

# The Fundamental Theorem of Calculus with Gossamer numbers

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## Abstract

Within the gossamer numbers  $*G$  which extend  $\mathbb{R}$  to include infinitesimals and infinities we prove the Fundamental Theorem of Calculus (FTC). Riemann sums are also considered in  $*G$ , and their non-uniqueness at infinity. We can represent the sum as a continuous function in  $*G$  by inserting infinitesimal intervals at the discontinuities, and threading curves between the sums discontinuities. As the FTC is a difference of integrals at the end points, the same is true for sums.

## 1 Introduction

1. Introduction
2. Riemann sums
3. Fundamental Theorem of Calculus (FTC)
4. Heavyside function and integration
5. FTC for sums

While the heavyside step function is well known, it can also be represented using the Iverson bracket notation (see [10, p.24]) where the logical statement evaluates to 1 if true, else 0. The prime-number function could be expressed as  $\pi(x) = [x \geq 2] + [x \geq 3] + [x \geq 5] + [x \geq 7] + \dots$ . As the name “step” suggests, a discontinuity exists between the levels.

We consider a step function’s continuous representation with the gossamer numbers [1]. The discontinuity in  $\mathbb{R}$  (or  $*G$ ) is replaced by a continuous function in  $*G$  over an infinitesimal interval.

A discontinuous curve in  $\mathbb{R}$  can become a continuous curve in  $*G$ . Since we have theory for continuous curves such as FTC, having a theory that allows the transfer from the discontinuous to the continuous and back is useful. A continuous representation is advantageous as we gain the benefits of continuity and classical calculus in  $*G$ .

The next benefit of working in  $*G$  is the extension of the interval to include infinitesimals and infinities, thereby bypassing the need to extend  $\mathbb{R}$ , and potentially representing the theory in a simpler way. For example, theory may encompass both the finite and the infinite intervals and not need or reduce to special cases.

The fundamental theorem of calculus in  $\mathbb{R}$  is stated for a finite interval, and extended to

improper integrals. The Lebesgue measure and integral are a generalization, and are not limited by the real numbers not having infinitely small and large numbers, well defined. We see  $*G$  as possibly another way. Our understanding is that Non-Standard Analysis (NSA) is also a two-tiered calculus and also has the benefits of the infinitesimals and infinities. Having an extended domain rather than extending the reals is beneficial.

Consequently, we believe the fundamental theorem of calculus in  $*G$  is better than otherwise extending calculus for specific cases.

## 2 Riemann sums

The typical way of visualising a Riemann sum is to partition a function into bars, where as the columns become infinitesimal, the sum of the bars evaluates to the area under the curve.

Contrast this method of integration with Archimedes' method of exhaustion [9]. We give an example where the area is underestimated, but at infinity is equal to the area of the circle.

**Example 2.1.** *Consider a function of straight lines through an ordered list of points on a circle about the origin (0 to  $2\pi$ ). At infinity, having the points cover the circumference, the function length equals the circle's circumference.*

*An iteration of the sum follows. A point is inserted into the ordered sequence of points about origin, and the function is subsequently updated.*

*With the insertion of each additional point, a construction of two triangles' area is added to a cumulative sum. In this way the circle's area is being integrated. Starting with a circle and an inscribed square, points are added indefinitely and the area of the circle is calculated.*

**Example 2.2.** *A Riemann sum of the above, for any line segment of the function forms a triangle with the origin.*

*From a numerical point of view, in the Riemann sum the thin isosceles triangles will have roundoff errors associated with the sum. However, if the sum can be evaluated symbolically then this may not be an issue. [ In contrast, the method of exhaustion maintains the triangles shape, even for the infinitely small ]*

The above are examples of definite integrals, of which the Riemann sum was investigated as a theory of the sum of integrals. We consider without loss of generality, positive Riemann sums in  $*G$ .

**Definition 2.1.** *A uniform Riemann sum in  $*G$ ;  $\nu \in \mathbb{N}_\infty$ ;*

$$\sum_{j=1}^{\nu} f\left(\frac{j}{\nu}\right) \frac{1}{\nu} \Big|_{\nu=\infty}$$

As a uniform partition, with arbitrary infinitely small partitions, can divide any interval with an infinitely small width. Since there is no smallest number, and there is no smallest interval, then we can change the interval width.

