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S-Transform on a Generalized Quotient Space

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Abstract. This paper investigates the S-transform on the space of tempered Boehmians. Inversion and properties are also discussed.

Keywords: Distribution spaces, Boehmians, Fourier transform, Wavelet transform, S-transform

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1. Introduction

Boehmian, which is a class of generalized functions, have been introduced by Boehme [1]. The construction of Boehmians was given by Mikusiński and Mikusiński ([5], [6]). Boehmian spaces have been studied for various classical integral transforms ([5], [6], [7]). In this paper we establish the S-transform for tempered Boehmians.

The S-transform is variable window of short time Fourier transform or an extension of wavelet transform ([2], [11]). It is based on a scalable localizing Gaussion window and supplies the frequency dependent resolution. The S-transform was proposed by Stockwell and et al. [10]. The properties of S-transform are that it has a frequency dependent resolution of time-frequency domain and entirely refer to local phase information. Also the S-transform has been successfully used to analyses in numerous applications, such as seismic recordings, ground vibrations, hydrology, gravitational waves, and power system analysis. The S-transform of $f \in L^2(\mu)$ is defined as the inner product

$$
\begin{aligned}
Sf(\xi) &:= (\bar{f}, :\exp(\langle ., \xi\rangle):)_{L^2(\mu)} \\
&= \exp(-\frac{1}{2}|\xi|^2)\int_{\mathbb{R}} f(x)\exp(\langle x, \xi\rangle)d\mu(x).
\end{aligned}
$$

[1] The corresponding author

The extension of the S-transform to generalized functions needs to consider standard Gel'fand triple [3]

$$\mathcal{D} \subset \mathbf{H} \subset \mathcal{D}'$$

$\mathbf{H}$ is a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$ and $\mathcal{D}$ is a separable nuclear space densely topologically embedded in $\mathbf{H}$. The dual space $\mathcal{D}'$ is the inductive limit of the corresponding dual space $(\mathbf{H}_{-p})_{p\in\mathbb{N}}$. We denote by $<\cdot,\cdot>$ the dual pairing between $\mathcal{D}$ and $\mathcal{D}'$ given by the extension of the inner product $(\cdot,\cdot)$ on $\mathbf{H}$. Chain of the spaces (which are used) is given by in this order

$$(\mathcal{D})^1 \subset (\mathcal{D})^\eta \subset (\mathcal{D}) \subset L^2(\mu) \subset (\mathcal{D})' \subset (\mathcal{D})^{-\eta} \subset (\mathcal{D})^{-1}.$$

The integral $\int_{\mathcal{D}'} f(x) d_\mu(x)$ of a measurable function f define on $\mathcal{D}'$ is called the expectation of f if f is integrable. i.e., the integral $\int_{\mathcal{D}'} |f(x)| d_\mu(x)$ is finite. The space of the integrable functions is denoted by $L^1(\mu) = L^1(\mathcal{D}', C_{\sigma,\mu}(\mathcal{D}'), \mu)$ and the expectation of the function f w.r.t. μ is denoted by $\mathrm{E}_\mu(f)$. The central space in our set-up is the space of complex valued functions which are square integrable w.r.t. this measure

$$L^2(\mu) = L^2(\mathcal{D}', C_{\sigma,\mu}(\mathcal{D}'), \mu).$$

While constructing an orthogonal system in $L^2(\mu)$, the definition of Wick powers is required. For any $\xi \in \mathcal{D}$, the function

$$:\exp(\langle x,\xi\rangle): := \frac{\exp(\langle x,\xi\rangle)}{\mathrm{E}_\mu(\exp(\langle \cdot,\xi\rangle))}$$

$$= \exp(\langle x,\xi\rangle - \frac{1}{2}\langle \xi,\xi\rangle), \quad x \in \mathcal{D}'.$$

is called Wick exponential.

