\delta\}\leq\max\{(F(\varepsilon)\circ(F(\varepsilon)\circ G(\varepsilon))(a),\gamma\}$. This implies that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F(\varepsilon)\circ(F(\varepsilon)\circ G(\varepsilon))$.

Therefore in any case, we have $K_1(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}K_2(\varepsilon)$. Hence

$$\langle F,I\rangle\tilde{\cap}\langle G,B\rangle\subset_{(\gamma,\delta)}(\langle F,I\rangle\odot(\langle F,I\rangle)\odot\langle G,B\rangle).$$

$(iv)\implies(iii)$ is obvious.

$(iii)\implies(ii)$

Assume that I and B are ideal and bi-ideal of S, respectively, then $\Sigma(I,E)$ and $\Sigma(B,E)$ are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal over S,respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(I,E)\tilde{\cap}\Sigma(B,E)\subset_{(\gamma,\delta)}(\Sigma(I,E)\odot(\Sigma(I,E)\odot\Sigma(B,E)).$$

So,

$$\begin{aligned}
\chi^\delta_{\gamma(I\cap B)} &= {}_{(\gamma,\delta)}\chi^\delta_{\gamma I}\cap\chi^\delta_{\gamma B} \\
&\subseteq \vee q_{(\gamma,\delta)}(\chi^\delta_{\gamma I}\circ(\chi^\delta_{\gamma I}\circ\chi^\delta_{\gamma B})) \\
&= \vee q_{(\gamma,\delta)}(\chi^\delta_{\gamma I})\circ(\chi^\delta_{\gamma IB}) \\
&= {}_{(\gamma,\delta)}\chi^\delta_{\gamma I(IB)}.
\end{aligned}$$

Thus $I\cap B\subseteq I(IB)$.

$(ii)\Rightarrow(i)$

$I[a]=aS\cup Sa$, and $B[a]=a\cup a^2\cup(aS)\,a$ are principle ideal and principle bi-ideal of S generated by a respectively. Thus by (ii), left invertive law, paramedial law and using law $a(bc) = b(ac)$, we have,

$$\begin{aligned}
\{(aS) \cup (Sa)\} \cap (Sa) &\subseteq [\{(aS) \cup (Sa)\}S](Sa) \\
&= [\{(aS)S\} \cup \{(Sa)S\}](Sa) \\
&= [\{S(Sa)\} \cup \{(Sa)S\}](Sa) \\
&= [Sa \cup \{(Sa)S\}](Sa) \\
&= [(Sa)(Sa)] \cup \{(Sa)S\}(Sa) \\
&\subseteq (aS)\, a.
\end{aligned}$$

Hence S is regular. ■

Theorem 157 *For an AG-groupoid S, with left identity, the following are equivalent.*
(i) S is regular.
(ii) For an ideal I and left ideal L of S, $I \cap L \subseteq (IS)\, L$.
(iii) $\langle F, I\rangle \tilde{\cap} \langle G, L\rangle \subset_{(\gamma,\delta)} (\langle F, I\rangle \odot \Sigma \langle S, E\rangle \odot \langle G, L\rangle)$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal $\langle F, A\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle H, B\rangle$ of S.
(iv) $\langle F, I\rangle \tilde{\cap} \langle G, L\rangle \subset_{(\gamma,\delta)} (\langle F, I\rangle \odot \langle S, E\rangle \odot \langle G, L\rangle)$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft ideal $\langle F, A\rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft left ideal $\langle H, B\rangle$ of S.

Proof. $(i) \Rightarrow (iv)$
Let $\langle F, I\rangle$ and $\langle G, L\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft left ideal over S, respectively. Let a be any element of S, $\langle F, I\rangle \tilde{\cap} \langle G, L\rangle = \langle K_1, I \cup L\rangle$.For any $\varepsilon \in I \cup L$. We consider the following cases,
Case 1: $\varepsilon \in I - L$. Then $K_1(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $K_2(\varepsilon) = (F \circ G)(\varepsilon)$, so we have $K_1(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} K_2(\varepsilon)$
Case 2: $\varepsilon \in L - I$. Then $K_1(\varepsilon) = H(\varepsilon)$ and $K_1(\varepsilon) = H(\varepsilon) = K_2(\varepsilon)$
Case 3: $\varepsilon \in I \cap L$. Then $K_1(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $K_2(\varepsilon) = F(\varepsilon) \circ G(\varepsilon)$. Now we show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ G(\varepsilon))$. Now since S is regular AG-groupoid, so for $a \in S$ there exist $x \in S$ such that using medial law, we have,

