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Solving Systems of Linear Volterra Integro-Differential Equations by Using Sinc-Collocation Method

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Abstract. This paper presents meshfree method for solving systems of linear Volterra integro-differential equations with initial conditions. This approach is based on collocation method using Sinc basis functions. It's well-known that the Sinc approximate solution converges exponentially to the exact solution. Some numerical results are included to show the validity of this method.

Keywords: Sinc function, Integro-differential equation, Volterra equation, Collocation method.

1 Introduction

Integral and integro-differential equations are among an active research field for many years. They arise in many scientific and engineering applications such as glass forming process, nano-hydrodynamics and wind ripple in the desert [6]. Thus applications of numerical methods for solving these equations are attractive. Systems of Volterra integro-differential equations have been solved by several numerical approaches such as Variational iteration method [9], Differential transform method [1], Radial basis functions [2] and so on. Another approach is based on the Sinc functions. In recent years, Sinc functions are used widely for obtaining the approximate solution of ODE, PDE and integral equation [3-5, 10]. In [7], the authors used Sinc-collocation method for solving Volterra integral equations. Also Volterra integro-differential equations are solved by Sinc-collocation method in [11]. In this study we present Sinc-collocation method to approximate the solution of system of linear Volterra integro-differential equations given by

$$u_i^{(n)}(x) = f_i(x) + \int_a^x \left(\sum_{j=1}^{l} k_{ij}(x,t)u_j(t) \right) dt,\ a \le x \le b, 1 \le i \le l, \tag{1.1}$$

for which the initial conditions are given as the value of functions $u_i(x)$ and their derivatives (up to $n-1$th order derivative) at point a. The kernels $k_{ij}(x,t)$ and functions $f_i(x)$ are given real-valued functions and the unknown functions $u_i(x)$ that will be determined.

The layout of the paper is as follows. In section 2, some properties of the Sinc function that is necessary for the formulation of the discrete system are given. Section 3 is concerned with the Sinc-collocation discretization for system (1.1). Some numerical examples are presented in section 4. And finally the last section is a brief conclusion.

2 Preliminaries

In this section, some properties of the Sinc function and the Sinc method are given. They are discussed thoroughly in [4,10]. The Sinc function is defined on the whole real line, $-\infty < x < \infty$, by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0, \\ 1 & x = 0. \end{cases} \quad (2.1)$$

For each integer k and the mesh size h, the translated Sinc basis functions are defined by

$$s(k,h)(x) = sinc\left(\frac{x-kh}{h}\right). \quad (2.2)$$

The Sinc function for the interpolating points $x_j = jh$, is given by

$$s(k,h)(jh) = \delta_{kj} = \begin{cases} 1 & k = j, \\ 0 & k \neq j. \end{cases} \quad (2.3)$$

If a function $f(x)$ is defined on the real axis, then for any $h > 0$, the Whittaker cardinal expansion of $f(x)$ is as follows:

$$c(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\, sinc\left(\frac{x-kh}{h}\right), \quad (2.4)$$

whenever this series converges. The properties of the Whittaker cardinal expansion have been extensively studied in [4]. These properties are derived in the infinite strip D_S of the complex w-planes where for $d > 0$,

$$D_S = \left\{ w = t + is : |s| < d \leq \frac{\pi}{2} \right\}. \quad (2.5)$$

Approximation can be constructed for infinite, semi-infinite and finite intervals. To construct approximations on the finite interval (a,b), which is used in this paper, we consider the one-to-one conformal map $w = \emptyset(z) = \ln\left(\frac{z-a}{b-z}\right)$, which maps the eye-shaped domain

$$D_E = \left\{z = x + iy: \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d \le \frac{\pi}{2}\right\}, \tag{2.6}$$

onto the infinite strip D_S. The basis functions on (a, b) are taken to be the composite translated Sinc function

$$s_k(x) = s(k,h)o\emptyset(x) = sinc\left(\frac{\emptyset(x) - kh}{h}\right), \tag{2.7}$$

Where $s(k,h)o\emptyset(x)$ is defined by $s(k,h)(\emptyset(x))$.

Also we define the range of $\psi = \emptyset^{-1}$ on the real line as $\Gamma = \{\psi(u) \in D_E: -\infty < u < \infty\}$. For the uniform grid $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \psi(kh) = \frac{a + be^{kh}}{1 + e^{kh}} \quad k = 0, \pm 1, \pm 2, \cdots. \tag{2.8}$$

For discretizing the problem we need the following definition and theorems.

Definition 2.1. Let $L_\alpha(D_E)$ be the set of all analytic functions, for which there exist a constant, C, such that

$$|u(z)| \le C\frac{|\rho(z)|^\alpha}{[1 + |\rho(z)|]^{2\alpha}}, \qquad z \in D_E, \qquad 0 < \alpha \le 1$$

where $\rho(z) = e^{\emptyset(z)}$.

