Proof of Infinite Number of Fibonacci Primes

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Abstract

This paper presents a complete and exhaustive proof of that an infinite number of Fibonacci Primes exist . The approach to this proof uses same logic that Euclid used to prove there are an infinite number of prime numbers. Then we prove that if $p > 1$ and $d > 0$ are integers, that p and $p + d$ are both primes if and only if for integer n (see reference 1 and 2):

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d} d!}{p + d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

We use this proof for $p = F_{y-1}$ and $d = F_{y-2}$ to prove the infinitude of Fibonacci prime numbers.

The author would like to give many thanks to the authors of *1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004, Exercise Number 161* (see Reference 1). The proof provided in Exercise 6 is the key to making this paper on the finitude or Fibonacci Primes possible.

Introduction

The Fibonacci prime conjecture, was made by Alphonse de Fibonacci in 1849. Alphonse de Fibonacci (1826 – 1863) was a French mathematician whose father, Jules de Fibonacci (1780-1847) was prime minister of Charles X until the Bourbon dynasty was overthrown in1830. Fibonacci attended the École Polytechnique (commonly known as Polytechnique) a French public institution of higher education and research, located in Palaiseau near Paris. In 1849, the year Alphonse de Fibonacci was admitted to Polytechnique, he made what's known as Fibonacci's conjecture:

For every positive integer k, there are infinitely many prime gaps of size 2k.

Alphonse de Fibonacci made other significant contributions to number theory, including the de Fibonacci's formula, which gives the prime factorization of n!, the factorial of n, where n ≥ 1 is a positive integer.

Proof of Infinite Number of Fibonacci Primes

In number theory, a Fibonacci prime is a Fibonacci number that is also prime. It is not known whether there are infinitely many Fibonacci primes; this poof shall prove their infinitude. Fibonacci numbers or Fibonacci series or Fibonacci sequence are the numbers in the following integer sequence:

The sequence F_n of Fibonacci numbers is defined by the following reoccurring relation:

$$F_n = F_{n-1} + F_{n-2},$$

with seed values,

$$F_0 = 0,\ F_1 = 1.$$

By definition, the first two numbers in the Fibonacci sequence are 0 and 1, and each subsequent number is the sum of the previous two. The first several Fibonacci numbers are:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ….

Of these Fibonacci numbers the following are Fibonacci primes:

2, 3, 5, 13, and 89

We shall use Euclid's logic that he used to prove there are an infinite number of prime numbers to prove there are an infinite number Fibonacci primes.

First we shall assume there are only a finite number of n Fibonacci's primes for all Fibonacci numbers, specifically;

1) Say, 2, 3, 5, 13, $F_{11,}$ . . . . , $F_x = F_{x-1} + F_{x-2}$, $F_n = F_{n-1} + F_{n-2}$

Where, F_x is the next to the last Fibonacci prime, x is used instead of n-1, because the last Fibonacci prime may not be the next Fibonacci number in the Fibonacci sequence.

2) Let N = 2*3*5*13*(F_{11}) . . . . , $(F_{x-1} + F_{x-2})(F_{n-1} + F_{n-2})$ +1

By the fundamental theorem of arithmetic, N is divisible by some prime q. Since N is the product of all existing Fibonacci primes plus 1, then this prime q cannot be among the F_i that make up the n Fibonacci primes since by assumption these are all the Fibonacci primes that exist and N is not evenly divisible by any of the, F_i Fibonacci primes. N is clearly seen not to be divisible by any of the F_i Fibonacci primes. Therefore, q must be another prime number that does not exist in the finite set of Fibonacci prime numbers.

The only thing left to prove that there are an infinite number of Fibonacci primes is to prove that $q = F_y = F_{y-1} + F_{y-2}$ where F_y is Fibonacci prime that is not in the set of finite Fibonacci primes, since q is not in that set.