**Example 2.3.**  $\sum_{j=1}^n f(j) \frac{1}{n} |_{n=\infty}$  has  $1 \dots n - 1$  steps and width  $\frac{1}{n} |_{n=\infty}$ .  $\sum_{j=1}^{n^2} f(\frac{j}{n^2}) \frac{1}{n^2} |_{n=\infty}$  has a  $1 \dots n^2 - 1$  steps and width  $\frac{1}{n^2} |_{n=\infty}$ .  $\sum_{j=1}^{n!} f(\frac{j}{n!}) \frac{1}{n!} |_{n=\infty}$  are uniform Riemann sums.

Another visual way to understand a Riemann sum and integral is to consider the Riemann sum *not* over a finite interval  $[a, b]$ , but as an infinite positive interval in  $*G$ . The columns and area under the graph are asymptotic, which we describe with the asymptotic relation .

**Definition 2.2.** We say a uniform Riemann sum (Definition 2.1) is uniform Riemann integrable;  $\nu \in \mathbb{N}_\infty$ :

$$\int_j^{j+1} f\left(\frac{x}{\nu}\right) dx \sim f\left(\frac{j}{\nu}\right)$$

$$\int_0^\nu f\left(\frac{x}{\nu}\right) dx \sim \sum_{j=1}^\nu f\left(\frac{j}{\nu}\right) |_{\nu=\infty}$$

**Remark: 2.1.** Consider  $\int_0^\nu f\left(\frac{x}{\nu}\right) \frac{1}{\nu} dx \sim \sum_{j=1}^\nu f\left(\frac{j}{\nu}\right) \frac{1}{\nu} |_{\nu=\infty}$ ,  $\int_0^\nu f\left(\frac{x}{\nu}\right) \frac{d\left(\frac{x}{\nu}\right)}{dx} dx \sim \sum_{j=1}^\nu f\left(\frac{j}{\nu}\right) \frac{1}{\nu} dj |_{\nu=\infty}$  where  $dj$  is a change in integers (see [5]),  $dj = (j + 1) - j = 1$ ; we see the one to one correspondence of the Riemann sum to the integral, as the columns are asymptotic to the integral between integer values. With this we can interpret between a continuous change and a discrete change of variable.

These conditions arise from transforming the definite integral to the uniform Riemann series, and vice versa.

However, in the transformation, while the definite integral  $\int_0^1 f(x) dx$  is used with the chain rule, this does not exclude indefinite integral evaluation as the following example shows.

**Example 2.4.** Integrate the divergent integral  $\int n^2 dn = \frac{1}{3} n^3 |_{n=\infty}$  with a Riemann sum.

We will use the closed form formula for the sum of squares,  $\sum_{k=1}^n k^2 = \frac{2n^3 + 3n^2 + n}{6}$ , for convenience we can find this using a symbolic maths package: in Maxima ‘sum(k<sup>2</sup>, k, 1, n), simpsum;’.

$\int_1^n x^2 dx |_{n=\infty} = \int_{\frac{1}{n}}^1 (nx)^2 d(nx) |_{n=\infty} = n^3 \int_0^1 x^2 dx |_{n=\infty}$ . Evaluate the definite integral with a finite uniform Riemann sum,  $\int_0^1 x^2 dx = \sum_{k=1}^\nu \left(\frac{k}{\nu}\right)^2 \frac{1}{\nu} |_{\nu=\infty} = \frac{1}{\nu^3} \sum_{k=1}^\nu k^2 |_{\nu=\infty} = \frac{1}{\nu^3} \frac{2\nu^3 + 3\nu^2 + \nu}{6} |_{\nu=\infty} = \frac{1}{\nu^3} \frac{2\nu^3}{6} |_{\nu=\infty} = \frac{1}{3}$ , then  $\int_1^n x^2 dx |_{n=\infty} = \frac{1}{3} n^3 |_{n=\infty}$ .

To understand Riemann sums better, we consider some of the limitations of the Riemann sum and integral [8].