For $\xi \in \mathcal{D}$, consider the Wick exponential

$$:\exp(\langle x,\xi\rangle) := \sum_{n=0}^{\infty} \frac{1}{n!} \langle :x^{\otimes n}:, \xi^{\otimes n}\rangle, \quad x \in \mathcal{D}'. \tag{1}$$

Calculating its p,q,η-norm, we find

$$\|:\exp(\langle \cdot,\xi\rangle):\|_{p,q,\eta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\eta} 2^{nq} \left|\frac{1}{n!}\xi^{\otimes n}\right|_p^2 = \sum_{n=0}^{\infty} (n!)^{\eta-1} 2^{nq} |\xi|_p^{2n} \tag{2}$$

For $\eta < 1$ the norm in (2) is finite which shows that $:\exp(<\cdot,\xi>):\in (\mathcal{D})^\eta$. Thus, we can define the S-transform of a distribution $\Phi \in (\mathcal{D})^{-\eta}, \eta < 1$, at $\xi \in \mathcal{D}$ as follows

$$S\Phi(\xi) := \left\langle\left\langle \Phi, :\ \exp(\langle\cdot,\xi\rangle):\right\rangle\right\rangle = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \xi^{\otimes n}\rangle \quad . \tag{3}$$

For $\eta = 1$ the norm in (2) is finite if and only if $2^q |\xi|_p^2 < 1$.

Nevertheless, it is still possible to define S-transform in the space $(\mathrm{D})^{-1}$ because every distribution is of finite order. Let $\Phi \in (\mathrm{D})^{-1}$. Then there exists $p, q \in \mathrm{N}$ such that $\Phi \in (\mathbf{H}_{-p})_{-q}^{-1}$. For all $\xi \in \mathrm{D}$, with $2^q |\xi|_p^2 < 1$, we can define the S-transform of Φ as

$$S\Phi(\xi) := \left\langle\left\langle \Phi, :\ \exp(\langle\cdot,\xi\rangle):\right\rangle\right\rangle = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \xi^{\otimes n}\rangle \quad . \tag{4}$$

The one dimensional continuous S-transform of $u(t)$ is defined by [8]:

$$S(\tau, f) = \int_{\mathrm{R}} u(t) w(\tau - t,\ f) e^{-i2\pi ft} dt, \tag{5}$$

where the window w is assumed to satisfy

$$\int_{\mathrm{R}} w(t, f) dt = 1 \quad \text{for all} \quad f \in \mathrm{R} \setminus \{0\}. \tag{6}$$

The most common window w is the Gaussion

$$w(t, f) = \frac{|f|}{k\sqrt{2\pi}} e^{-f^2 t^2 / 2k^2}, \quad k > 0, \tag{7}$$

in which f is the frequency, t the time variable, and k is a scaling factor that controls the number of oscillation in the window. Then Equation (5) can be rewritten as a convolution

$$S(\tau, f) = (u(\cdot) e^{-i2\pi f\cdot} * w(\cdot, f))(\tau). \tag{8}$$

Applying the convolution property for the Fourier transform, we obtain

$$S(\tau, f) = \mathbf{F}^{-1}\{\hat{u}(\cdot + f)\hat{w}(\cdot,\ f)\}(\tau), \tag{9}$$

where $\mathbf{F}^{-1}$ is the inverse Fourier transform, for the Gaussian window suggested by (3),

$$\hat{w}(\alpha, f) = \int_{\mathrm{R}} e^{-i2\pi t\alpha} \frac{|f|}{k\sqrt{2\pi}} e^{-f^2 t^2 / 2k^2} dt$$

$$= \frac{|f|}{k\sqrt{2\pi}} \int_{\mathrm{R}} e^{-(f^2 t^2 / 2k^2 + i2\pi t\alpha)} dt = e^{-2(\pi k\alpha / f)^2} \tag{10}$$

Thus, the S-transform can be written as

$$S(\tau, f) = \int_{\mathrm{R}} \hat{u}(\alpha + f) e^{-2(\pi k\alpha / f)^2} e^{i2\pi\alpha\tau}\, d\alpha. \tag{11}$$