$$a = (ax)\, a = (ax)\{(ax)\, a\} = \{a(ax)\}(xa) \in (IS)L.$$

Thus we have,

$$\begin{aligned}
& \max\left\{((F(\varepsilon)\circ\mathcal{X}_{\gamma S}^{\delta})\circ G(\varepsilon))(a),\gamma\right\} \\
= & \max\left\{\bigvee_{a=pq}(F(\varepsilon)\circ\mathcal{X}_{\gamma S}^{\delta})(p)\wedge G(\varepsilon)(q),\gamma\right\} \\
\geq & \max\{(F(\varepsilon)\circ\mathcal{X}_{\gamma S}^{\delta})\{a(ax)\}\wedge G(\varepsilon))(xa),\gamma\} \\
= & \max\{\bigvee_{\{a(ax)\}=uv}(F(\varepsilon)(u)\wedge\mathcal{X}_{\gamma S}^{\delta}(v))\wedge G(\varepsilon)(xa)\},\gamma\} \\
\geq & \max\left\{F(\varepsilon)(a)\wedge\mathcal{X}_{\gamma S}^{\delta}(ax)\wedge G(\varepsilon)(xa),\gamma\right\} \\
= & \max\{F(\varepsilon)(a)\wedge 1\wedge G(\varepsilon)(xa),\gamma\} \\
= & \min\{\max\{F(\varepsilon)(a),\gamma\},\max\{G(\varepsilon)(xa),\gamma\}\} \\
\geq & \min\{\min\{(F(\varepsilon)(a),\delta\},\min\{G(\varepsilon)(xa),\delta\}\} \\
= & \min\{(F(\varepsilon)\wedge G(\varepsilon))(a),\delta\} \\
= & \min\{(F(\varepsilon)\cap G(\varepsilon))(a),\delta\}
\end{aligned}$$

Thus $\min\{(F(\varepsilon)\cap G(\varepsilon))(a),\delta\}\leq\max\left\{((F(\varepsilon)\circ\mathcal{X}_{\gamma S}^{\delta})\circ G(\varepsilon))(a),\gamma\right\}$. This implies that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F(\varepsilon)\circ\mathcal{X}_{\gamma S}^{\delta})\circ G(\varepsilon)$.

Therefore in any case, we have $K_1(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}K_2(\varepsilon)$. Hence

$$\langle F,I\rangle\tilde{\cap}\langle G,L\rangle\subset_{(\gamma,\delta)}(\langle F,I\rangle\odot\Sigma\langle S,E\rangle)\odot\langle G,L\rangle.$$

$(iv)\implies(iii)$ is obvious.

$(iii)\implies(ii)$

Assume that I and L are ideal and left ideal of S, respectively, then $\Sigma(I,E)$ and $\Sigma(L,E)$ are $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy soft ideal and $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy soft left ideal over S,respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(I,E)\tilde{\cap}\Sigma(L,E)\subset_{(\gamma,\delta)}(\Sigma(I,E)\odot\Sigma(S,E))\odot\Sigma(L,E).$$

So,

$$\begin{aligned}
\chi_{\gamma(I\cap L)}^{\delta} &= {}_{(\gamma,\delta)}\chi_{\gamma I}^{\delta}\cap\chi_{\gamma L}^{\delta} \\
&\subseteq \vee q_{(\gamma,\delta)}(\chi_{\gamma I}^{\delta}\circ\chi_{\gamma S}^{\delta})\circ\chi_{\gamma L}^{\delta} \\
&= {}_{(\gamma,\delta)}\chi_{\gamma(IS)L}^{\delta}.
\end{aligned}$$

Thus $I\cap L\subseteq(IS)L$.