First we shall show that if $q = F_y = F_{y-1} + F_{y-2} = p + d$, where $F_{y-1} = p$ and $F_{y-2} = d$, then if q is prime it cannot be in the set of finite p_i Fibonacci primes above. Since q is a prime number that does not exist in the set of finite p_i Fibonacci primes, then if there exists a prime number equal to $F_{y-1} + F_{y-2}$ that is also prime, it would be a Fibonacci prime; therefore a prime $F_{y-1} + F_{y-2}$ cannot be in the set of finite n Fibonacci primes otherwise q would be in the set of n finite Fibonacci primes and we have proven that q is not in the set of fine Fibonacci primes, therefore if p + d is prime it cannot be in the finite set of Fibonacci primes since it would be Fibonacci to q.

Now we shall proceed to prove $p + d = F_{y-1} + F_{y-2}$ is prime as follows:

We use the proof, provided in Reference 1, that if $p > 1$ and $d > 0$ are integers, that p and $p + d$ are both primes if and only if for positive integer n:

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d}\mathrm{d}!}{\mathrm{p} + \mathrm{d}}\right) + \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{p}+\mathrm{d}}$$

For our case $p = F_{y-1}$ and $d = F_{y-2}$ are Fibonacci numbers, and, $F_y = F_{y-1} + F_{y-2}$, our objective is to prove that if $p = F_{y-1}$ and $d = F_{y-2}$, then $p + d = F_{y-1} + F_{y-2}$ = prime number = F_y = Fibonacci prime. Since, as discussed earlier F_y is outside our finite set of Fibonacci Prime, therefore, if we prove F_y is prime then we will prove the infinitude of Fibonacci Primes.

$$n = (F_{y-1} - 1)!\left(\frac{1}{F_{y-1}} + \frac{(-1)^{F_{y-2}}(F_{y-2})!}{F_{y-1} + F_{y-2}}\right) + \frac{1}{F_{y-1}} + \frac{1}{F_{y-1} + F_{y-2}}$$

Multiplying by F_{y-1} ,

$$n(F_{y-1}) = (F_{y-1})!\left(\frac{1}{F_{y-1}} + \frac{(-1)^{F_{y-2}}(F_{y-2})!}{F_{y-1} + F_{y-2}}\right) + 1 + \frac{F_{y-1}}{F_{y-1} + F_{y-2}}$$

Multiplying by $(F_{y-1} + F_{y-2})$ and since F_{y-1} , is greater than 2 and is prime, then F_{y-1} is prime and therefore F_{y-2} must be even for F_y to be prime:

$$n(F_{y-1} + F_{y-2})(F_{y-1}) = (F_{y-1} + F_{y-2})(F_{y-1})!\left(\frac{1}{F_{y-1}} + \frac{(F_{y-2})!}{F_{y-1} + F_{y-2}}\right) + F_{y-1} + F_{y-2} + F_{y-1}$$

Reducing,

$$n(F_{y-1} + F_{y-2})(F_{y-1}) = (F_{y-1})!\left(\frac{F_{y-1} + F_{y-2}}{F_{y-1}} + (F_{y-2})!\right) + 2F_{y-1} + F_{y-2}$$

Reducing again,

$$n(F_{y-1} + F_{y-2})(F_{y-1}) = \left((F_{y-1)} - 1)\right)!\left(F_{y-1} + F_{y-2} + F_{y-1}(F_{y-2})!\right) + 2F_{y-1} + F_{y-2}$$

We already know F_{y-1} and F_{y-2} are Fibonacci numbers therefore; F_{y-1} and F_{y-2} are both positive integers. We also know that by the definition of Fibonacci numbers that the sum of F_{y-1} and F_{y-2} is a Fibonacci number, specifically, $F_y = F_{y-1} + F_{y-2}$, therefore F_y is also a positive integer. Since F_{y-1} and F_{y-2} are integers the right hand side of the above equation is an integer. Since the right hand side of the above equation is an integer and F_{y-1} and F_{y-2} are integers on the left hand side of the equation, then there are only 4 possibilities (see 1, 2a, 2b, and 2c below) that can hold for n so the left hand side of the above equation is an integer, they are as follows.