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The S -transform is defined as a continuous wavelet transform with a specific mother wavelet multiplied by the phase factor

$$S(\tau, f) = e^{-j2\pi f\tau} W(\tau, d) \tag{12}$$

where the mother wavelet is defined as

$$w(t, f) = \frac{|f|}{\sqrt{2\pi}} e^{-t^2 f^2/2} e^{-j2\pi ft}, \tag{13}$$

and the factor d in (12) is the inverse of the frequency f. However, the mother wavelet in (13) does not satisfy the property of zero mean, (12) is not absolutely a continuous wavelet transform, it is given by

$$S(\tau, f) = \int_{-\infty}^{\infty} u(t) \frac{|f|}{\sqrt{2\pi}} e^{-\frac{(\tau-t)^2 f^2}{2}} e^{-j2\pi ft}\, dt \tag{14}$$

If the S -transform is a representation of the local spectrum, then we can show the relation between the S -transform and Fourier transform as

$$\int_{-\infty}^{\infty} S(\tau, f) d\tau = \hat{u}(f) \tag{15}$$

where $\hat{u}(f)$ is the Fourier transform of $u(t)$, which is given by

$$u(t) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S(\tau, f) d\tau \right) e^{j2\pi ft}\, df. \tag{16}$$

The relation between the S -transform and Fourier transform can also be written as (see Equation (11))

$$S(\tau, f) = \int_{-\infty}^{\infty} \hat{u}(\alpha + f) e^{\frac{-2\pi^2\alpha^2}{f^2}} e^{j2\pi\alpha\tau} d\alpha, \quad f \neq 0. \tag{17}$$

By taking the advantage of the efficiency of the Fast Fourier transform and the convolution theorem, the discrete analog of (17) can be used to compute and discrete S -transform.

The distribution of slow growth S', for the Fourier transform is defined in terms of the Parseval equation, by [12]

$$\langle \hat{f}, \hat{\varphi} \rangle = 2\pi \langle f, \varphi \rangle \tag{18}$$

and

$$\langle f, \overline{\hat{\varphi}} \rangle = 2\pi \langle f, \overline{\varphi} \rangle,$$

respectively. Similarly, using relation between the Fourier transform and the S -transform, the tempered distribution space S' for the S -transform is given by

$$\langle \tilde{g}(\rho), \tilde{\varphi}(\rho) \rangle = \langle g(r), \varphi(r) \rangle. \tag{19}$$

Lemma 1 (*Linearity*) [8] : If $u(t), v(t) \in L^1(\mathbb{R})$ and a,b are any scalars, then

$$(S(au+bv))(\tau, f) = (au+bv)^{\sim}(\tau, f) = a(\tilde{u})(\tau, f) + b(\tilde{v})(\tau, f) \ .$$

Lemma 2 [8]: If $u(t) \in L^1(\mathrm{R})$, then the following results hold :

(a) (*Shifting*) $(\tilde{T}_a(u))(\tau, f) = e^{i2\pi fa}\tilde{(u)}(\tau - a, f)$,

(b) (*Conjugation*) $(\tilde{\bar{u}})(\tau, f) = \overline{(u)(\tau - t, f)}$,

(c) $(S\delta)(\tau, f) = w(\tau, f) = 1.$

2. *S*- Transformation of Tempered Boehmians

The pair of sequence (f_a, φ_a) is called a quotient of sequence, denoted by f_a / φ_a, whose numerator belongs to some set $\mathbf{A}$ and the denominator is a delta sequence such that

$$f_a * \varphi_b = f_b * \varphi_a, \ \forall a, b \in \mathrm{N}. \tag{20}$$

Two quotients of sequence f_a / φ_a and g_a / ψ_a are said to be equivalent if

$$f_a * \psi_a = g_a * \varphi_a, \ \forall a \in \mathrm{N}. \tag{21}$$

The equivalence classes are called the Boehmians. The space of Boehmians is denoted by β, an element of which is written as $x = f_a / \varphi_a$. Applications of construction of Boehmians to function spaces with the convolution product yields various spaces of generalized functions. The spaces, so obtained, contain the standard spaces of generalized functions defined as dual spaces. For example, if $\mathbf{A} = C(\mathrm{R}^N)$ and a delta sequence defined as sequence of functions $\varphi_n \in \mathbf{D}$ such that