$(ii)\Rightarrow(i)$

$I[a]=aS\cup Sa$, and $L[a]=a\cup Sa$ are principle ideal and principle left ideal of S generated by a respectively. Thus by (ii), left invertive law, paramedial law and using law $a(bc) = b(ac)$, we have,

$$\begin{aligned}
\{(aS) \cup (Sa)\} \cap (Sa) &\subseteq [\{(aS) \cup (Sa)\}S](Sa) \\
&= [\{(aS)S\} \cup \{(Sa)S\}](Sa) \\
&= [\{S(Sa)\} \cup \{(Sa)S\}](Sa) \\
&= [Sa \cup \{(Sa)S\}](Sa) \\
&= [(Sa)(Sa)] \cup \{(Sa)S\}(Sa) \\
&\subseteq (aS)\, a.
\end{aligned}$$

Hence S is regular. ■

Theorem 158 *For an AG-groupoid S, with left identity, the following are equivalent.*
(i) S is regular.
(ii) For left ideal L of S, $L \subseteq \{L\,(LS)\}L$.
(iii) $\langle F, L\rangle \subset_{(\gamma,\delta)} \{\langle F, L\rangle \odot (\langle F, L\rangle \odot \Sigma\langle S, E\rangle\} \odot \langle F, L\rangle$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle H, B\rangle$ of S. $(iv)\langle F, L\rangle \subset_{(\gamma,\delta)} \{\langle F, L\rangle \odot (\langle F, L\rangle \odot \Sigma\langle S, E\rangle\} \odot \langle F, L\rangle$ $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft generalized left ideal $\langle F, B\rangle$ of S.

Proof. $(i) \Rightarrow (iv)$
Let $\langle F, L\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized soft left ideal over S. Now we show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F(\varepsilon) \circ G(\varepsilon))$. Now since S is regular AG-groupoid, so for $a \in S$ there exist $x \in S$ such that using medial law,paramedial law,left invertive law and also using law $a(bc) = b(ac)$, we have,

$$\begin{aligned}
a &= (ax)\,a = [\{(ax)a\}x]a = \{(xa)(ax)\}a = \{a(ax)\}(xa) \\
&= \{(ea)(ax)\}(xa) = \{(xa)(ae)\}(xa) = [\{(ae)a\}x](xa) \\
&= x[\{((ae)a)x\}a] = (ex)[\{((ae)a)x\}a] \\
&= [a\{((ae)a)x\}](xe) = [a\{((ae)a)x\}]\acute{x} = [\{(ae)a\}(ax)]\acute{x} \\
&= [\acute{x}(ax)][(ae)a] = [a(\acute{x}x)][(ae)a] = [a(ae)][(\acute{x}x)a] \\
&= (\acute{x}x)[\{a(ae)\}a] = (\acute{x}x)[\{a(ae)\}(ea)] \\
&= (\acute{x}x)[(ae)\{(ae)a\}] = (ae)[(\acute{x}x)\{(ae)a\}] \\
&= [\{(ae)a\}(\acute{x}x)](ea) = [\{(ae)a\}x_1]a = [(x_1a)(ae)]a \\
&= [(ea)(ax_1)]a = \{a(ax_1)\}a \in [L(LS)]L, \text{where } \acute{x} = (xe) \\
\text{and } x_1 &= (\acute{x}x)
\end{aligned}$$

