Rearrangements of convergence sums at infinity

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Abstract

Convergence sums theory is concerned with monotonic series testing. On face value, this may seem a limitation but, by applying rearrangement theorems at infinity, non-monotonic sequences can be rearranged into monotonic sequences. The resultant monotonic series are convergence sums. The classes of convergence sums are greatly increased by the additional versatility applied to the theory.

1 Introduction

The premise of the paper is that convergence sums [3] order of terms affects convergence. Surprisingly, the most simple rearrangement of bracketing terms of the sum differently (addition being associative) is profoundly useful for sums at infinity, as these sums have an infinity of terms, and an order.

We believe there is still much that is unknown regarding convergence. In fact, historically the discussions and difference of opinions were and perhaps are far apart. Pringsheim says:

> Since in a series of positive terms the order in which the terms come has nothing to do with convergence or divergence of the series... [8, p.9]

F. Cajori addresses this; however the above is a real problem. That such a basic fact was not accepted may explain why the sums order has not previously been incorporated into convergence theory. From our perspective, the existence of infinite integers has opened the possibilities, yet in general, the infinitesimal or infinity still has not been accepted as a number.

Knopp on "Infinite sequences and series" [5] does not refer to infinitesimals, or infinity as a partition. Theorems are generally stated from a finite number to infinity; however he does state theorems from a certain point onwards: something is true, an infinity in disguise. This is of course classical mathematics. The concept of infinity creeps in through subtle arguments, the use of null sequences, which effectively are at infinity. For an example, see Theorems 3.5 and 3.6 which are the same theorem, said in a different way.

Having said this, we find Knopp's exposition and communication exceptional. So we do not necessarily agree with the content, but for the rearrangement theorems in this paper we look to Knopp for both mathematical depth and the presentation. If we do not succeed, this is our and not Knopp's fault.

However, we do find classical mathematics applied at infinity to be extremely useful, if anything, often extending the original concept.

In an interesting way, similarly to topology that stretches or deforms shapes, converting a coffee cup into a doughnut, we can stretch a monotonic sequence to a strictly monotonic sequence for convergence testing, where criteria E3 does not allow for plateaus.

A rearrangement is a reordering. $(1, 2, 3)$ can be rearranged to $(2, 3, 1)$. For an infinite sequence, we can partition the sequence into other infinite sequences.

Example 1.1. *Partition the natural numbers into odd and even sequences.* $(1, 2, 3, 4, 5, \ldots) = (1) + (2) + (3) + (4) + (5) + \ldots$, *select every second element to generate two sequences,* $(1, 3, 5, \ldots)$ *and* $(2, 4, 6, \ldots)$.

By partitioning an infinite sequence into two or more other infinite sequences, we can construct rearrangements by taking (or by copying the whole and deleting) from the partition sequences.

Definition 1.1. *A 'subsequence' is a sequence formed from a given sequence by deleting elements without changing the relative position of the elements.*

Just as we have uses for empty sets, we define an empty sequence.

Definition 1.2. *Let* $()$ *define an empty sequence.*

We find it useful to consider partition sequences which are subsequences. (a_n) partitioned into subsequences (b_k) and (c_j). While these are only a subset of arrangements, they can be used in theory and calculation.

Proposition 1.1. *If* $a = (a_1, a_2, a_3, \ldots)$ *is partitioned into* $b = (b_1, b_2, b_3, \ldots)$ *and* $c = (c_1, c_2, c_3, \ldots)$, *where* b *and* c *are subsequences of* a. *Let an element* a_k *in* a *be in either* b *or* c. *We can form a rearrangement of* a, *by having positional counters in* b *and* c, *and sampling to a new sequence.*

Proof. Let d be an empty sequence. Start the counters at the first element, increment by one after each sample to sequence d by appending to d. Arbitrarily sample from a and b depending on the rearrangement choice. $\square$

What is interesting about infinity, is that you may iterate over the different partitions unevenly. For example, in an unequal ratio. In this case, we say that the partitions are being sampled at different rates.

Example 1.2. *Given infinite sequences* $(1, 3, 5, 7, \ldots)$ *and* $(2, 4, 6, 8, \ldots)$, *create a rearrangement that for every odd number, sample two even numbers. A ratio of* $1 : 2$.

$(1)+(2,4)+(3)+(6,8)+(5)+(10,12)+\ldots = (1,2,4,3,6,8,5,10,12,\ldots)$ *This is a rearrangement of* $(1,2,3,4,\ldots)$.

Another example, although the partition of the original sequence is partitioned in two, an infinitely small number of terms are sampled from one partition compared with the other partition.

Example 1.3. $(2,3,4,5,\ldots)$ *rearranged in a* $1:2^n$ *ratio between the odd and even numbers,* $((2),(3),(4,6),(5),(8,10,12,14),(7),(16,18,20,22,24,26,28,30),(9),\ldots)$
$= (2,3,4,6,5,8,10,12,14,7,16,18,20,22,24,26,28,30,9,\ldots)$.