Theorem 2.1. Let $u \in L_\alpha(D_E)$, N be a positive integer, and h be selected by the Formula

$$h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}, \tag{2.9}$$

then there exists a positive constant c_1, independent of N, such that

$$sup_{z\in\Gamma}\left|u(z) - \sum_{j=-N}^{N} u(z_j)s(j,h)o\emptyset(z)\right| \le c_1 e^{-(\pi d\alpha N)^{\frac{1}{2}}}.$$

Theorem 2.2. Let $\frac{u}{\emptyset'} \in L_\alpha(D_E)$, N be a positive integer, and h be selected by the formula (2.9), and let $\delta_{kj}^{(-1)}$ be defined as

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j} \frac{\sin(\pi t)}{\pi t}dt,$$

then there exist a positive constant c_2, independent of N, such that

$$\left|\int_a^{z_k} u(t)dt - h\sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{u(z_j)}{\emptyset'(z_j)}\right| \le c_3 e^{-(\pi d\alpha N)^{\frac{1}{2}}}.$$

Also derivatives of composite Sinc functions evaluated at the nodes are needed. So we recall the following theorem for this purpose.

Theorem 2.3. Let $\emptyset$ be a conformal one-to-one map of the simply connected domain D_E onto D_S. Then

$$\delta_{kj}^{(0)} = s_k(x)|_{x=x_j} = \begin{cases} 1 & k = j, \\ 0 & k \neq j. \end{cases}$$

$$\delta_{kj}^{(1)} = \frac{d}{d\emptyset}[s_k(x)]|_{x=x_j} = \frac{1}{h}\begin{cases} 0 & k = j, \\ \dfrac{(-1)^{j-k}}{j-k} & k \neq j. \end{cases}$$

$$\delta_{kj}^{(2)} = \frac{d^2}{d\emptyset^2}[s_k(x)]|_{x=x_j} = \frac{1}{h^2}\begin{cases} \dfrac{-\pi^2}{3} & k = j, \\ \dfrac{-2(-1)^{j-k}}{(j-k)^2} & k \neq j. \end{cases}$$

3 Method of solution

Consider the system (1.1) in the domain $[0,1]$ and let $u_i(x) \in L_\alpha(D_E)$. In order to discretize this system by using Sinc-collocation method, due to [8] and using theorem (2.1) $u_i(x)$ is approximated as follows

$$u_i(x) = y_i(x) + P_i(x), \tag{3.1}$$

where

$$y_i(x) = \sum_{k=-N}^{N} c_k^i w(x) s_k(x), \quad P_i(x) = a_0^i + a_1^i x + \cdots + a_n^i x^n, \tag{3.2}$$

so

$$\int_0^x u_i(t)dt = \int_0^x y_i(x)dt + \int_0^x P_i(t)dt, \tag{3.3}$$

and

$$u_i^{(n)}(x) = y_i^{(n)}(x) + P_i^{(n)}(x), \tag{3.4}$$

where $y_i^{(n)}(x) = \sum_{k=-N}^{N} c_k \frac{d^n(w(x)s_k(x))}{dx^n}$.

Notice that according to the system (1.1),$u_i^{(n)}(0) = f_i(0)$ so we take $w(x) = x^n(x-1)^n$ in order to equaling the first, second... and nth derivatives of the modified Sinc basis functions to 0 as x approaches 0 or 1. Now by substituting relations (3.3) and (3.4) into system (1.1), evaluating the result at the Sinc points $x_j = \frac{e^{jh}}{1+e^{jh}}$ $, j = -N-1, \cdots, N,$ and using theorems (2.2) and (2.3), a system of

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algebraic equations is obtained. Then we can solve it to obtain unknown coefficients $\{c_k^i\}_{k=-N}^{N}$ and $\{a_j^i\}_{j=0}^{n}$.

4 Numerical Examples

In this section, some numerical examples are presented to show the validity of Sinc-collocation method for solving systems of linear Volterra integro-differential equations. In the following examples, we choose $\alpha = \frac{1}{2}$ and $d = \frac{\pi}{2}$.

Example 4.1. In this example we consider the following system of Volterra integro-differential equations on $[0,1]$

$$u'(x) = 1 - x^2 + e^x + \int_0^x \big(u(t) + v(t)\big)dt, \quad u(0) = 1,$$

$$v'(x) = 3 - 3e^x + \int_0^x \big(u(t) - v(t)\big)dt, \quad v(0) = -1,$$

whose exact solution is $u(x) = x + e^x, v(x) = x - e^x$. Using the Sinc-collocation method, the approximate solution with $N = 20$ and absolute error with $N = 10$ and $N = 20$ are plotted in Figure 1 and Figure 2 respectively.