Thus we have,

$$\begin{aligned}
& \max\left\{\mathcal{X}^{\delta}_{\gamma S}\circ(F(\varepsilon)\circ F(\varepsilon))(a),\gamma\right\} \\
= & \max\left\{\bigvee_{a=pq}F(\varepsilon)((F(\varepsilon)\circ\mathcal{X}^{\delta}_{\gamma S})(p)\wedge F(\varepsilon)(q),\gamma\right\} \\
\geq & \max[\{F(\varepsilon)(F(\varepsilon)\circ\mathcal{X}^{\delta}_{\gamma S})\}\{a(ax)\}\wedge F(\varepsilon))(a),\gamma] \\
= & \max[\min[\{F(\varepsilon)(F(\varepsilon)\circ\mathcal{X}^{\delta}_{\gamma S})\}\{a(ax)\}],F(\varepsilon))(a),\gamma] \\
= & \max[\min[\{\bigvee_{a(ax)=uv}F(\varepsilon)(F(\varepsilon)\wedge\mathcal{X}^{\delta}_{\gamma S})\}(uv)],F(\varepsilon))(a),\gamma] \\
\geq & \max[\min[\min\{\{F(\varepsilon)(u)(F(\varepsilon)\circ\mathcal{X}^{\delta}_{\gamma S})\}(v)\},F(\varepsilon))(a)],\gamma] \\
= & \max[\min[\min\{F(\varepsilon)(a)\{(F(\varepsilon)\circ\mathcal{X}^{\delta}_{\gamma S})\}\}(ax)],F(\varepsilon))(a),\gamma] \\
= & \max[\min[\min\{F(\varepsilon)(a)\{\bigvee_{ax=lm}(F(\varepsilon)\wedge\mathcal{X}^{\delta}_{\gamma S})\}(lm)],F(\varepsilon))(a),\gamma] \\
\geq & \max[\min[\min\{F(\varepsilon)(a)\{\min(F(\varepsilon)(a),\mathcal{X}^{\delta}_{\gamma S})(x)\}],F(\varepsilon))(a)\}],\gamma] \\
= & \max[\min[\min\{F(\varepsilon)(a)\{\min(F(\varepsilon)(a),1)(x)\}],F(\varepsilon))(a)\}],\gamma] \\
= & \max[\min[\min\{F(\varepsilon)(a),(F(\varepsilon)(a),F(\varepsilon))(a)\}],\gamma] \\
= & \max[\min\{F(\varepsilon)(a),\gamma\}] \\
= & \min[\max\{F(\varepsilon)(a),\gamma\}] \\
\geq & \min[\min\{F(\varepsilon)(a),\delta\}].
\end{aligned}$$

Thus $\min\{(F(\varepsilon)(a),\delta\}\leq\max\left\{\mathcal{X}^{\delta}_{\gamma S}\circ(F(\varepsilon)\circ F(\varepsilon))(a),\gamma\right\}$. This implies that $F(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(\mathcal{X}^{\delta}_{\gamma S}\circ(F(\varepsilon)\circ F(\varepsilon))$.

Therefore in any case, we have $K_1(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}K_2(\varepsilon)$. Hence

$$\langle F,L\rangle\subset_{(\gamma,\delta)}(\Sigma\langle S,E\rangle\odot(\langle F,L\rangle\odot\langle F,L\rangle).$$

$(iv)\implies(iii)$ is obvious.

$(iii)\implies(ii)$

Assume that L is left ideal of S, respectively, then $\Sigma(L,E)$ is $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S. Now we have assume that (iv) holds, then we have

$$\Sigma(F,L)\subset_{(\gamma,\delta)}(\Sigma(I,E)\odot\Sigma(S,E))\odot\Sigma(L,E).$$

So,

$$\begin{aligned}
\chi^{\delta}_{\gamma(I\cap L)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma I}\cap\chi^{\delta}_{\gamma L} \\
&\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma I}\circ\chi^{\delta}_{\gamma S})\circ\chi^{\delta}_{\gamma L} \\
&= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma(IS)L}.
\end{aligned}$$

Thus $I\cap L\subseteq(IS)L$.

$(ii)\Rightarrow(i)$

$I[a] = aS \cup Sa$, and $L[a] = a \cup Sa$ are principle bi-ideal and principle left ideal of S generated by a respectively. Thus by (ii), left invertive law, paramedial law and using law $a(bc) = b(ac)$, we have,

$$\begin{aligned}
\{(aS) \cup (Sa)\} \cap (Sa) &\subseteq [\{(aS) \cup (Sa)\}S](Sa) \\
&= [\{(aS)S\} \cup \{(Sa)S\}](Sa) \\
&= [\{S(Sa)\} \cup \{(Sa)S\}](Sa) \\
&= [Sa \cup \{(Sa)S\}](Sa) \\
&= [(Sa)(Sa)] \cup \{(Sa)S\}(Sa) \\
&= [(Sa)(Sa)] \cup [(aS)(Sa)] \\
&\subseteq (Sa^2)S \cup (aS)\, a.
\end{aligned}$$