That the usability for an infinite domain is limited. This may well be true, as a Riemann sum is a primitive sum. However, the Riemann sum can be constructed in  $*G$ , extending the domain to include infinitesimals and infinities. Example 2.4 shows that with scalings, a divergent integral can be summed. Thus, the possibility of addressing the limitation of the infinite bounds exists.

Another limitation of the Riemann sum and integral such as not being able to integrate over the enumeration of rational numbers in  $[0, 1]$ , does not invalidate the sum for other purposes.

Concerns regarding the functions being summed, are problem or domain dependent.

If we restrict the Riemann sum to continuous functions in  $\mathbb{R}$  and  $*G$  the scope is still large, particularly in light of transference [2, Part 4].

For example, consider a Riemann sum to a finite value. While we consider the Riemann sum native to  $*G$  (because this includes infinitesimals and infinities), as a two-tiered calculus we can have a uniform Riemann sum equal to a Riemann sum at infinity after transference (see Conjecture 2.1).

$$\sum_j \int f_j(x) dx \sim \int \sum_j f_j(x) dx$$

Interchanging limit processes we believe is more fundamental. However, we see mathematics with infinity, as the necessary way forward. Orderings of variables reaching infinity before others is a much larger question, but one that has to be asked (See [4, Part 6 A two-tiered calculus]). See our use of interchanging point evaluations in the proof of Theorem 3.1.

Assuming continuous functions, uniform Riemann sums are preferred, because they are easier to calculate with. Hence, while the Riemann sum may appear more general, for the infinite partitions which are important, they are actually equivalent. That is, their definition is no more general for the infinite case than the uniform partition.

**Conjecture 2.1.** *A uniform Riemann sum of a continuous curve is asymptotic to a Riemann sum with infinite limit.*

In forming a Riemann sum [7] advised to express the sum in a particular way and apply Corollary 2.1.

**Proposition 2.1.** *With the Riemann integral Definition 2.2, there exists  $\nu|_{n=\infty} \in \mathbb{N}_\infty$ ;  $c \in *G$ ;  $c \prec \sum_{j=1}^\nu f(\frac{j}{\nu})\frac{1}{\nu}|_{n=\infty}$  and  $c \prec \int_0^1 f(x) dx$ :*

$$\sum_{j=1}^\nu f(\frac{j}{\nu})\frac{1}{\nu}|_{n=\infty} = \int_0^1 f(x) dx + c$$

*Proof.* Follows  $a \sim b$  then there exists  $c$ :  $a + c = b$ ,  $a \succ c$  and  $b \succ c$  [3, Part 5, Proposition 2.1]. □

**Corollary 2.1.** *A Riemann sum may be evaluated by the following.*

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n f\left(\frac{j}{n}\right) = \int_0^1 f(x) dx$$

However, this does not describe the transition between sum and integral, although it is the most practical approach to evaluating a Riemann sum. We discuss the transition between Riemann sums and integrals in both directions, in the context of sum and integral scaling and shifting.

**Proposition 2.2.**  *$c, x \in *G$ ; When  $c$  is constant,  $d(x + c) = dx$*

*Proof.* Let  $u = c + x$ ,  $\frac{du}{dx} = 1$ ,  $d(c + x) = du = \frac{du}{dx} dx = dx$  □

**Proposition 2.3.** *When  $\alpha$  is constant,  $d(\alpha x) = \alpha dx$*

*Proof.* Let  $v = \alpha x$ ,  $dv = \frac{dv}{dx} dx = \alpha dx$  □

If we apply the inverse operation in the integrand to within the integral, for operators  $+$  and  $\times$  we can shift and scale the integral. This can be shown by substitution.