For finite sums, a sum rearrangement does not change the sum. However, for an infinite sum the situation can become very different. The given order of terms affects convergence.

Different orderings/rearrangements on the same partition of infinite terms can radically change the sum's value and convergence or divergence result.

We develop rearrangement theorems at infinity. We also construct theorems at infinity then transfer these back to known theorems via the transfer principle [1]. Hence, we establish the usefulness of infinity at a point.

With the rearrangement of sums at infinity, it was found that a conditionally convergent sum ($\sum_{\mathbb{N}_\infty} a_j = 0$ converges but $\sum_{\mathbb{N}_\infty} |a_j| = \infty$ diverges), can be rearranged into a divergent sum [5, Theorem 5 p.80]. We will encounter examples of this with convergence sums [3], when considering sum rearrangements independently.

That is, consider a partition of a sum at infinity. A rearrangement of the sum at infinity could unevenly sample one partition compared with the other. Since the index is still iterating over infinity, all elements are still summed.

Partition $(a_n)|_{n=\infty}$ into $(b_n)|_{n=\infty}$ and $(c_n)|_{n=\infty}$. $\sum a_n|_{n=\infty} = \sum b_n|_{n=\infty} + \sum c_n|_{n=\infty}$

Example 1.4. *The elementary proof of the harmonic series divergence. Choose a rearrangement with a 'variable period' of powers of two.* $\sum_{k=1}^{\infty} \frac{1}{k}$, *group in powers of two,* $\sum_{k=2}^{\infty} \frac{1}{k}$ $= \frac{1}{2} + (\frac{1}{3}+\frac{1}{4}) + (\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}) + \ldots \geq \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ *diverges. As a sequence rearrangement,* $(\frac{1}{2}, \frac{1}{4}+\frac{1}{6}, \frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\frac{1}{14}, \ldots) \geq (\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots)$

$b_n = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k}$, $b_k \geq \frac{1}{2}$, $0 \leq \sum \frac{1}{2}|_{n=\infty} \leq \sum b_n|_{n=\infty}$, $\sum b_n|_{n=\infty} = \infty$ *diverges,* $\sum \frac{1}{n}|_{n=\infty} = \infty$.

Example 1.5. $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots$ *is conditionally convergent.* $\sum (-1)^n \frac{1}{n}|_{n=\infty} = 0$. *At infinity, partition* $((-1)^n\frac{1}{n})|_{n=\infty}$ *into* $(\frac{1}{2n})|_{n=\infty}$ *and* $(-\frac{1}{2n+1})|_{n=\infty}$. *As this is a conditionally convergent series, we can find a rearrangement of the series which diverges.*

By considering the even numbers of the sum, we can construct a divergent harmonic series.

$\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\ldots=\frac{1}{2}(\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+\ldots)=\frac{1}{2}(b_0+b_1+b_2+\ldots)$ *As in the previous example,* $b_k \geq \frac{1}{2}$.

Choose an arrangement: for every odd term summed, sum 2^n *even* a_n *terms; for a ratio of* $1:2^n$.

$\sum a_n|_{n=\infty}=\frac{1}{2}+\sum_{k=0}^{n}(\frac{b_k}{2}-\frac{1}{2k+3})|_{n=\infty}$ *For convergence or divergence, consider the sum at infinity then* $\sum(\frac{b_n}{2}-\frac{1}{2n+3})|_{n=\infty}=\sum\frac{b_n}{2}|_{n=\infty}\geq\sum\frac{1}{4}|_{n=\infty}=\infty$, $\sum a_n|_{n=\infty}=\infty$ *diverges.*

The problem of concern above, with a conditionally convergent sum, is that summing in different unequal rates of the partitions affects the sum's result.

Consideration of different arrangements leads to Riemann's rearrangement theorem (see Theorem 3.4), where the same sum converges to a chosen value, or, for another rearrangement, diverges (see Theorem 3.6).

To get around this, simply do not consider sum rearrangements independently, but as a contiguous sum, hence when summing consider the order of the sum's terms.

Since a sum's convergence or divergence is determined at infinity, we need only consider a contiguous sum at infinity.

Since a sequence is a more generic structure than a sum, and a sum can be constructed from a sequence by applying a plus operation to adjacent sequence terms, we describe the partitioning of a sequence, and an application to sums will follow.

2 Periodic sums

A sequence is a more primitive structure than a set and retains its order. We first need to develop some notation to partition sequences, and sequences at infinity. This involves the generalization of the period on a contiguous sequence.

Once this is done, we can in a sweeping move present the most general rearrangement theorem for convergence sums at infinity, which we call the first rearrangement theorem.