(i) $\int \varphi_a dx = 1$, $\forall a \in \mathrm{N}$

(ii) $\int |\varphi_a| dx \le C$, for some constant C and $\forall a \in \mathrm{N}$,

(iii) supp $\varphi_a(x) \to 0$, as $n \to \infty$

then the space of Boehmian that is obtained, contains properly the space of Schwartz distributions. Similarly, this space of Boehmians also contains properly the space of tempered distributions S' , when $\mathbf{A}$ is the space of slowly increasing functions with delta sequence. The *S*-transform of tempered Boehmian form a proper subspace of Schwartz distribution $\mathbf{D}'$. Boehmian space have two types of convergence, namely, the δ - and Δ - convergences, which are stated as:

(i) A sequence of Boehmians (x_a) in the Boehmian space B is said to be δ - convergent to a Boehmian x in B, which is denoted by $x_a \xrightarrow{\delta} x$ if there exists a delta sequence (δ_a) such that $(x_a * \delta_a), (x * \delta_a) \in \mathbf{A}, \forall a \in \mathrm{N}$ and $(x_a * \delta_k) \to (x * \delta_k)$ as $a \to \infty$ in $\mathbf{A}$, $\forall k \in \mathrm{N}$.

(ii) A sequence of Boehmians (x_a) in B is said to be Δ - convergent to a Boehmian x in B , denoted by $x_a \xrightarrow{\delta} x$ if there exists a delta sequence $(\delta_a) \in \Delta$ such that $(x_a - x) * \delta_a \in \mathbf{A}, \forall n \in \mathrm{N}$ and $(x_a - x) * \delta_a \to 0$ as $a \to \infty$ in $\mathbf{A}$.

For details of the properties and convergence of Boehmians one can refer to [5]. We have employed following notations and definitions.

A complex valued infinitely differentiable function f , defined on R^N , is called rapidly decreasing, if

$$\sup_{|\alpha| \le m} \sup_{x \in R^N} \left(1 + x_1^2 + x_2^2 + \Lambda + x_N^2\right)^m \left|D^\alpha f(x)\right| < \infty,$$

for every non-negative integer m . Here $|\alpha| = |\alpha_1| + \Lambda + |\alpha_N|$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \Lambda \ \partial x_N^{\alpha_N}}.$$

The space of all rapidly decreasing functions on R^N is denoted by S . The delta sequence, i.e., sequence of real valued functions $\varphi_1, \varphi_2, \mathrm{K} \in S,$ is such that

(i) $\int \varphi_a dx = 1$, $\forall a \in \mathrm{N}$

(ii) $\int |\varphi_a| dx \le C$, for some constant C and $\forall a \in \mathrm{N}$,

(iii) $\lim_{a \to \infty} \int_{\|x\| \ge \varepsilon} \|x\|^k |\varphi_a| dx = 0$, for every $k \in \mathrm{N}$, $\varepsilon > 0$.

If $\varphi \in S$ and $\int \varphi = 1$, then the sequence of functions φ_a is a delta sequence.

A complex-valued function f on R^N is called slowly increasing if there exists a

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polynomial p on R^N such that $f(x)/p(x)$ is bounded. The space of all increasing continuous functions on R^N is denoted by $\mathbf{I}$. If $f_a \in \mathbf{I}$, $\{\varphi_a\}$ is a delta sequence under usual notion, then the space of equivalence classes of quotients of sequence will be denoted by $\mathbf{B_I}$, elements of which will be called tempered Boehmians.

For $F=[f_a/\varphi_a]\in \mathbf{B_I}$, define $D^\alpha F=[(f_a * D^\alpha \varphi_a)/(\varphi_a * \varphi_a)]$. If F is a Boehmian corresponding to differentiable function, then $D^\alpha F \in \mathbf{B_I}$.