**Theorem 2.1.** *Scaling the integral;  $a, b, \alpha, \nu, x, f \in *G$ ;*

$$\int_a^b f(x) dx = \int_{\frac{a}{\alpha}}^{\frac{b}{\alpha}} f(\alpha \nu) d(\alpha \nu)$$

*Proof.*  $\int_a^b f(x) dx$ . Let  $x = \alpha \nu$ .  $x = a$  then  $\nu = \frac{a}{\alpha}$ ,  $x = b$  then  $\nu = \frac{b}{\alpha}$ .  $\int_a^b f(x) dx = \int_{\frac{a}{\alpha}}^{\frac{b}{\alpha}} f(\alpha \nu) \frac{dx}{d\nu} d\nu = \alpha \int_{\frac{a}{\alpha}}^{\frac{b}{\alpha}} f(\alpha \nu) d\nu = \int_{\frac{a}{\alpha}}^{\frac{b}{\alpha}} f(\alpha \nu) d(\alpha \nu)$  □

**Theorem 2.2.** *Shifting the integral;  $a, b, \alpha, \nu, x, f \in *G$ ;*

$$\int_a^b f(x) dx = \int_{a-c}^{b-c} f(x + c) dx$$

*Proof.*  $\int_a^b f(x) dx$ . Let  $x = v + c$ ,  $x = a$  then  $v = a - c$ ,  $x = b$  then  $v = b - c$ .  $\int_a^b f(x) dx = \int_{a-c}^{b-c} f(v + c) \frac{dx}{dv} dv = \int_{a-c}^{b-c} f(v + c) dv$

The same result could be achieved by shifting the integrand arguments, and shifting in the opposite direction the variable in the body.  $\int_a^b f(x) dx = \int_{a-c}^{b-c} f(x + c) d(x + c) = \int_{a-c}^{b-c} f(x + c) dx$  □

**Proposition 2.4.** *A definite integral of a continuous function in  $*G$  can be transformed to a Riemann sum, and vice versa, if uniform Riemann integrable.  $\nu \in \mathbb{N}_\infty$ ;  $f, x \in *G$ ;*

$$\begin{aligned}
& \int_0^1 f(x) dx && \text{(Scale to an improper integral)} \\
&= \int_0^\nu f\left(\frac{x}{\nu}\right) d\left(\frac{x}{\nu}\right) \Big|_{\nu=\infty} \\
&= \int_0^\nu f\left(\frac{x}{\nu}\right) \frac{1}{\nu} dx \Big|_{\nu=\infty} && \text{(Proposition 2.1; scaling out the infinitesimal change)} \\
&= \sum_{j=1}^\nu f\left(\frac{j}{\nu}\right) \frac{1}{\nu} dj \Big|_{\nu=\infty} + c && \text{(the magnitude of the sum dominates)} \\
&= \sum_{j=1}^\nu f\left(\frac{j}{\nu}\right) \frac{1}{\nu} \Big|_{\nu=\infty}
\end{aligned}$$

*Reversing the above operations transforms the Riemann sum back into the definite integral.*

### 3 Fundamental Theorem of Calculus (FTC)

The following proofs are a derivative from those given by D. Joyce [6], but in  $*G$  number system. Since this number system includes infinitesimals and infinities, the domain and space are significantly increased. For example, FTC is proved for finite bounds, but in  $*G$ , the bounds  $[a, b]$  can represent improper integrals.

**Theorem 3.1.** *FTC<sup>-1</sup> the derivative of an integral of a function is the function.*

*$f, F, a, b, x, t \in *G$ ; If  $f$  is a continuous function on the closed interval  $[a, b]$ , and  $F$  is its accumulation function defined by*

$$F(x) = \int_a^x f(t) dt$$

*for  $x \in [a, b]$ , then  $F$  is differentiable on  $[a, b]$  and its derivative is  $f$ , that is,  $F'(x) = f(x)$ .*

*Proof.*  $h \in *G$ ;  $F'(x) = \frac{F(x+h) - F(x)}{h} \Big|_{h=0} = \frac{1}{h} (\int_a^{x+h} f(t) dt - \int_a^x f(t) dt) \Big|_{h=0} = \frac{1}{h} \int_x^{x+h} f(t) dt \Big|_{h=0}$   
 $= \frac{1}{h} \int_0^h f(t+x) d(t+x) \Big|_{h=0} = \frac{1}{h} \int_0^h f(t+x) dt \Big|_{h=0}$  as  $x$  is constant with respect to  $t$ .

Use is made of a one variable reaching its value before another, hence justifying interchanging point evaluations. By sending  $h \rightarrow 0$  before performing integration, we are investigating infinitely close to the point  $x$ .