Definition 2.1. *A 'contiguous subsequence' is a subsequence with no deleted elements between its start and end elements.*

Definition 2.2. *A partition of a sequence is contiguous if partitioned into continuous subsequences which when joined form the original sequence.*

Example 2.1. $(1,2,3,4,5,\ldots)\mapsto((1,2),(3,4),(5,6)\ldots)$ *is a contiguous partition.*

A contiguous partition has the property of reversibility. If the subsequences are joined together, the formed sequence is the original sequence.

Definition 2.3. $(a_n) = (b_n)$ *when* (b_n) *is a contiguous partition of* (a_n)

Definition 2.4. *A periodic sequence has fixed length subsequences.*

Definition 2.5. *A contiguous periodic sequence is a periodic sequence of a contiguous sequence.*

Example 2.2. $(1, 2, 3, 4, 5, \dots) = ((1, 2), (3, 4), (5, 6) \dots)$ *is a contiguous periodic sequence.*

We can consider the sequence itself as a contiguous periodic sequence with a period of 1. This can then be partitioned into other contiguous periodic sequences.

Given $(a_n)|_{n=\infty}$ then $(a_{2n}, a_{2n+1})|_{n=\infty}$. Since at infinity, we start counting down by finite integers, both sequences can be put into one-one correspondence.

Proposition 2.1. $(a_n)|_{n=\infty} = (\dots, a_{n-2}, a_{n-1}, a_n, a_{n+1}, a_{n+2}, \dots)|_{n=\infty}$

Proof. At infinity can iterate both forwards and backwards as no greatest or least element. $(a_n)|_{n=\infty} = (a_n, a_{n+1}, a_{n+2}, \dots)|_{n=\infty} = (\dots, a_{n-2}, a_{n-2}, a_n, a_{n+1}, a_{n+2}, \dots)|_{n=\infty}$ □

Proposition 2.2. $(a_n)|_{n=\infty}$ *can be partitioned with a fixed period* τ*, and a contiguous partition can be formed.*

$$(a_n)|_{n=\infty} = (a_{\tau n}, a_{\tau n+1}, a_{\tau n+2}, \dots, a_{\tau n+\tau-1})|_{n=\infty}$$

Proof. Expand $(a_{\tau n}, a_{\tau n+1}, a_{\tau n+2} + \dots + a_{\tau n+\tau-1})|_{n=\infty} = \dots + (a_{\tau n}, a_{\tau n+1}, a_{\tau n+2} + \dots + a_{\tau n+\tau-1}) + (a_{\tau(n+1)}, a_{\tau(n+1)+1}, a_{\tau(n+1)+2} + \dots + a_{\tau(n+1)+\tau-1}) + \dots |_{n=\infty} = \dots + (a_{\tau n}, a_{\tau n+1}, a_{\tau n+2} + \dots + a_{\tau n+\tau-1}) + (a_{\tau n+\tau}, a_{\tau n+\tau+1}, a_{\tau n+\tau+2} + \dots + a_{\tau n+2\tau-1}) + \dots |_{n=\infty}$ $= (\dots, a_{\tau n-2}, a_{\tau n-1}, a_{\tau n}, a_{\tau n+1}, a_{\tau n+2}, \dots)|_{n=\infty}$. Apply Proposition 2.1. □

The concept of the period is extended to include arbitrary contiguous sequences. The period is described by a function $\tau(n)$ on the sequence index.

Definition 2.6. *Let* $\tau(n)$ *describe a 'variable periodic sequence'.*
$\tau(n) \geq 1$
$\tau(n+1) - \tau(n) \geq 1$
$\tau(n)$ *contiguously partitions the sequence* (a_n), $(a_{\tau(n)}, a_{\tau(n)+1}, \dots, a_{\tau(n+1)-1})$

Proposition 2.3. $(a_n)|_{n=\infty} = (a_{\tau(n)}, a_{\tau(n)+1}, \dots, a_{\tau(n+1)-1})|_{n=\infty}$

Proof. $(a_n)|_{n=\infty} = ((a_{\tau(n)}, a_{\tau(n)+1}, \dots, a_{\tau(n+1)-1}), (a_{\tau(n+1)}, a_{\tau(n+1)+1}, \dots, a_{\tau(n+2)-1}), \dots)|_{n=\infty}$ $= (a_{\tau(n)}, a_{\tau(n)+1}, \dots)|_{n=\infty} = (\dots, a_{\tau(n)-2}, a_{\tau(n)-1}, a_{\tau(n)}, a_{\tau(n)+1}, \dots)|_{n=\infty}$ □

We now have two ways to classify the partitioning, periodic with fixed τ or a variable period with $\tau(n)$, both of which describe a contiguous partition.

Construct sums with the same definitions as their associated sequences. A series is by definition sequential, applying a sum operator to a sequence.