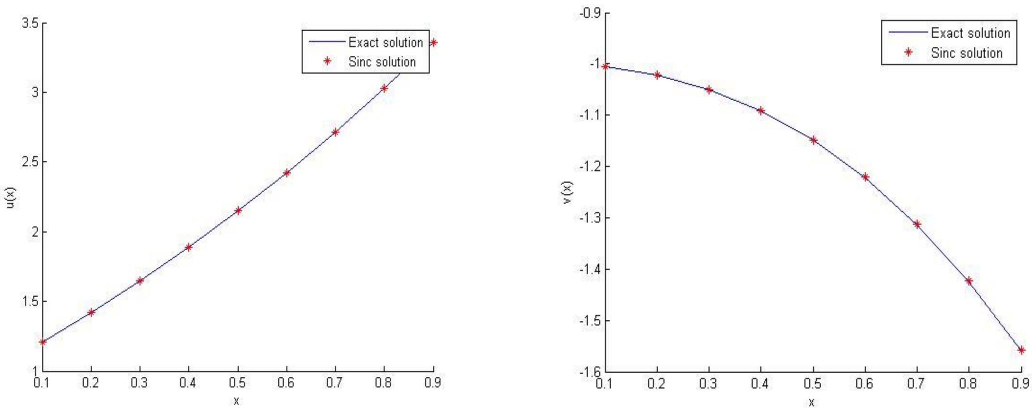

Figure 1. Plot of analytic and Sinc solutions of example 4.1 with $N = 20$.

Example 4.2. Consider the following system of Volterra integro-differential equations on $[0,1]$ with $u(0) = 1, \quad v(0) = 1.$

$$u'(x) = 2x^2 + \int_0^x \big((x-t)u(t) + (x-t)v(t)\big)dt\,,$$
$$v'(x) = -3x^2 - \frac{1}{10}x^5 + \int_0^x \big((x-t)u(t) - (x-t)v(t)\big)dt,$$

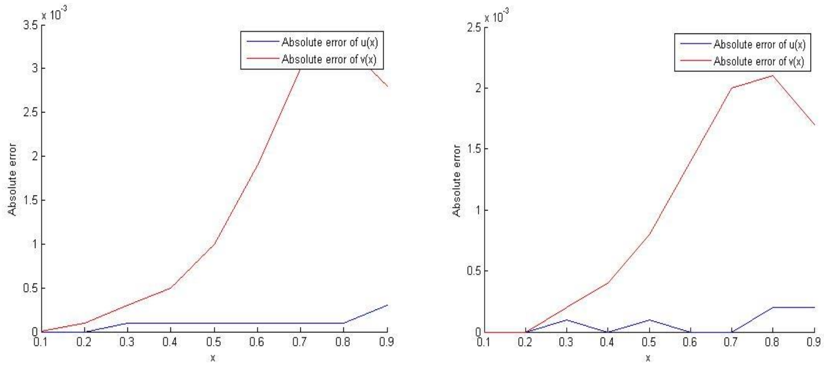

Figure 2. Plot of the absolute error with $N = 10$ (left) and $N = 20$ (right) for example 4.1.

The exact solution of this system is $u(x) = 1 + x^3$, $v(x) = 1 - x^3$. In Tables 1 and 2, the results of the present method with $N = 10$ and $N = 25$ are compared with the exact solution. The absolute error is also plotted in Figure 3.

Table 1. Comparison between the exact and Sinc values of u(x) for example 4.2

x_j	Exact solution u(x)	Sinc solution(N=10)	Sinc solution(N=25)
0.1	1.0010	1.0011	1.0010
0.2	1.0080	1.0079	1.0080
0.3	1.0270	1.0268	1.0270
0.4	1.0640	1.0642	1.0640
0.5	1.1250	1.1254	1.1249
0.6	1.2160	1.2161	1.2160
0.7	1.3430	1.3427	1.3430
0.8	1.5120	1.5119	1.5120
0.9	1.7290	1.7292	1.7290