By constructing a Riemann sum, integrating infinitesimally close to the point about  $x$ ,  $h \rightarrow 0$  before  $dt \rightarrow 0$ ,  $\int_0^h f(x+t) dt \Big|_{h=0} = \int_0^1 f(x+ht) h dt \Big|_{h=0} = h \sum_{k=1}^n f(x + h\frac{k}{n}) \frac{1}{n} \Big|_{n=\infty} \Big|_{h=0}$

$= h \sum_{k=1}^n f(x + h\frac{k}{n})|_{h=0} \frac{1}{n}|_{n=\infty}|_{h=0} = h \sum_{k=1}^n f(x + 0)|_{n=\infty} \frac{1}{n}|_{n=\infty}|_{h=0} = hf(x)|_{h=0}$ . Since  $h$  is arbitrarily small.

$$F'(x) = \frac{1}{h}hf(x)|_{h=0} = f(x) \quad \square$$

**Remark: 3.1.** *If the order of variables reaching infinity is interchanged,  $n \rightarrow \infty$  before  $h \rightarrow 0$ , we obtain a different sum. This is a consequence of non-uniqueness at infinity, corresponding with the different possibilities there.*

$$h \sum_{k=1}^n f(x+h\frac{k}{n}) \frac{1}{n}|_{n=\infty}|_{h=0} = h \sum_{k=1}^n f(x+h\frac{k}{n})|_{n=\infty} \frac{1}{n}|_{n=\infty}|_{h=0} = h \sum_{k=1}^n f(x+\frac{k}{n})|_{n=\infty} \frac{1}{n}|_{n=\infty}|_{h=0} = h \int_0^1 f(x) dx|_{h=0}$$

**Theorem 3.2.** *FTC*

*If  $F'$  is continuous on  $[a, b]$ , then*

$$\int_a^b F'(x) dx = F(b) - F(a)$$

*Proof.*  $G(x) = \int_a^x F'(t) dt$ . By FTC<sup>-1</sup>  $G'(x) = F'(x)$ , integrate then  $G(x) = F(x) + c$  where  $c \in *G$  is constant. But  $G(a) = 0$ ,  $0 = F(a) + c$ ,  $c = -F(a)$ .

$$G(x) = F(x) - F(a), G(b) = F(b) - F(a), \int_a^b F'(t) dt = F(b) - F(a) \quad \square$$

## 4 Integrating the heavyside step function

Given that between any two real numbers, there exists another real number, may give the impression that the real number line is dense and complete. However, this is not the case.

At a finer layer, the gossamer numbers  $*G$  fill the gaps between the real numbers, with infinitesimals, as well as extending number line.

**Lemma 4.1.** *If  $a_i \in \mathbb{R}$ ,  $h \in \Phi$ ; then  $\sum_{k=1}^{\infty} a_i h^k \in \Phi$ .*

*Proof.*  $\exists \beta \in \mathbb{R} : \forall k, |a_k| \leq \beta, \sum_{k=1}^{\infty} a_i h^k \leq \sum_{k=1}^{\infty} \beta h^k$

$$\theta = h + h^2 + h^3 + \dots, \theta h = h^2 + h^3 + h^4 + \dots, \theta(1 - h) = h, \theta = \frac{h}{1-h} \in \Phi$$

$$\sum_{k=1}^{\infty} a_i h^k \leq \beta \theta \in \Phi \quad \square$$

**Theorem 4.1.** Given  $x \in \mathbb{R}$  and  $y \in *G$ :  $x \simeq y$ , there exist and infinity of numbers infinitesimally close to  $x$  which transfer  $*G \mapsto \mathbb{R}$  to the unique  $x$ .

*Proof.*  $h, \delta \in \Phi$ ;  $a_k \in \mathbb{R}$ ;  $\delta = \sum_{k=1}^{\infty} a_k h^k$ ,  $y = x + \delta$ , taking the standard part  $\text{st}(y) = x$  as from Lemma 4.1  $\delta \in \Phi$ ,  $\text{st}(\delta) = 0$ .  $\square$

The heavyside step function, discontinuous in  $\mathbb{R}$  can, with infinitesimals, be representative of a continuous function in  $*G$ . There is no contradiction, because during the transfer  $*G \mapsto \mathbb{R}$  the continuous function becomes discontinuous as the infinitesimals are projected to 0.

