On Integral Representations for Harmonic Number and Digamma Function

BY EDIGLES GUEDES

March 3, 2015

"Then Simon Peter answered him, Lord, to whom shall we go? thou hast the words of eternal life." - John 6:68.

Abstract. In this paper, we demonstrate some integral representations for harmonic number and digamma function.

1. Introduction

Leonhard Euler established the following integral representation [1] for harmonic number

\[ H_n = \int_0^1 \frac{1-x^n}{1-x} \, dx, \]

and, accordingly, we can deduce the integral representation for the digamma function [2]

\[ \psi(s+1) = -\gamma + \int_0^1 \frac{1-x^s}{1-x} \, dx. \]

In this article, we prove that

\[ H_n = 2 \int_0^1 \frac{(1-x^n)(1-x^{n+1})}{1-x^2} \, dx; \]

\[ H_n = 2m \int_0^1 \frac{x^{m-1}(1-x^m)(1-x^{(n+1)m})}{1-x^{2m}} \, dx \]

and

\[ \psi(s+1) = -\gamma - 2 \int_0^1 \frac{(1-x)^s - (1-x^2)^s}{x} \, dx. \]

(1)

2. Integral Representations for Harmonic Number

Lemma 1. If \( k = 1, 2, 3, \ldots \), then

\[ \frac{1}{k} = 2\int_0^1 x^{k-1}(1-x^k) \, dx. \]

(2)

Proof. We know the elementary identity

\[ \frac{1}{k} = \frac{a}{ak-1} - \frac{1}{k(a-k-1)} \]

and the integral representation

\[ \frac{1}{k} = \int_0^1 x^{k-1} \, dx, \]

for \( \text{Re}(k) > 0 \).

From (2) and (3), it follows that

\[ \frac{1}{k} = \int_0^1 x^{\frac{ak-1}{n}} \, dx - \int_0^1 x^{k(a-k-1)} \, dx \]

\[ = \int_0^1 \left[ x^{\frac{ak-1}{n}} - x^{k(a-k-1)} \right] \, dx. \]

(5)
We take $a = \frac{2}{k}$ in Eq. (4)
\[
\frac{1}{k} = \int_{0}^{1} \frac{k^2 - x^k}{x} \, dx.
\]
(6)

Let $k \to 2k$ in Eq. (5)
\[
\frac{1}{k} = 2\int_{0}^{1} \frac{x^k - x^{2k}}{x} \, dx
= 2\int_{0}^{1} x^{k-1}(1 - x^k) \, dx,
\]
which is the desired result.

**Theorem 2.** If $n = 1, 2, 3, \ldots$, then
\[
H_n = 2\int_{0}^{1} \frac{(1 - x^n)(1 - x^{n+1})}{1 - x^2} \, dx,
\]
where $H_n$ denotes the harmonic number.

**Proof.** We knew the definition [1]
\[
H_n = \sum_{k=1}^{n} \frac{1}{k}
\]
From (1) and (6), we conclude that
\[
H_n = \sum_{k=1}^{n} 2\int_{0}^{1} x^{k-1}(1 - x^k) \, dx
= 2\int_{0}^{1} \sum_{k=1}^{n} x^{k-1}(1 - x^k) \, dx
= 2\int_{0}^{1} \frac{(1 - x^n)(1 - x^{n+1})}{1 - x^2} \, dx,
\]
which is the desired result.

**Theorem 3.** If $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$, then
\[
H_n = 2m \int_{0}^{1} \frac{x^{m-1}(1 - x^m)(1 - x^{(n+1)m})}{1 - x^{2m}} \, dx,
\]
where $H_n$ denotes the harmonic number.

**Proof.** We use the induction. Let $m = 1$, thereafter,
\[
H_n = 2 \cdot 1 \cdot \int_{0}^{1} \frac{1}{x} \sum_{k=1}^{n} (x^k - x^{2k}) \, dx
= 2\int_{0}^{1} \frac{1}{x} \cdot \frac{x(1 - x^n)(1 - x^{n+1})}{1 - x^2} \, dx
= 2\int_{0}^{1} \frac{(1 - x^n)(1 - x^{n+1})}{1 - x^2} \, dx,
\]
as shown in previous theorem. Assume that the formula is valid for $m = j$. Therefore, for $m = j + 1$, we have
\[
H_n = 2(j + 1) \int_{0}^{1} \frac{1}{x} \sum_{k=1}^{n} (x^{(j+1)k} - x^{2k(j+1)}) \, dx
= 2(j + 1) \int_{0}^{1} \frac{1}{x} \cdot \frac{x^{(j+1)}(1 - x^{(j+1)n}(1 - x^{(n+1)(j+1)})}{1 - x^{2(j+1)}} \, dx
= 2(j + 1) \int_{0}^{1} \frac{1}{x} \cdot \frac{x^{(j+1)}(1 - x^{(n+1)(j+1)})}{1 - x^{2(j+1)}} \, dx,
\]
consequently, setting $j + 1 \rightarrow m$, we obtain

$$H_n = 2m \int_0^1 \frac{x^{m-1}(1 - x^m)(1 - x^{(n+1)m})}{1 - x^{2m}} \, dx,$$

which is the desired result. \qed

3. Integral Representation for Digamma Function

**Theorem 4.** If $\text{Re}(s) > -1$, then

$$\psi(s + 1) = -\gamma - 2 \int_0^1 \frac{(1 - x)^s - (1 - x^2)^s}{x} \, dx,$$

where $\psi(s)$ denotes the digamma function and $\gamma$ denotes the Euler-Mascheroni constant.

**Proof.** We consider the Newton series [2] for the digamma function

$$\psi(s + 1) = -\gamma - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \binom{s}{k}.$$

Substitute

$$\psi(s + 1) = -\gamma - \sum_{k=1}^{\infty} (-1)^k 2 \int_0^1 x^{k-1}(1 - x^k) \, dx \binom{s}{k},$$

$$= -\gamma - 2 \int_0^1 \sum_{k=1}^{\infty} (-1)^k x^{k-1}(1 - x^k) \binom{s}{k} \, dx,$$

which is the desired result. \qed

**Remark 5.** With the aid of Theorem 4, we evaluate some special values for digamma functions:

$$\psi\left(\frac{5}{2}\right) = -\gamma + \frac{8}{3} - 2 \ln 2,$$

$$\psi\left(\frac{7}{3}\right) = -\gamma + \frac{15}{4} - \frac{\pi \sqrt{3}}{6} - 3 \ln 3,$$

$$\psi\left(\frac{9}{4}\right) = -\gamma + \frac{24}{5} - \frac{\pi}{2} - 3 \ln 2,$$

$$\psi\left(\frac{11}{5}\right) = -\gamma + \frac{35}{6} - \frac{\pi}{2} \sqrt{1 + \frac{2}{\sqrt{5}}} - \frac{\sqrt{5}}{4} \ln \left(\frac{\sqrt{5} + 1}{\sqrt{5} - 1}\right) - \frac{15 \ln 5}{12},$$

$$\psi\left(\frac{13}{6}\right) = -\gamma + \frac{48}{7} - \frac{\pi \sqrt{3}}{2} - 2 \ln 2 - 3 \ln 3.$$

**References**