If $F=[f_a/\varphi_a]\in \mathbf{B_I}$ and $f_a \in S$, for all $a\in \mathrm{N}$, then F is called a rapidly decreasing Boehmian. The space of all rapidly decreasing Boehmian is denoted by $\mathbf{B}_S$. If $F=[f_a/\varphi_a]\in \mathbf{B_I}$ and $G=[g_a/\gamma_a]\in \mathbf{B}_S$, then the convolution is

$$F*G=[(f_a * g_a)/(\varphi_a * \gamma_a)]\in \mathbf{B_I}.$$

The convolution quotient is denoted by f/φ and $\frac{f}{\varphi}$ denotes a usual quotient. Let $f\in \mathbf{I}$. Then the S-transformation of f, denoted as $\tilde{S}$, is defined for distribution spaces of slowly increasing function f as

$$\langle \tilde{f},\tilde{\varphi}\rangle=\langle f,\varphi\rangle,\quad \varphi\in S$$

and

If $\hat{u}(f)$ and $(\tilde{u})(\tau,f)$ are the Fourier transform and the S-transform of u, respectively, then

$$\hat{u}(f)=\int_{\mathrm{R}}(\tilde{u})(\tau,f)d\tau,$$

so that

$$u(t)=\mathbf{F}^{-1}\left(\int_{\mathrm{R}}(\tilde{u})(\tau,\cdot)\,d\tau\right)(t).$$

Theorem 1 : If $[f_a/\varphi_a]\in \mathbf{B_I}$, then the sequence $\{\tilde{f}_a\}$ in $\mathbf{D}'$. Moreover, if $[f_a/\varphi_a]=$ $[g_a/\gamma_a]$, then $\{\tilde{f}_a\}$ and $\{\tilde{g}_a\}$ converges to the same limit for the S-transformation of tempered Boehmians.

Proof : Let $\varphi\in \mathbf{D}$ (testing function space) and $p\in \mathrm{N}$ be such that $\tilde{\varphi}_p>0$ on the support of φ. Since

$$f_a * \varphi_b = f_b * \varphi_a \,,\ \forall a,b \in \mathrm{N}$$

thus,

$$(\tilde{f}_a)\cdot(\tilde{\varphi}_b) = (\tilde{f}_b)\cdot(\tilde{\varphi}_a),\ \forall a,b \in \mathrm{N}.$$

We, thus, write the following

$$\left\langle (\tilde{f}_a),(\tilde{\varphi}_a)\right\rangle = \left\langle \tilde{f}_a, \tilde{\varphi}_a \cdot \frac{\tilde{\varphi}_p}{\tilde{\varphi}_p}\right\rangle$$

$$= \left\langle \tilde{f}_a \cdot \tilde{\varphi}_p, \frac{\tilde{\varphi}_a}{\tilde{\varphi}_p}\right\rangle$$

$$= \left\langle \tilde{f}_p \cdot \tilde{\varphi}_a, \frac{\tilde{\varphi}_a}{\tilde{\varphi}_p}\right\rangle$$

$$= \left\langle \tilde{f}_p, \frac{\tilde{\varphi}_a \cdot \tilde{\varphi}_a}{\tilde{\varphi}_p}\right\rangle \quad .$$

Since the sequence $\left\{\frac{\tilde{\varphi}_a\cdot\tilde{\varphi}_a}{\tilde{\varphi}_p}\right\}$ converges to $\frac{\tilde{\varphi}_a}{\tilde{\varphi}_p}$ in $\mathbf{D}$, the sequence $\{\tilde{f}_a, \tilde{\varphi}_a\}$ converges in $\mathbf{D}$. This proves that the sequence $\{\tilde{f}_a\}$ converges in $\mathbf{D}'$ (dual of space $\mathbf{D}$). Now to prove the second part of the theorem, we assume that $[f_a/\varphi_a] = [g_a/\gamma_a] \in \mathbf{B_I}$. Define

$$h_a = \begin{cases} f_{\frac{a+1}{2}} * \gamma_{\frac{a+1}{2}} & \text{, if } a \text{ is odd} \\ g_{\frac{a}{2}} * \varphi_{\frac{a}{2}} & \text{, if } a \text{ is even} \end{cases} \quad .$$

and

$$\delta_a = \begin{cases} \varphi_{\frac{a+1}{2}} * \gamma_{\frac{a+1}{2}} & \text{, if } a \text{ is odd} \\ \varphi_{\frac{a}{2}} * \gamma_{\frac{a}{2}} & \text{, if } a \text{ is even} \end{cases} \quad .$$