Hence S is regular. ■

Theorem 159 *If S is an AG-groupoid with left identity then the following are equivalent*
(i) S is regular,
(ii) $L \cap B \subseteq LB$ for left ideal I and bi-ideal B,
(iii) $\langle F, L\rangle \cap \langle G, B\rangle \subseteq_{(\gamma,\delta)} (\langle F, L\rangle \circ \langle G, B\rangle)$, where f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal and bi-ideal of S.

Proof. $(i) \Longrightarrow (iii)$ Let $a \in S$, then since S is regular so using left invertive law we get

$$\begin{aligned}
a &= (ax)a = (ax)\{(ax)a\} = \{a(ax)\}(xa) = \{(xa)(ax)\}a \\
&= \{(xa)(ax)\}\{(ax)a\} = \{a(ax)\}\{(ax)(xa)\}. \\
\max\{f \circ (\mathcal{X}^\delta_{\gamma S} \circ g), \gamma\} &= \max\left[\left\{\bigvee_{a=pq} f(p) \wedge (\mathcal{X}^\delta_{\gamma S} \circ g)(q)\right\}, \gamma\right] \\
&\geq \max\left[f(ax) \wedge (\mathcal{X}^\delta_{\gamma S} \circ g)[x\{(ae)a\}], \gamma\right] \\
&= \max\left[\min\{f(ax), (\mathcal{X}^\delta_{\gamma S} \circ g)\{x((ae)a)\}\}, \gamma\right] \\
&= \max\left[\min\{f(ax), \{\bigvee_{x((ae)a)\}=st} (\mathcal{X}^\delta_{\gamma S}(s) \wedge g(t), \gamma\right] \\
&\geq \max\left[\min\{f(ax), \{(\mathcal{X}^\delta_{\gamma S}(x) \wedge g((ae)a)\}, \gamma\right] \\
&= \max\left[\{\min\{f(ax), \min\{(\mathcal{X}^\delta_{\gamma S}(x), g((ae)a)\}\}, \gamma\right] \\
&= \max\left[\{\min\{f(ax), \min\{1, g((ae)a)\}\}, \gamma\right] \\
&= \max\left[\{\min\{f(ax), g((ae)a)\}, \gamma\right] \\
&= \min\left[\max\{f(ax), \gamma\}, \max\{g((ae)a), \gamma\}\right] \\
&\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\} \\
&= \min\{f \cap g(a), \delta\}.
\end{aligned}$$

Thus $f\cap g\subseteq \vee q_{(\gamma,\delta)} f\circ(\mathcal{X}^{\delta}_{\gamma S}\circ g)$.

$(iii)\Longrightarrow(ii)$ Let L and B are ideal and bi-ideal of S respectively. Then $\mathcal{X}^{\delta}_{\gamma L}$ and $\mathcal{X}^{\delta}_{\gamma B}$ are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideal and bi-ideal of S respectively. Now by (iii)

$$\begin{aligned}\mathcal{X}^{\delta}_{\gamma L\cap B} &= \mathcal{X}^{\delta}_{\gamma L}\cap\mathcal{X}^{\delta}_{\gamma B}\subseteq\vee q_{(\gamma,\delta)}\mathcal{X}^{\delta}_{\gamma L}\circ(\mathcal{X}_{\gamma S}\circ\mathcal{X}^{\delta}_{\gamma B})\\ &= {}_{(\gamma,\delta)}\vee q_{(\gamma,\delta)}\mathcal{X}^{\delta}_{\gamma\{L(SB)\}}.\end{aligned}$$

Thus $L\cap B\subseteq L(SB)$.

$(ii)\Longrightarrow(i)$Using $\{S(Sa)\}\subseteq(Sa)$ and we get

$$a\in(Sa)\cap\{a\cup a^2\cup(aS)a\}\subseteq(Sa)\{(aS)a\}\subseteq(aS)a.$$

Hence S is regular. ■

Theorem 160 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) S is regular.