Let  $g$  be the function over the discontinuities in  $*G$ . Let  $f_2 = f + g$ .  $(*G, f_2) \mapsto (\mathbb{R}, f)$ ,  $(*G, \int f_2 dx) \mapsto (\mathbb{R}, \int f dx)$ . Construct the integral summing the discontinuities, which is also an infinitesimal  $\int g dx \in \Phi$ .

The transfer from  $*G$  to  $\mathbb{R}$  would discard the infinitesimal sum as  $\Phi \mapsto 0$ .

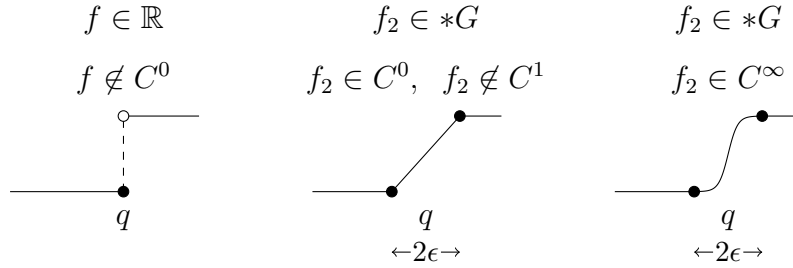


Figure 1: Discontinuous heavyside step function continuous in  $*G$

Let  $f$  be continuous,  $(*G, f) \mapsto (\mathbb{R}, f_2)$  is not unique. The continuous function in  $*G$  can transfer back to include or not include the discontinuous points. The transferred curve may not be a function.

If we construct a straight line in  $*G$  between the discontinuities in  $\mathbb{R}$ , we can show that the  $*G$  representation possesses the same area when the curve is transferred back. For the heavyside step function, at the discontinuity, the infinitesimal triangle area is  $\frac{1}{2}(2\epsilon) \cdot 1 = \epsilon$ .

**Proposition 4.1.** For any function with finite or countable discontinuities  $(\mathbb{R}, f)$  :  $\exists f_2 : (*G, f_2) \mapsto (\mathbb{R}, f)$

*Proof.* Since  $\mathbb{R}$  is embedded in  $*G$  this is always true.  $\square$

**Proposition 4.2.** For any function  $(\mathbb{R}, f)$  :  $\exists f_2 : (*G, f_2) \mapsto (\mathbb{R}, f)$  and  $f_2 \in \mathbb{C}^w$

*Proof.* Between any discontinuous ordered  $x \in \mathbb{R}$  values, construct a function in  $*G$  that is continuous, to the desired continuity condition  $\mathbb{C}^w$ . For example, at a singularity, construct an s-curve with infinitesimals.  $\square$



**Condition:** Without loss of generality, consider the curves greater than or equal to zero and on a positive domain.  $x \in *G : 0 < x < \infty$

**Proposition 4.3.** *For any piecewise continuous function on a positive interval with a finite number of discontinuities, in  $*G$  a continuous curve can be constructed, that transferred back represents the original curve and has the same area.*

*Proof.* By Proposition 4.2 we can construct a continuous curve.

The area under the discontinuities in  $*G$  is a trapezoid. Let  $y_k$  be the points of discontinuity, let the area at the discontinuity be described as a sum of the square and triangle, then let  $\epsilon \in \Phi$ ,  $2\epsilon$  be the interval width, the trapezoid area between the discontinuities  $a_k = |(y_{k+1} - y_k)|\epsilon + \min(y_{k+1}, y_k)2\epsilon$ .

Area of infinitesimal discontinuities  $\sum_{k=0}^{w-1} a_k = \sum_{k=0}^{w-1} \epsilon(|(y_{k+1} - y_k)| + 2 \min(y_{k+1}, y_k))|_{\epsilon=0} \in \Phi$   
 $= \sum_{k=0}^{w-1} \epsilon|_{\epsilon=0} \in \Phi, (*G, \Phi) \mapsto (\mathbb{R}, 0)$ . □

The transfer principle is not only between different number systems, but can be applied within  $*G$  as it is a ‘realization’ - the truncation of terms.