Then $[h_a/\delta_a] = [f_a/\varphi_a] = [g_a/\gamma_a]$. This proves that $\{\tilde{h}_a\}$ converges in $\mathbf{D}'$. Moreover,

$$\lim_{a\to\infty}\{\tilde{h}_{2a-1}\}\{\tilde{\varphi}_a\} = \lim_{a\to\infty}\{\tilde{f}_a * \tilde{\gamma}_a\}\cdot\{\tilde{\varphi}_a\} = \lim_{a\to\infty}\{\tilde{f}_a\}\cdot\{\tilde{\gamma}_a\}\cdot\{\tilde{\varphi}_a\} = \lim_{a\to\infty}\{\tilde{f}_a\}\cdot\{\tilde{\varphi}_a\}.$$

Thus, $\{\tilde{f}_a\}$ and $\{\tilde{h}_a\}$ have the same limit. Similarly, it can be shown that $\{\tilde{h}_a\}$ and $\{\tilde{g}_a\}$ will also have the same limit.

This completes the proof of the theorem .

Theorem 2 : Let $F = [f_a/\varphi_a] \in \mathbf{B_I}$ and $G = [g_a/\gamma_a] \in \mathbf{B}_S$.
Then (i) $(\tilde{G})$ is an infinitely differentiable function
(ii) $[\tilde{F} * \tilde{G}] = [\tilde{F}][\tilde{G}]$
and (iii) $(\tilde{F}) \cdot (\tilde{\varphi}_p) = (\tilde{f}_p)$, $p \in N$

Proof : (i) Let $G = [g_a/\gamma_a] \in \mathbf{B}_S$ and let U be the bounded open subset of R^N . Then there exists $a \in N$ such that $\{\tilde{\gamma}_a\} > 0$ on U . We have, thus

$$\begin{aligned}[\tilde{G}] &= \lim_{a\to\infty} \{\tilde{g}_a\} = \lim_{a\to\infty} \frac{\{\tilde{g}_a\}\{\tilde{\gamma}_p\}}{\{\tilde{\gamma}_p\}} \\ &= \lim_{a\to\infty} \frac{\{\tilde{g}_p\}\{\tilde{\gamma}_a\}}{\{\tilde{\gamma}_p\}} = \frac{\{\tilde{g}_p\}}{\{\tilde{\gamma}_p\}} \lim_{a\to\infty} \{\tilde{\gamma}_a\} \\ &= \frac{\{\tilde{g}_p\}}{\{\tilde{\gamma}_p\}} \quad \text{on } U .\end{aligned}$$

Since $\{g_p\}, \{\tilde{\gamma}_p\} \in S$ and $\{\tilde{\gamma}_p\} > 0$ on U , thus $\{\tilde{G}\}$ is an infinitely differentiable function on U.

(ii) Let $F = [f_a/\varphi_a] \in \mathbf{B_I}$ and $G = [g_a/\gamma_a] \in \mathbf{B}_S$. If $\varphi \in \mathbf{D}$, then there exists $p \in N$ such that $\{\tilde{\gamma}_p\} > 0$ on the support of e . We have

$$\begin{aligned}\{\tilde{F} * G\}\{\varphi\} &= \lim_{a\to\infty} (\tilde{f}_a * g_a)(\varphi) \\ &= \lim_{a\to\infty} (\tilde{f}_a g_a)(\varphi) = \lim_{a\to\infty} (\tilde{f}_a)(\tilde{g}_a, \varphi) \\ &= \lim_{a\to\infty} (\tilde{f}_a) \left\{ \frac{\tilde{g}_a \cdot \tilde{\gamma}_p}{\tilde{\gamma}_p} \cdot \varphi \right\} \quad , \quad p \in \mathrm{N} \\ &= \lim_{a\to\infty} (\tilde{f}_a) \left\{ \frac{\tilde{g}_p \cdot \tilde{\gamma}_a}{\tilde{\gamma}_p} \cdot \varphi \right\}\end{aligned}$$