Definition 2.7. *A 'contiguous series' from a series is defined by applying addition to a contiguous subsequence.*

Definition 2.8. *A 'periodic sum' is obtained by applying addition to a periodic sequence.*

Definition 2.9. *A contiguous periodic sum is obtained by applying addition to a contiguous periodic sequence.*

Definition 2.10. *A variable periodic sum is obtained by applying addition to a variable periodic sequence.*

Definition 2.11. *A contiguous variable periodic sum is obtained by applying addition to a contiguous variable periodic sequence.*

A periodic sum and variable periodic sum are rearrangements of sums. The prepending of 'contiguous' can be omitted, for convergence sums will require monotonicity.

Consider the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$. We notice that the series oscillates, continually rising and falling with the positive and negative terms added respectively.

For convergence sums, the criteria requires a monotonic series, which clearly the above is not. However by considering the order of terms, taking two terms at a time, the above series is monotonic. Put another way, by considering the order in the rearrangement of the series, if we can transform the series to a monotonic series, this can be used to determine convergence/divergence.

Example 2.3. *Does* $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$ *converge or diverge? Consider a sum rearrangement* $(1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+\ldots$ *and the successive terms, then* $(\frac{1}{1}-\frac{1}{2}, \frac{1}{3}-\frac{1}{4}, \frac{1}{5}-\frac{1}{6}, \ldots)$, $(\frac{1}{2}, \frac{1}{3\cdot 4}, \frac{1}{5\cdot 6}, \ldots)$, *with a fixed period* $\tau = 2$ *we see the sequence is strictly monotonically decreasing, and can be tested by our convergence criteria.*

Test the sum at infinity, $\sum(-1)^{n+1}\frac{1}{n}|_{n=\infty} = \sum(\frac{1}{2n}-\frac{1}{2n+1})|_{n=\infty} = \sum \frac{1}{2n(2n+1)}|_{n=\infty} = \frac{1}{4}\sum \frac{1}{n^2}|_{n=\infty}$ $= 0$ *converges, and by a contiguous rearrangement Theorem 2.1,* $\sum(-1)^n\frac{1}{n}|_{n=\infty} = 0$ *converges too.*

Example 2.4. *A period of three convergence example. Does* $\frac{1}{1} - \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots$ *converge or diverge?*

Consider the sum at infinity, the denominators have a sequence $(4k+1, 4k+3, 2k+2)$. $\sum(\frac{1}{4k+1} + \frac{1}{4k+3} - \frac{1}{2k+2})|_{k=\infty} = \sum \frac{8k+5}{32k^3+64k^2+38k+6}|_{k=\infty} = \sum \frac{1}{k^2}|_{k=\infty} = 0$ *converges.*

If we consider any index sum, brackets can be placed about any contiguous series of terms. Let $s = a_1 + a_2 + a_3 + \ldots$, $s = (a_1 + a_2) + (a_3 + a_4 + a_5) + (a_6) + (a_7 + \ldots + a_{11}) + \ldots$.

Definition 2.12. *A contiguous sum rearrangement constructs another sum by bracketing sequential terms.*

Definition 2.13. *A contiguous series is a contiguous sum rearrangement.*

Proposition 2.4. *If s' is a contiguous sum rearrangement of s then $s = s'$.*

Proof. $s = a_1 + a_2 + \ldots$, $s' = b_1 + b_2 + \ldots$. Considering s', if we replace b_k with the original contiguous a_k series for each b_k, the original sum s is restored. □

This idea extends to sums at infinity, where-by preserving the order and bracketing the terms, we can determine a sum's convergence from the rearranged sum. This is necessary, as convergence sums require monotonic series as input.

Theorem 2.1. *A contiguous rearrangement theorem.*
If $(b_n)|_{n=\infty}$ is a contiguous arrangement of $(a_n)|_{n=\infty}$ and $\sum a_n|_{n=\infty}$ is a monotonic convergence sum then $\sum a_n = \sum b_n|_{n=\infty}$.

Proof. Replace b_n by the contiguous a_n terms restores $\sum a_n|_{n=\infty}$. □

The advantage of the theorem is that we need not consider all the different rearrangements of the series at infinity, one will suffice. Once a contiguous rearrangement is found which can be tested for convergence or divergence, all the other contiguous rearrangements have the same value. For a sum at infinity, by convergence criterion E3, this value is an infinireal, and the sum either converges or diverges.

A rearrangement of a non-monotonic series to a monotonic series greatly increases the class of functions that can be considered.