(ii) For bi-ideal B of S, $B\subseteq B^2S$.

(iii) $\langle F,B\rangle\subset_{(\gamma,\delta)}(\langle F,B\rangle\odot\langle F,B\rangle)\odot\Sigma\langle S,E\rangle$, for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal $\langle F,A\rangle$ of S.

(iv) $\langle F,B\rangle\subset_{(\gamma,\delta)}(\langle F,B\rangle\odot\langle F,B\rangle)\odot\Sigma\langle S,E\rangle$, for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized soft bi-ideal $\langle F,A\rangle$ of S.

Proof. $(i)\Rightarrow(iv)$

Let $\langle F,B\rangle$ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized soft bi-ideal. Let a be any element of S, Now since S is regular AG-groupoid, so for $a\in S$ there exist $x\in S$ such that using left invertive law, medial law, paramedial law and also using law $a(bc)=b(ac)$, we have,

$$\begin{aligned}a &= (ax)a=(ax)\{(ax)a\}\{a(ax)\}(xa)=x[\{a(ax)\}a]\\ &= x[\{(ea)(ax)\}a]=x[\{(xa)(ae)\}a]=(ex)[\{(xa)(ae)\}a]\\ &= [a\{(xa)(ae)\}](xe)=[a\{((ae)a)x\}](xe)=\{(xe)x\}[a\{(ae)a\}]\\ &= a[\{(xe)x\}\{(ae)a\}]=a[(ae)\{((xe)x)a\}]\\ &= (ea)[(ae)\{((xe)x)a\}]=[\{(ae)\{((xe)x)a\}\}a]e\\ &= [\{(ae)(ta)\}a]e=[\{(at)(ea)\}a]e=[\{(at)a\}a]e\in B^2S,\text{ where } t=\{(xe)x\}.\end{aligned}$$

Thus we have,

$$\begin{aligned}
& \max\left\{((F(\varepsilon)\circ F(\varepsilon))\circ(\mathcal{X}^{\delta}_{\gamma S}))(a),\gamma\right\} \\
= & \max\left\{\bigvee_{a=pq}(F(\varepsilon)\circ F(\varepsilon))(p)\wedge(\mathcal{X}^{\delta}_{\gamma S})(q),\gamma\right\} \\
\geq & \max\{(F(\varepsilon)\circ F(\varepsilon))\{((at)a)a\}\wedge(\mathcal{X}^{\delta}_{\gamma S})(e),\gamma\} \\
= & \max\{\bigvee_{\{((at)a)a\}=uv}\{F(\varepsilon)(u)\wedge F(\varepsilon)(v)\wedge\mathcal{X}^{\delta}_{\gamma S}(e)\},\gamma\} \\
\geq & \max\left\{F(\varepsilon)\{(at)a\}\wedge F(\varepsilon)(a)\wedge\mathcal{X}^{\delta}_{\gamma S}(e),\gamma\right\} \\
= & \max\left\{F(\varepsilon)\{(at)a\}\wedge F(\varepsilon)(a)\wedge 1,\gamma\right\} \\
= & \min\left\{\max\{F(\varepsilon)\{(at)a\},\gamma\},\max\{F(\varepsilon)(a),\gamma\}\right\} \\
\geq & \min\min\{F(\varepsilon)(a),\delta\},\min\{F(\varepsilon)(a),\delta\}\} \\
= & \min\left\{(F(\varepsilon)(a),\delta\right\} \\
= & \min\left\{(F(\varepsilon)(a),\delta\right\}
\end{aligned}$$

Thus $\min\left\{(F(\varepsilon)(a),\delta\right\}\leq\max\left\{((F(\varepsilon)\circ F(\varepsilon))\circ(\mathcal{X}^{\delta}_{\gamma S}))(a),\gamma\right\}$. This implies that $F(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}((F(\varepsilon)\circ F(\varepsilon))\circ\mathcal{X}^{\delta}_{\gamma S})$.

Therefore in any case, we have $K_1(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}K_2(\varepsilon)$. Hence

$$\langle F,B\rangle\subset_{(\gamma,\delta)}(\langle F,B\rangle\odot\langle F,B\rangle)\odot\Sigma\langle S,E\rangle.$$

$(iv)\implies(iii)$ is obvious.