**Theorem 4.2.** *For a given function we can always construct a continuous function in  $*G$ , which can transfer back to the given function. If  $A$  is the area of the original function, and  $A'$  is the area of the continuous function,  $A' \simeq A$  can be found.*

*Proof.* Let  $a_k(x)$  be a continuous function with an infinitesimal width that describes the discontinuities of a function in  $*G$ . Consequently  $a_k(x)$  has the property that  $\sum_{k=0}^{w-1} a_k(x) \in \Phi$ .

The condition required to be satisfied is  $\sum_{k=0}^{w-1} a_k(x) \in \Phi$ , as the infinitesimal realized becomes an additive identity.

The domain need not be finite, for example an improper Riemann sum. The area  $A$  could be an infinity.

With infinitesimals, using the linear approximation between discontinuities, the area of infinitesimal discontinuities  $\sum_{k=0}^{w-1} a_k = \sum_{k=0}^{w-1} \epsilon(|(y_{k+1} - y_k)| + 2\min(y_{k+1}, y_k))|_{\epsilon=0} \in \Phi$  since there is no smallest infinitesimal, we can always choose  $\epsilon$  to satisfy this condition. □

**Corollary 4.1.** *If the given curve is differentiable on one side of the discontinuity then a curve can be constructed that is differentiable.*

*Proof.* Have the joining function between the discontinuities be differentiable everywhere in its infinitesimal domain and at the joins. □

**Example 4.1.**  $f(x) = [x > q]$  in  $\mathbb{R}$

Let  $\epsilon \in \Phi$  be an arbitrarily small infinitesimal. Let  $f(x) \in *G$ .  $y = \frac{1}{2\epsilon}x + \frac{1}{2}(\frac{q}{\epsilon} + 1)$   
 $f(x) = [x \in (q - \epsilon, q + \epsilon)]y + [x > q + \epsilon]$

## 5 FTC for sums

If we consider the FTC, for a known function  $f(x)$  it was found that the integral over the interval  $[a, b]$  was the difference of two integrals at the end points.

**Definition 5.1.** *Integration at a point*

$$F(a) = \int f(x) dx$$

Hence, a view that  $\int f(x) dx$  is a function. Then we can consider the FTC as a difference of functions, the integrals at a point. FTC Theorem 3.2 from a point's perspective.

$$\int_a^b f(x) dx = \int f(x) dx - \int_a^a f(x) dx$$

From a functional perspective, the integrand arguments are the variable parameters. That we can separate the parameters, the difference, allows the integral to be effectively integrated at the points and be considered separately.

$$F(a, b) = F(b) - F(a)$$

We can similarly consider a sum at a point. For example  $1 + 2 + 3 + \dots + n$  at point  $n$  is equal to  $G(n)$ ,  $G(n) = \frac{n(n+1)}{2}$ . A sum at a point is described by a function.

**Definition 5.2.** *Summation at a point*

$$G(a) = \sum g_j$$

Summation and integration at a point could both be considered as having a fixed starting point, and then the integration/summation at a point could be a unique function.

$$G(a, b) = G(b) - G(a)$$

We introduce notation for sums similar to notation for integrals.

**Definition 5.3.** *Given  $\sum_a^b g_k$  then  $\sum^b g_j$  and  $\sum_a g_j = -\sum^a g_j$*

**Theorem 5.1.** *A sum representation of the fundamental theorem of calculus.*  
 $a, b \in \mathbb{J}$  or  $\mathbb{J}_\infty$ ;

$$\sum_{k=a}^b g_k = \sum^b g_k - \sum^a g_k$$

*Proof.* Transfer the sum to  $*G$ , by Theorem 4.2 construct a continuous function,  $\sum_{k=a}^b g_k = \int_a^{b+1} f(x) dx$ . Assume the fundamental theorem of calculus is true in  $*G$ .  $\int_a^{b+1} f(x) dx = \int^{b+1} f(x) dx - \int^a f(x) dx$ . Since both the sum and integral at a point are defined generally as functions,  $\sum^b g_j = \int^{b+1} f(x) dx$ ,  $\sum^a g_j = \int^a f(x) dx$ .  $\square$

## References

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