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A note on the solutions of a kind of matrix polynomial equation

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Abstract. In this note, the nonlinear matrix polynomial equation $X^2 = A$ over complex field C are investigated. All solutions of this matrix polynomial equation of order 2 are presented.

Keywords: diagonal matrix; similar matrix; eigenvalue; Jordan canonical form.

1 Introduction

The matrix polynomial equations with degree n,

$$A_n X^n + A_{n-1} X^{n-1} + \cdots + A_1 X + A_0 = 0,$$

Where A_i is a known complex or real matrix of order n for $i = 0, 1, \ldots, n$, have been widely applied in many fields of scientific computation and engineer science. Although there are many literatures that address on the symmetric matrix polynomial equations; see, for example, [1-4], there is little work on the nonsymmetric case. As we know, it is very difficult and challenging to find the general solution of the above matrix polynomial equation.

In this short note, as a preliminary study, the matrix polynomial equation $X^2 = A$ over complex field C are investigated. All solutions of the above matrix polynomial matrix equation are presented.

Note that all definitions throughout this note can be referenced in [5], [6] and references therein.

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2 Main results

Our main result is about the solutions of the matrix polynomial equation

$$X^2 = A. \tag{1}$$

Since matrix A is of order 2, it follows from [1, Theorem 3.1.11] that A must be similar to the diagonal matrix $X^2 = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$,

or its Jordan canonical form $X^2 = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$,

where λ_1, λ_2 and λ are eigenvalues of matrix A. It is obvious that the matrix equations (1) and

$$(PXP^{-1})^2 = A \tag{2}$$

have the same solutions, where P is a nonsingular complex matrix. So it suffices to consider the following two matrix polynomial equations'

$$X^2 = \begin{pmatrix} a & \\ & b \end{pmatrix}, \tag{3}$$

and

$$X^2 = \begin{pmatrix} a & 1 \\ & a \end{pmatrix}, \tag{4}$$

respectively, where a and b are complex numbers.

The following two theorems are our main results.

Theorem 1. *There exists at least a solution of the matrix polynomial equation (3). Furthermore, if $a = b = 0$, then all the nonzero solutions of equation (3) have the form*

$$P\begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}P^{-1}$$

and if there exists at least nonzero entry in $\{a, b\}$, then all the solutions of (3) have the form

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

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where $\lambda_1^2 = a$ *and* $\lambda_2^2 = b$.

Theorem 2. *There exists at least a solution of the matrix polynomial equation (4) for* $a = 0$. *However, if* $a \neq 0$, *then all the solutions of equation (4) have the form*

$$\begin{pmatrix} \lambda & \frac{1}{2\lambda} \\ & \lambda \end{pmatrix},$$

where $\lambda^2 = a$.

3 Proofs of main results

Proof of Theorem 1. It is clear that matrix polynomial equation (3) has at least one solution. If $a = b = 0$, then we claim that the nonzero solution X of equation (3) cannot be diagonalizable. In fact, the nonzero solution X has an eigenvalue 0 for its determinant is zero. If follows that X is similar to $\begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$. Because $\begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$ is a solution of equation (3), all solutions of matrix polynomial equation (3) are of the form $P\begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}P^{-1}$, where P is an arbitrary nonsingular complex matrix of order 2.

For the case $a \neq 0$ or $b \neq 0$, we first claim that the solution X of equation (3) is diagonalizable. In fact, if the solution X is not diagonalizable, then X must be similar to matrix $\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$, where $\lambda \neq 0$. Then for some nonsingular matrix P, we have

$$P^{-1}\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}^2 P = \begin{pmatrix} a & \\ & b \end{pmatrix}.$$

If follows that $\begin{pmatrix} \lambda & 2 \\ & \lambda \end{pmatrix}$ is diagonalizable; a contradiction. So X is similar to $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$. For some nonsingular matrix $P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}$, we have

$$\begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^2 \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}^{-1} = \begin{pmatrix} a & \\ & b \end{pmatrix}.$$

Consequently,

$$\begin{pmatrix} \lambda_1^2 p_1 & \lambda_2^2 p_2 \\ \lambda_1^2 p_3 & \lambda_2^2 p_4 \end{pmatrix} = \begin{pmatrix} ap_1 & ap_2 \\ bp_3 & bp_4 \end{pmatrix}.$$

It can be verified readily that all solutions of equation (3) are of the form $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, where $\lambda_1^2 = a$ and $\lambda_2^2 = b$.

Proof of Theorem 2. It suffices to consider the following two cases.

Case 1. $a = 0$.

In this case, we claim that there exist no solution in equation (4). By a way of contradiction, let the solution of equation (4) be X. Then $|X| = 0$ and X has 0 as its an eigenvalue.

Subcase 1.1. *0 is an eigenvalue of* X *with algebraical multiplicity 1.*

In this case, it follows from [5, Theorem 1.3.9] that X is diagonalizable. Without loss of generality, let X be similar to $\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$. So we get

$$P^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}^2 P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

for some nonsingular matrix P. It follows that matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is diagonalizable; a contradiction.

Subcase 1.2. *0 is an eigenvalue of* X *with algebraical multiplicity 2.*

In this case, the solution X must be similar to matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Consequently,

$$X^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Clearly, it is a contradiction.

Case 2. $a \neq 0$.

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In this case, the solution X of equation (4) is not diagonalizable. Without loss of generality, let the Jordan canonical form of X be $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Then for some nonsingular matrix $P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}$, we have

$$\begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^2 \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}^{-1} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

Consequently,

$$\begin{pmatrix} \lambda^2 p_1 & 2\lambda p_1 + \lambda^2 p_2 \\ \lambda^2 p_3 & 2\lambda p_3 + \lambda^2 p_4 \end{pmatrix} = \begin{pmatrix} ap_1 + p_3 & ap_2 + p_4 \\ ap_3 & ap_4 \end{pmatrix}.$$

By computation, we get $p_3 = 0$, $\lambda^2 = a$ and $2\lambda p_1 = p_4$. It follows that

$$X = \begin{pmatrix} p_1 & p_1 \\ 0 & 2\lambda p_1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} p_1 & p_1 \\ 0 & 2\lambda p_1 \end{pmatrix}^{-1} = \begin{pmatrix} \lambda & \frac{1}{2\lambda} \\ 0 & \lambda \end{pmatrix}.$$

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