A note on minimal extensions of the Fibonacci sequence

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Abstract

We propose minimal extensions (modifications) of the Fibonacci sequence that preserve its crucial characteristics and yet are quite different from the original.

The Fibonacci sequence is one of the most famous integer sequences (see [1]), perhaps second in popularity only to the sequences of natural and prime numbers and their simple derivatives.

Many modifications of this sequence have been entertained since it was first proposed centuries ago, some, if not most of them, departing rather significantly from its key properties. It is not the goal of this brief note to review such modifications. On the contrary, here we want to focus on least invasive modifications of this sequence, minimal in the sense elaborated below (properties 1-5).

The Fibonacci sequence is defined by the following recurrence formula,

F(n) = F(n-1) + F(n-2),

with the initial terms stipulated to be **F(0)=0, and F(1)=1**.

The first terms of this sequence are **0,1,1,2,3,5,8,13,21,34,55,89**, …, and the numbers of this sequence are often referred to as the Fibonacci numbers.

We propose the following minimal modification of this famous sequence, defined in a recursive way as follows (^ denotes the power raising operation),

FMPM(n) = FMPM(n-1) + FMPM(n-2) + (1+ (-1)^ (FMPM(n-1) + FMPM(n-2)),

with the initial two terms as in the original sequence, **FMPM(0)=0, and FMPM(1)=1**.

Using Mathematica, the following first 50 terms of this sequence have been obtained,

0,1,1,4,5,9,16,25,41,68,109,177,288,465,753,1220,1973,3193,5168,8361,13529,21892,35421,57313,92736,150049,242785,392836,635621,1028457,1664080,2692537,4356617,7049156,11405773,18454929,29860704,48315633,78176337,126491972,204668309,331160281,535828592,866988873,1402817465,2269806340,3672623805,5942430145,9615053952,15557484097,25172538049.

One can easily note the following properties of the sequence just proposed,

1) *the defining recurrence relation of it is a function of the sum of its two previous terms, as is the case in the original sequence, and not the function of individual previous terms, as could be the case in general*,
2) *the ratio of its two consecutive terms tends fast to the Golden Mean (for the last two terms out the 50 presented above the difference between their ratio and the Golden Mean is less than 10 to the negative power of 10)*,
3) *the initial terms of the sequence are the same as in the original Fibonacci sequence, which, in general, does not have to be the case*,
4) *the only constants used are 1 and –1, two simplest integers one can think of (save for 0, which is trivial),*
5) *these constants are combined through addition, the defining arithmetic operation of the Fibonacci sequence*.

If these two constants are combined through subtraction, one obtains yet another modification of the Fibonacci sequence,

FMNM(n) = FMNM(n-1) + FMNM(n-2) + (1- (-1)^ (FMNM(n-1) + FMNM(n-2)),

with the initial two terms as in the original sequence, **FMNM(0)=0, and FMNM(1)=1**.

Using Mathematica, the following first 50 terms of this sequence have been obtained,

0,1,3,4,9,15,24,41,67,108,177,287,464,753,1219,1972,3193,5167,8360,13529,21891,35420,57313,92735,150048,242785,392835,635620,1028457,1664079,2692536,4356617,7049155,11405772,18454929,29860703,48315632,78176337,126491971,204668308,331160281,535828591,866988872,1402817465,2269806339,3672623804,5942430145,9615053951,15557484096,25172538049,40730022147.

Some, though relatively few, of these numbers are the same as the numbers in the other sequence proposed here.

This modification shares properties 1-4 listed above, but not 5, and, as such, it is a less bit minimal than FMPM; it departs a bit more from the properties of the original Fibonacci sequence.

FMPM and **FMNPM** stand for **Fibonacci Minimal Positive Modification** and **Fibonacci Minimal Negative Modification**, respectively.

Yet another modification of the Fibonacci sequence can be produced through the following recurrence relation,

FSM(n) = FSM(n-1) + FSPM(n-2) - (-1)^ (FSPM(n-1) + FMSM(n-2)),

with the initial two terms as in the original sequence, **FSM(0)=0, and FSM(1)=1.**

Here **FSM** stands for **Fibonacci Simplest Modification.**

Using Mathematica, the following first 50 terms of this sequence have been obtained,

0,1,2,4,5,10,16,25,42,68,109,178,288,465,754,1220,1973,3194,5168,8361,13530,21892,35421,57314,92736,150049,242786,392836,635621,1028458,1664080,2692537,4356618,7049156,11405773,18454930,29860704,48315633,78176338,126491972,204668309,331160282,535828592,866988873,1402817466,2269806340,3672623805,5942430146,9615053952,15557484097,25172538050.

This modification shares properties 1-3 listed above, and since it uses only one constant, which is -1, it is even simpler than the other two proposed above, while being minimal in the same spirit as they are.

Moreover, this modification does not possess a positive, nontrivial counterpart, which would make it even more unique. More precisely, when the first negative sign is replaced with a positive one in the defining formula for FSM, the formula produces a regular sequence of 1s and 0s, which in no way resembles the original version of the Fibonacci sequence or those proposed above.

Some of the terms of FSM are equal to those found in FMPM and (to a lesser extent) in FMNM.

Despite their quite apparent simplicity, none of the modifications proposed in this note belongs yet to the Online Encyclopedia of Integer Sequences.

References

[1] https://oeis.org/A000045