$(iii)\implies(ii)$

Assume that B is bi-ideal of S, then $\Sigma(B,E)$ is $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal over S,respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(B,E)\subset_{(\gamma,\delta)}(\Sigma(B,E)\odot\Sigma(B,E))\odot\Sigma(S,E).$$

So,

$$\begin{aligned}
\chi^{\delta}_{\gamma B} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma B^2}\cap\chi^{\delta}_{\gamma S} \\
&\subseteq \vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma B}\circ\chi^{\delta}_{\gamma B})\circ\chi^{\delta}_{\gamma S} \\
&= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma B^2 S}.
\end{aligned}$$

Thus $B\subseteq B^2S$.

$(ii)\Rightarrow(i)$

$B\,[a]=a\cup a^2\cup(aS)\,a$, and $L\,[a]=a\cup Sa$ are principle bi-ideal and principle left ideal of S generated by a respectively. Thus by (ii), left invertive law, paramedial law and using law $a(bc)=b(ac)$, we have,

$$\begin{aligned}
Sa &\subseteq [(Sa)(Sa)]S=[S(Sa)](Sa)=(SS)[a(Sa)] \\
&= S[a(Sa)]\subseteq(aS)\,a.
\end{aligned}$$

Hence S is regular. ■

5.3 References

[1] S. K. Bhakat and P. Das, On the definition of a fuzzy subgroup, Fuzzy Sets and Systems, 51(1992), $235-241$.

[2] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroups, Fuzzy Sets and Systems, 80(1996), $359-368$.

[3] A.H. Clifford and G.B. Preston, The algebraic theory of semigroups, Volume II, Amer. Math. Soci., 1967.

[4] B. Davvaz, $(\in, \in \vee q)$*-fuzzy subnear-rings and ideals*, Soft Computing, 10 (2006), 206-211.

[5] B. Davvaz and O. Kazanci, *A new kind of* $(\in, \in \vee q)$*-fuzzy sublattice (ideal, filter) of a lattice*, International Journal of Fuzzy Systems, 13(1) (2011), 55-63.

[6] B. Davvaz and A. Khan, *Characterizations of regular ordered semigroups in terms of* (α, β)*-fuzzy generalized bi-ideals*, Information Sciences, 181 (2011), 1759-1770.

[7] B. Davvaz and A. Khan, *Generalized fuzzy filters in ordered semigroups*, Iranian Journal of Science and Technology, Transaction A, 36 (A1) (2012), 77-86.

[8] B. Davvaz and M. Khan, S. Anis and S. Haq, *Generalized fuzzy quasi-ideals of an intra-regular Abel-Grassmann's groupoid*, Journal of Applied Mathematics, Volume 2012, Article ID 627075, 16 pages.

[9] F. Feng, C.X. Li, B. Davvaz, M.I. Ali, Soft sets combined with fuzzy sets and rough sets: a tenta approach, Soft Computing 14 (2010), 899–911.

[10] F. Feng, X.Y. Liu, V. Leoreanu-Fotea, Y.B. Jun, Soft sets and soft rough sets, Inform. Sci., 181 (2011), 1125–1137.

[11] I.R. Goodman, Fuzzy sets as equivalence classes of random sets, in: Recent Developments in Fuzzy Sets and Possibility Theory (R. Yager, Ed.), Pergamon, New York (1982).

[12] K. Iseki, A characterization of regular semigroups, Proc. Japan Acad., 32 (1965), $676-677$.

[13] Y. B. Jun, Generalizations of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebra, Comput. Math. Appl., 58 (2009) , 1383-1390.

[14] Y.B. Jun and S.Z. Song, Generalized fuzzy interior ideals in semigroups, Inform. Sci., 176(2006), 3079-3093.

[15] O. Kazancı and S. Yamak, Generalized fuzzy bi-ideals of semigroups, Soft Comput., 12(2008), $1119 - 1124$.

[16] M. A. Kazim and M. Naseeruddin, On almost semigroups, The Alig. Bull. Math., 2 $(1972), 1 - 7$.