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$$= \lim_{a\to\infty}(\tilde{f}_a)\left\{\frac{\tilde{g}_p}{\tilde{\gamma}_p}\cdot\varphi(\tilde{\gamma}_a)\right\}$$

$$= \lim_{a\to\infty}(\tilde{f}_a)\{(\tilde{G})\varphi(\tilde{\gamma}_a)\} \quad ; \quad \text{from (i)}$$
$$=(\tilde{G})\lim_{a\to\infty}\{(\tilde{f}_a)(\tilde{\gamma}_a)\}(\varphi)$$
$$=(\tilde{G})\lim_{a\to\infty}(\tilde{f}_a*\gamma_a)(\varphi)$$
$$=(\tilde{F})(\tilde{G})(\varphi)=\{\tilde{F}\}\cdot\{\tilde{G}\}\{\varphi\}$$

The last equality follows from Theorem 1 and due to the fact that $[f_a/\varphi_a]=$ $[(f_a*\varphi_a)/(\varphi_a*\gamma_a)]$.

(iii) Let $\varphi\in\mathbf{D}$. Then

$$\{\tilde{F}\cdot\tilde{\varphi}_p\}\{\varphi\}=\{\tilde{F}\}\{(\tilde{\varphi}_p)\varphi\} \quad , \quad p\in \mathrm{N}$$
$$=\lim_{a\to\infty}\{\tilde{f}_a\}\{(\tilde{\varphi}_p)\varphi\}=\lim_{a\to\infty}(\tilde{f}_a\cdot\tilde{\varphi}_p)(\varphi)$$
$$=\lim_{a\to\infty}(\tilde{f}_p\cdot\tilde{\varphi}_a)(\varphi)=\lim_{a\to\infty}\tilde{f}_p(\tilde{\varphi}_a\cdot\varphi)$$
$$=(\tilde{f}_p)(\varphi)=\tilde{f}_p$$

The theorem is, therefore, completely proved.

Theorem 3 : The distribution f is the S -transform of a tempered Boehmian if and only if there exists a delta sequence $\{\delta_n\}$ such that $(f\tilde{\delta}_n)$ is a tempered distribution for every $n\in N$.

Proof : From Theorem 2 (iii), if $F=[f_n/\varphi_n]\in B_{\mathbf{I}}$, then $f\cdot\tilde{\varphi}_n=\tilde{F}\cdot\tilde{\varphi}_n=\tilde{f}_n$. Thus, $f\cdot\tilde{\varphi}_n$ is a tempered distribution. ***Conversely***, if $f\in\mathbf{D}'$ and $\{\delta_n\}$ is delta sequence, then $(f\tilde{\delta}_n)$ is a tempered distribution for every $n\in N$. Thus, we have

$$F=\left[\frac{(\mathbf{F}^{-1}(f\tilde{\delta}_n)*\delta_n)}{(\delta_n*\delta_n)}\right] ,$$

where $\mathbf{F}^{-1}(f\tilde{\delta}_n)$ is the inverse S -transform of $(f\tilde{\delta}_n)$. As $\tilde{F}=f$, F is, therefore, proved to be a tempered Boehmian.

Theorem 4 (*Inversion*) : Let F be a tempered Boehmian and $\tilde{F}=f$. Then

$$F = \left[\frac{(\mathbf{F}^{-1}(f\tilde{\delta}_n) * \delta_n)}{(\delta_n * \delta_n)} \right] ,$$

where $\{\delta_n\}$ is delta sequence such that $(f\tilde{\delta}_n)$ is a tempered distribution for every $n \in N$.

Proof: If we consider $F = [f_n / \varphi_n] \in B_{\mathbf{I}}$ and $\delta_n = \varphi_n$, then the inversion formula is given by

$$F = \left[\frac{(\mathbf{F}^{-1}(f\tilde{\varphi}_n))}{(\varphi_n)} \right] .$$

The detailed analysis is avoided for its similarity to that of the Theorem 3.

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