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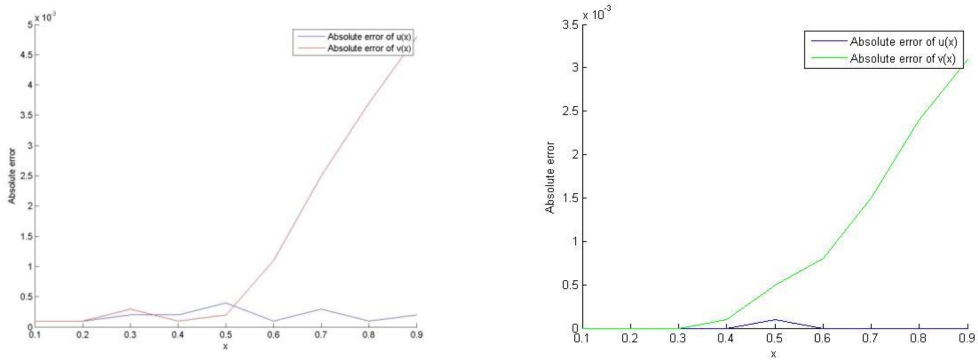

Figure 3. Plot of the absolute error with N=10 (left) and N=25 (right) for example 4.2.

Table 2. Comparison between the exact and Sinc values of v(x) for example 4.2

x_j	Exact solution v(x)	Sinc solution(N=10)	Sinc solution(N=25)
0.1	0.9990	0.9989	0.9990
0.2	0.9920	0.9921	0.9920
0.3	0.9730	0.9733	0.9730
0.4	0.9360	0.9361	0.9361
0.5	0.8750	0.8752	0.8755
0.6	0.7840	0.7851	0.7848
0.7	0.6570	0.6595	0.6585
0.8	0.4880	0.4917	0.4904
0.9	0.2710	0.2758	0.2741

The above results show that the accuracy of the present method can be improved by increasing N.

Example 4.3. As the last example, we consider the following system of linear Volterra integro-differential equations

$$u''(x) = -1 - x^2 - \sin x + \int_0^x \big(u(t) + v(t)\big)dt\ ,$$

$$u(0) = 1, u'(0) = 1,$$

$$v''(x) = 1 - 2\sin x - \cos x + \int_0^x \big(u(t) - v(t)\big)\, dt,$$

$$v(0) = 0, v'(0) = 2.$$

The exact solution of this system is $u(x) = x + cosx$, $v(x) = x + sinx$. The absolute errors of this example with $N = 10$ and $N = 20$ are plotted in Figure 4. Also the approximate solution is plotted in Figure5.

5 Conclusion

In this work we used the Sinc-collocation method to approximate the solution of systems of linear Volterra integro-differential equations with initial conditions. However it can be used for such systems for which the boundary conditions are given as the values of $u_i(x)$ or its derivatives or combination of them at m points $(m \leq n)$ in the domain $[a,b]$. By using the Sinc function properties the system of integro-differential equations is reduced to a system of algebraic equations. The obtained results show the efficiency of this method. Note that this method provides solutions at any point not only at mesh points.

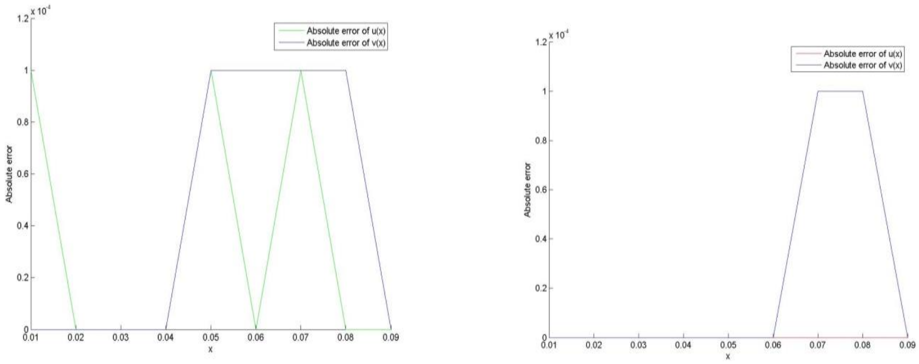

Figure 4. Plot of the absolute error with $N = 10$ (left) and $N = 20$ (right) for example 4.3.

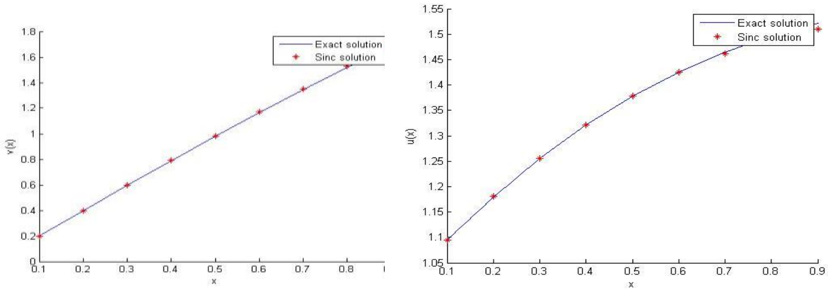

Figure 5. Plot of analytic and Sinc solutions of example 4.3 with $N = 20$.

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