Example 2.5. *In [2] we showed, by comparison of sequential terms, $(\frac{(-1)^n}{n^{\frac{1}{2}}-(-1)^n})|_{n=\infty}$ to be a non-monotonic sequence, and hence the Alternating Convergence Theorem(ACT) cannot be used to determine if $\sum \frac{(-1)^n}{n^{\frac{1}{2}}-(-1)^n}|_{n=\infty}$ converges or diverges. However, the fixed period contiguous rearrangement theorem can be used for just such an event.*

$$\sum \frac{(-1)^n}{n^{\frac{1}{2}}-(-1)^n}|_{n=\infty} = \sum \left(\frac{(-1)^{2n}}{(2n)^{\frac{1}{2}}-(-1)^{2n}} + \frac{(-1)^{2n+1}}{(2n+1)^{\frac{1}{2}}-(-1)^{2n+1}} \right) |_{n=\infty} = \sum \left(\frac{1}{(2n)^{\frac{1}{2}}-1} - \frac{1}{(2n+1)^{\frac{1}{2}}+1} \right) |_{n=\infty}$$
$$= \sum \left(\frac{(2n+1)^{\frac{1}{2}}+1-((2n)^{\frac{1}{2}}-1)}{((2n)^{\frac{1}{2}}-1)((2n+1)^{\frac{1}{2}}+1)} \right) |_{n=\infty} = \sum \left(\frac{((2n+1)^{\frac{1}{2}}-(2n)^{\frac{1}{2}})+2}{((2n)^{\frac{1}{2}}-1)((2n+1)^{\frac{1}{2}}+1)} \right) |_{n=\infty} = \sum \left(\frac{2}{((2n)^{\frac{1}{2}}-1)((2n+1)^{\frac{1}{2}}+1)} \right) |_{n=\infty}$$
$$= \sum \frac{2}{2n}|_{n=\infty} = \infty \textit{ diverges.}$$

Using the binomial theorem to determine the asymptotic result, $(2n+1)^{\frac{1}{2}} - (2n)^{\frac{1}{2}}|_{n=\infty} = (2n)^{\frac{1}{2}}(1+\frac{1}{2n})^{\frac{1}{2}} - (2n)^{\frac{1}{2}}|_{n=\infty} = (2n)^{\frac{1}{2}}(1+\frac{1}{2}\frac{1}{2n}+\ldots-1)|_{n=\infty} = (2n)^{\frac{1}{2}}\frac{1}{4n}|_{n=\infty} = \frac{1}{2^{\frac{3}{2}}n^{\frac{1}{2}}}|_{n=\infty} = 0.$

The following example is not essential, but an example of what the theory can do.

Example 2.6. *Partition* $(2,3,4,\ldots) = ((2),(3,4,\ldots,7),(8,9,\ldots 20),\ldots) = (b_n)|_{n=1,2,\ldots}$ *where* $b_n = (\lfloor e^{n-1}\rfloor + 1, \ldots, \lfloor e^n \rfloor)$. *We observe if* $y \in b_n$, $\lfloor \ln y \rfloor = n-1$. *To prove this consider the following.*

$k \in \mathbb{J}$*; since* e *is transcendental,* $e^k \notin \mathbb{J}$. *Consider intervals* $e^1, e^2, \ldots, e^n$*, then the least integer before* e^n *is* $\lfloor e^n \rfloor$. *The next integer after* e^n *is* $\lfloor e^n \rfloor + 1$. *Then we can form the sequence* $((\lfloor e \rfloor), e, (\lfloor e \rfloor + 1, \ldots \lfloor e^2 \rfloor), e^2, (\lfloor e^2 \rfloor + 1, \ldots \lfloor e^3 \rfloor), e^3, (\lfloor e^3 \rfloor + 1, \ldots \lfloor e^4 \rfloor), e^4, \ldots)$ *By considering* $e^{n-1} < \lfloor e^{n-1} \rfloor + 1, \ldots, \lfloor e^n \rfloor < e^n$*, apply* $\lfloor \ln(x) \rfloor$ *to the previous sequence,* $n-1 \leq \lfloor \ln(\lfloor e^{n-1} \rfloor + 1) \rfloor, \ldots, \lfloor \ln(\lfloor e^n \rfloor) \rfloor < n$.

[6, 3.4.9] By considering a variable period between powers of e*, the following inner sum is simplified.* $\sum \frac{(-1)^{\lfloor \ln n \rfloor}}{n}|_{n=\infty} = \sum (-1)^n \sum_{k=\lfloor e^{n-1}+1 \rfloor}^{e^n} \frac{1}{k}|_{n=\infty}$*, focussing on the inner sum,* $\sum_{k=\lfloor e^{n-1}+1 \rfloor}^{e^n} \frac{1}{k}|_{n=\infty} = \int_{e^{n-1}+1}^{e^n} \frac{1}{k}\,dk|_{n=\infty} = \ln k|_{e^{n-1}+1}^{e^n}|_{n=\infty} = n - (n-1)|_{n=\infty} = 1$. *Then* $\sum \frac{(-1)^{\lfloor \ln n \rfloor}}{n}|_{n=\infty} = \sum (-1)^n 1 = \infty$ *diverges.*

3 Tests for convergence sums

The following discussion concerns our second rearrangement theorem, converting monotonic functions to strictly monotonic functions.