[17] N. Kehayopulu, and M. Tsingelis, Regular ordered semigroups in terms of fuzzy subsets, Inform. Sci. 176 (2006), 3675-3693.

[18] N. Kuroki, Fuzzy semiprime quasi ideals in semigroups, Inform. Sci., $75(3)(1993)201 - 211$.

[19] A. Khan, N.H. Sarmin, F.M. Khan and B. Davvaz, *Regular AG-groupoids characterized by* $(\in, \in \vee q_k)$*-fuzzy ideas*, Iranian Journal of Science and Technology, Transaction A, 36A2 (2012), 97-113.

[20] A. Khan, N.H. Sarmin, B. Davvaz and F.M. Khan, *New types of fuzzy bi-ideals in ordered semigroups*, Neural Computing & Applications, 21 (2012), 295-305.

[21] M. Khan and N. Ahmad, Characterizations of left almost semigroups by their ideals, Journal of Advanced Research in Pure Mathematics, 2 (2010), $61 - 73$.

[22] M. Khan, Y.B. Jun and K. Ullah, Characterizations of right regular Abel-Grassmann's groupoids by their $(\in, \in \vee q_k)$-fuzzy ideals, submitted.

[23] M. Khan and V. Amjid, On some classes of Abel-Grassmann's groupoids, Journal of Advanced Research in Pure Mathematics, 3, 4 (2011), 109-119.

[24] S. Lajos, A note on semilattice of groups, Acta Sci. Math. Szeged, $31\,(1982)\,, 179 - 180$.

[25] J. Maiers and Y. S. Sherif, Applications of fuzzy set theory, IEEE Transactions on Systems, Man and Cybernetics, 15(1) (1985), 175-189.

[26] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (2001), no. 3, 589–602.

[27] J. N. Mordeson, D. S. Malik and N. Kuroki, *Fuzzy semigroups*, Springer-Verlag, Berlin, Germany, 2003.

[28] Q. Mushtaq and S. M. Yusuf, On LA-semigroupss, The Alig. Bull. Math., 8 $(1978), 65 - 70$.

[29] V. Murali, Fuzzy points of equivalent fuzzy subsets, Inform. Sci., 158 (2004), 277-288.

[30] M. Petrich, *Introduction to Semigroups*, Charles.E. Merrill, Columbus, 1973.

[31] P. V. Protić and N. Stevanović, On Abel-Grassmann's groupoids, Proc. Math. Conf. Priština, (1994), $31-38$.

[32] P. V. Protić and N. Stevanović, AG-test and some general properties of Abel-Grassmann's groupoids, PU. M. A., 4(6)(1995), $371-383$.

[33] P.M. Pu and Y.M. Liu, Fuzzy topology I, neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., 76 (1980), 571-599.

[34] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35(1971), $512-517$.

[35] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2007) 412–418.

[36] M. Shabir, Y. B. Jun and Y. Nawaz, Semigroups characterized by $(\in, \in \vee q_k)$-fuzzy ideals, Comput. Math. Appl., 60(2010), 1473-1493.

[37] M. Shabir, Y.B. Jun and Y. Nawaz, Characterizations of regular semigroups by (α, β)-fuzzy ideals, Comput. Math. Appl., 59(2010), 161-175.

[38] F. Smarandache, Neutrosophy. Neutrosophic Probability, Set, and Logic, Amer. Res. Press, Rehoboth, USA, 105 p., 1998.

[39] F. Smarandache & J. Dezert, editors, Advances and Applications of DSmT for Information Fusion", American Res. Press, Vols. 1-4, 2004, 2006, 2009, 2015.

[40] J. Zhan and Y.B. Jun, Generalized fuzzy interior ideals of semigroups, Neural Comput. Applic., 19(2010), $515-519$.

[41] Y.G. Yin and J. Zhan, Characterization of ordered semigroups in terms of fuzzy soft ideals, Bull. Malays. Math. Sci. Soc., (2)35(4)(2012), $997-1015$.

[42] L. A. Zadeh, Fuzzy sets. Inform. Control, 8 (1965), $338-353$.

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