1) n is an integer, or
2) n is a rational fraction that is divisible by F_{y-1} . This implies that n = $\frac{x}{F_{y-1}}$ where, F_{y-1} is prime and x is an integer. This results in the following three possibilities:
    a. Since n = $\frac{x}{F_{y-1}}$, then F_{y-1} = $\frac{x}{n}$, since F_{y-1} is prime, then F_{y-1} is only divisible by F_{y-1} and 1, therefore, the first possibility is for n to be equal to F_{y-1} or 1 in this case, which are both integers, thus n is an integer for this first case.

b. Since n = $\frac{x}{F_{y-1}}$, and x is an integer, then x is not evenly divisible by F_{y-1} unless x = F_{y-1}, or x is a multiple of F_{y-1}, where x = zF_{y-1}, for any integer z. Therefore n is an integer for x = F_{y-1} and x = zF_{y-1}.

c. For all other cases of, integer x, n = $\frac{x}{F_{y-1}}$, n is not an integer.

To prove there is a Fibonacci Prime, outside our set of finite Fibonacci Primes, we only need to prove that there is at least one value of n that is an integer, outside our finite set. There can be an infinite number of values of n that are not integers, but that will not negate the existence of one Fibonacci Prime, outside our finite set of Fibonacci Primes.

First the only way that n cannot be an integer is if every n satisfies paragraph 2.c above, namely, n = $\frac{x}{F_{y-1}}$, where x is an integer, x ≠ F_{y-1}, x ≠ yF_{y-1}, n ≠ F_{y-1}, and n ≠ 1 for any integer z. To prove there exists at least one Fibonacci Prime outside our finite set, we will assume that no integer n exists and therefore no Fibonacci Primes exist outside our finite set. Then we shall prove our assumption to be false.

Proof: Assumption no values of n are integers, specifically, every value of n is n = $\frac{x}{F_{y-1}}$, where x is an integer, x ≠ F_{y-1}, x ≠ yF_{y-1}, n ≠ F_{y-1}, and n ≠ 1, for any integer z. Paragraphs 2.a, and 2.b prove cases where n can be an integer, therefore our assumption is false and **there exist values of n that are integers.**

It suffices to show that there is at least one integer n to prove there exists a Fibonacci Prime outside our set of finite set of Fibonacci Primes.

Since there exists an n = integer, we have proven that there is at least one p and p + 2k that are both prime. Since we proved earlier that if p + 2k is prime then it also is not in the finite set of p_i, p_i + 2k Fibonacci primes, therefore, since we have proven that there is at least one p+2k that is prime, then we have proven that there is a Fibonacci prime outside the our assumed finite set of Fibonacci primes. This is a contradiction from our assumption that the set of Fibonacci primes is finite; therefore, by contradiction the set of Fibonacci primes is infinite. Also this same proof can be repeated infinitely for each finite set of Fibonacci primes, in other words a new Fibonacci prime can added to each set of finite Fibonacci primes. This thoroughly proves that an infinite number of Fibonacci primes exist.

It suffices to show that at least one n exists that is an integer since we have proven that F_{y-1} and $F_{y-1} + F_{y-2} = F_y$ are both primes if and only if for integer n:

$$n = (F_{y-1} - 1)!\left(\frac{1}{F_{y-1}} + \frac{(-1)^{F_{y-2}}(F_{y-2})!}{F_{y-1} + F_{y-2}}\right) + \frac{1}{F_{y-1}} + \frac{1}{F_{y-1} + F_{y-2}}$$

Since we have proven by contradiction that at least on n = integer, we have proven that their exists at least one $F_{y-1} + F_{y-2}$ that must be prime, and since $F_y = F_{y-1} + F_{y-2}$, then F_y is also prime. Since we proved earlier that if $F_{y-1} + F_{y-2}$ is prime then it also is not in the finite set of p_i Fibonacci primes, therefore, since we have proven that $F_{y-1} + F_{y-2} = F_y$ is prime, then we have proven that there is a Fibonacci prime outside our assumed finite set of Fibonacci primes. This is a contradiction from our assumption that the set of Fibonacci primes is finite, therefore, by contradiction the set of Fibonacci primes is infinite. Also this same proof can be repeated infinitely for each finite set of Fibonacci primes, in other words a new Fibonacci prime can added to each set of finite Fibonacci primes. This thoroughly proves that an infinite number of Fibonacci primes exist.

References:

1) *TYCM, Vol. 19, 1988, p. 191*
2) *1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004*