Definition 3.1. *A 'subfunction' is a function formed from a given function by deleting intervals or points without changing the relative position of the intervals or points.*

Definition 3.2. *Let* $\mathrm{unique}(a)$ *define a function where input* a *is monotonic, returns a function with only unique values. If* $b = \mathrm{unique}(a)$ *then the returning function* b *is strictly monotonic.* a *can be a sequence or function.*

Example 3.1. $a = (1,3,3,5,7,7,9)$, $b = \mathrm{unique}(a)$ *then* $b = (1,3,5,7,9)$. a *is monotonically increasing,* b *is strictly monotonic increasing.*

If $b(n) = \mathrm{unique}(a(n))$ then plateaus in $a(n)$ would be removed in $b(n)$. For continuous $a(n)$ a function has the interval with equality collapsed to a single point. For sequence a_n, multiple values of a_n would be collapsed to a single unique a_n.

The following rearrangement theorem of a sum at infinity is given. It is important because it allows the reduction of a monotonic sequence to a strictly monotonic sequence for convergence testing.

This also makes theory easier, for example we can construct a strictly monotonic power series through a strictly monotonic sequence, but not through a monotonic sequence.

Theorem 3.1. *The second rearrangement theorem:*

If a is a monotonic function or sequence, $b = \text{unique}(a)$, then for series $\sum a_n = \sum b_n|_{n=\infty}$ or for integrals $\int a(n)\,dn = \int b(n)\,dn|_{n=\infty}$ when $\mathbb{R}_\infty \mapsto \{0, \infty\}$.

Proof. By E3.5 and E3.6 [3, Criterion E3], if a series or integral plateaus, this is a localized event.

E3.5 For divergence, $\int a(n)\,dn|_{n=\infty}$ or $\sum a_n|_{n=\infty}$ can be made arbitrarily large. [3, E3.5]

E3.6 For convergence, $\int a(n)\,dn|_{n=\infty}$ or $\sum a_n|_{n=\infty}$ can be made arbitrarily small. [3, E3.6]

Let $b = \text{unique}(a)$. For sums, $\sum a_n = \sum b_n + \sum c_n|_{n=\infty}$ where c_n are the deleted terms from a_n. $\sum c_n|_{n=\infty} \geq 0$.

$\sum b_n \geq \sum c_n|_{n=\infty}$ as placing the series on one-one correspondence, $b_n \geq c_n|_{n=\infty}$. Consequently if $\sum a_n|_{n=\infty} = \infty$ diverges then $\sum b_n|_{n=\infty} = \infty$ diverges. If $\sum a_n|_{n=\infty} = 0$ converges then $\sum b_n|_{n=\infty}$ converges.

The same argument is made for continuous $a(n)$ and integrals. If $b(n) = \text{unique}(a(n))$ then $b(n)$ is a subfunction of $a(n)$. $\int a(n)\,dn|_{n=\infty} = \int b(n)\,dn + \int c(n)\,dn|_{n=\infty}$ where $c(n)$ is a subfunction of $a(n)$ and $\int b(n)\,dn \geq \int c(n)\,dn|_{n=\infty}$, again a one-one correspondence argument. $\int c(n)\,dn \geq 0$. If $\int a(n)\,dn|_{n=\infty} = \infty$ diverges then $\int b(n)\,dn|_{n=\infty} = \infty$ diverges. If $\int a(n)\,dn|_{n=\infty} = 0$ converges then $\int b(n)\,dn|_{n=\infty} = 0$ converges. □

With convergence sums, we have, in the same way as [3], looked at existing tests and theorems, and asked how can we do this with partitioning at infinity, and using a our non-standard analysis.

For while [5, Knopp] is one of the best expositors, he does not use the infinite, but ϵ.

We raise this more as a point of difference than fact. A personal or subjective taste, than of necessity. However, partitioning at infinity, and a determination to reason there, have allowed us personally to find an alternative and easier way to reason. It may take another equally good expositor like Knopp to communicate this.

We argue that a better way of reasoning at infinity is possible, but that it may take time and some other technical developments. We have taken some of the known sum theorems described by Knopp, and applied our ideas of the infinite for alternative theorems at infinity. See Theorems 3.3 and 3.5.

Theorem 3.2. *The Absolute convergence test [3, 11.6]. If* $\sum |a_n||_{n=\infty} = 0$ *then* $\sum a_n|_{n=\infty} = 0$ *was found to be a sum rearrangement theorem.*

Theorem 3.3. *"If* $\sum a_\nu$ *is an absolutely convergent series, and if* $\sum a'_\nu$ *is an arbitrary rearrangement of it, then this series is also convergent, and both series have the same value." [5, p.79 Theorem 4]*

Proof. $n_0 = \min(+\Phi^{-1})$; $n_1 = \max(+\Phi^{-1})$; $\sum a_\nu = \sum a_\nu|_{\nu<\infty} + \sum_{n_0}^{n_1} a_\nu$. $\sum a'_\nu = \sum a'_\nu|_{\nu<\infty} + \sum_{n_0}^{n_1} a'_\nu$. By Theorem 3.2, $\sum_{n_0}^{n_1} a_\nu = 0$ then $\sum_{n_0}^{n_1} a'_\nu = 0$. Since a rearrangement of a finite series has the same value, $\sum a_\nu|_{\nu<\infty} = \sum a'_\nu|_{\nu<\infty}$. Let $\sum a_\nu|_{\nu<\infty} = s$. Then $\sum a_\nu = s+0 = s$. $\sum a'_\nu = s + 0 = s$. □

We emphasize rearrangements of infinite series relate to sampling a series at different rates, and their interest is also when the sampling rate does not matter. Hence the importance of the absolute convergence theorem.

Lemma 3.1. *If we partition a conditionally convergent sum,* $\sum a_n|_{n=\infty} = 0$, $\sum |a_n||_{n=\infty} = \infty$*, into positive and negative sums, then one of these sums will diverge.*

Proof. Let the two sums, of positive and negative terms, have sequences $(c_n)|_{n=\infty}$ and $(d_n)|_{n=\infty}$: $|\sum c_n| \leq |\sum d_n||_{n=\infty}$.

$\sum |c_n| \leq \sum |d_n||_{n=\infty}$, $\sum |c_n| + |\sum d_n| \leq \sum |d_n| + |\sum d_n||_{n=\infty}$, $\sum |c_n| + \sum |d_n| \leq 2\sum |d_n||_{n=\infty}$, $\sum |a_n| \leq 2\sum |d_n||_{n=\infty}$, $\infty \leq 2\sum |d_n||_{n=\infty}$, $|\sum d_n||_{n=\infty} = \infty$ diverges. □

Proposition 3.1. *If we partition a conditionally convergent sum,* $\sum a_n|_{n=\infty} = 0$, $\sum |a_n||_{n=\infty} = \infty$*, into positive and negative sums, then both of these sums will diverge.*

Proof. By Lemma 3.1, let $\sum d_n|_{n=\infty} = \pm\infty$ be the divergent sum, when $\sum a_n|_{n=\infty} = \sum c_n|_{n=\infty} + \sum d_n|_{n=\infty} = 0$ converges. Case $\sum d_n|_{n=\infty} = +\infty$, $+\infty + \sum c_n|_{n=\infty} = 0$. $\sum c_n|_{n=\infty} = -\infty$ then $\sum c_n|_{n=\infty}$ diverges. Similarly if $\sum d_n|_{n=\infty} = -\infty$, $\sum c_n|_{n=\infty} = +\infty$ diverges. □

Theorem 3.4. *Riemanns Rearrangement Theorem If* $\sum_{k=1}^{\infty} a_k$ *is conditionally convergent, and* α *a given real number. Then there exists a rearrangement of the terms in* $\sum_{k=1}^{n} a_n$ *whose terms sum to* α.

Proof. The proof is almost the same as [7], but with the $*G$ number system. Construct an algorithm that leads to a sum infinitesimally close to α, which, when transferred back to $\mathbb{R}$ is equal to α. Partition a_n respectively into positive and negative sequences, monotonically decreasing in magnitude, (c_n) and (d_n). Have integer variables i for the current index into (c_n) and j for the current index into (d_n). $s_0 = 0$; If $s_n < \alpha$ then $s_{n+1} = c_i + s_n$ and $i = i+1$

increments i; else $s_{n+1} = d_j + s_n$ and $j = j + 1$ increments j. $s_n|_{n=\infty} \simeq \alpha$ generates a sum infinitesimally close to α. ☐

Theorem 3.5. *"If $\sum a_\nu$ is a convergent, but not an absolutely convergent, series, then there are arrangements, $\sum a'_\nu$, of it that diverge." [5, p.80 Theorem 5]*

Theorem 3.6. *If $\sum a_n|_{n=\infty}$ is conditionally convergent, $\sum a_n|_{n=\infty} = 0$, $\sum |a_n||_{n=\infty} = \infty$, then there exists rearrangements of it such that $\sum a'_n|_{n=\infty} = \infty$ diverges.*

Proof. By construction. Using Proposition 3.1 we can partition (a_n) into (c_n) and (d_n) where $\sum d_n|_{n=\infty} = \pm\infty$ diverges and the sign is dependent on whether the negative or positive group diverges.

Choose $b_n = c_n + \sum_{k=k_0}^{k_1} d_k$, where $\sum_{k=k_0}^{k_1} d_k$ is a contiguous sum of d_n terms and $|b_n|$ has a lower real bound $\alpha \neq 0$. Since $\sum d_n = \pm\infty$ this is always possible. As we can arbitrarily increase k_1, many of these b_n rearrangements are possible. Have b_n either all positive or all negative.

Case $b_n > 0$. $b_n \geq \alpha$, $\sum b_n \geq \sum \alpha|_{n=\infty}$, $\sum b_n|_{n=\infty} \geq \infty$, $\sum b_n|_{n=\infty} = \infty$ diverges. Case $b_n < 0$. $b_n \leq \alpha$, $\sum b_n \leq \sum \alpha|_{n=\infty}$, $\sum b_n|_{n=\infty} \leq -\infty$, $\sum b_n|_{n=\infty} = -\infty$ diverges. Many rearrangements which diverge were found. ☐

We now explore the chain rule, which varies the rate of counting, dependent on the integration variable.

Example 3.2. *$\int \frac{1}{1+2n} dn|_{n=\infty}$, let $u = 1 + 2n$, $\frac{du}{dn} = 2$ then $\int \frac{1}{u} dn = \int \frac{1}{u}\frac{dn}{du} du = \frac{1}{2}\int \frac{1}{u} du$ $= \frac{1}{2}\ln u|_{n=\infty}$*

Consider the same integral, but integrate with another variable. We find the result of a change in unequal variables changed the integral result.

$\int \frac{1}{1+2n} dn = 2\int \frac{1}{2+4n} dn$, let $v = 2 + 4n$, $\frac{dv}{dn} = 4$, $2\int \frac{1}{v} dn = 2\int \frac{1}{v}\frac{dn}{dv} dv = \frac{1}{2}\ln v|_{n=\infty}$

However $\frac{1}{2}\ln u \neq \frac{1}{2}\ln v|_{n=\infty}$ as $1 + 2n \neq 2 + 4n|_{n=\infty}$.

Conjecture 3.1. *A periodic sum τ can have its sums interchanged, if the change of variable stays the same.*

$$\sum a_n = \sum \sum_{k=1}^{\tau} b_{k,n}|_{n=\infty} = \sum_{k=1}^{\tau} \sum b_{k,n} = \sum_{k=1}^{\tau} \int b_k(n)\, dn|_{n=\infty}$$

The sums instead of being summed contiguously, are rearranged into period columns. If these columns can be treated independently, this is advantageous as we can use integrals to evaluate them.

However, because of our usage of the chain rule, we find here an example where we cannot apply the chain rule to one part of the partition differently to the others. Where infinity is concerned, we need to be more cautious.

Example 3.3. *Does* $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \ldots$ *converge or diverge? [6, p.105 3.7.2] solves the sum; however we use this example to demonstrate theory.*

$\sum(\frac{1}{1+2n} - \frac{1}{2+4n} - \frac{1}{4+4n})|_{n=\infty} = \int \frac{1}{1+2n} dn - \int \frac{1}{2+4n} dn - \int \frac{1}{4+4n} dn|_{n=\infty} = \frac{1}{2}\ln(1+2n) - \frac{1}{4}\ln(2+4n) - \frac{1}{4}\ln(4+4n)|_{n=\infty} = \frac{1}{4}(2\ln(1+2n) - \ln(2+4n) - \ln(4+4n))|_{n=\infty} = \frac{1}{4}\ln\frac{(1+2n)^2}{8(1+2n)(n+1)}|_{n=\infty} = \frac{1}{4}\ln\frac{1+2n}{8(n+1)}|_{n=\infty} = \frac{1}{4}\ln\frac{1}{4} = -0.34657359027997\ldots \neq 0$ *or* ∞ *which contradicts Criterion E3.*

We noted in Example 3.2 a different change in variable produced a different result. Keeping all the sums with variable $4n$*, perform the same integration as before.*

Integrating with the same rate for all sums. $\sum(\frac{1}{1+2n} - \frac{1}{2+4n} - \frac{1}{4+4n})|_{n=\infty} = \sum(\frac{2}{2+4n} - \frac{1}{2+4n} - \frac{1}{4+4n})|_{n=\infty} = \sum(\frac{1}{2+4n} - \frac{1}{4+4n})|_{n=\infty} = \int \frac{1}{2+4n} dn - \int \frac{1}{4+4n} dn|_{n=\infty} = \frac{1}{4}(\ln(2+4n) - \ln(4+4n))|_{n=\infty} = \frac{1}{4}\ln\frac{2+4n}{4+4n}|_{n=\infty} = 0$ *converges. The sum agrees with the theory, and the correct result is found.*

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