From Newton's Theorem to a Theorem of the Inscribable Octagon

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In this article we'll prove the Newton's theorem relative to the circumscribed quadrilateral, we'll transform it through duality, and we obtain another theorem which is true for an inscribable quadrilateral, which transformed through duality, we'll obtain a theorem which is true for a circumscribable octagon.

Theorem 1 (I. Newton)
In a circumscribable quadrilateral its diagonals and the cords determined by the contact points of the opposite sides of the quadrilateral with the circumscribed circle are four concurrent lines.

Proof

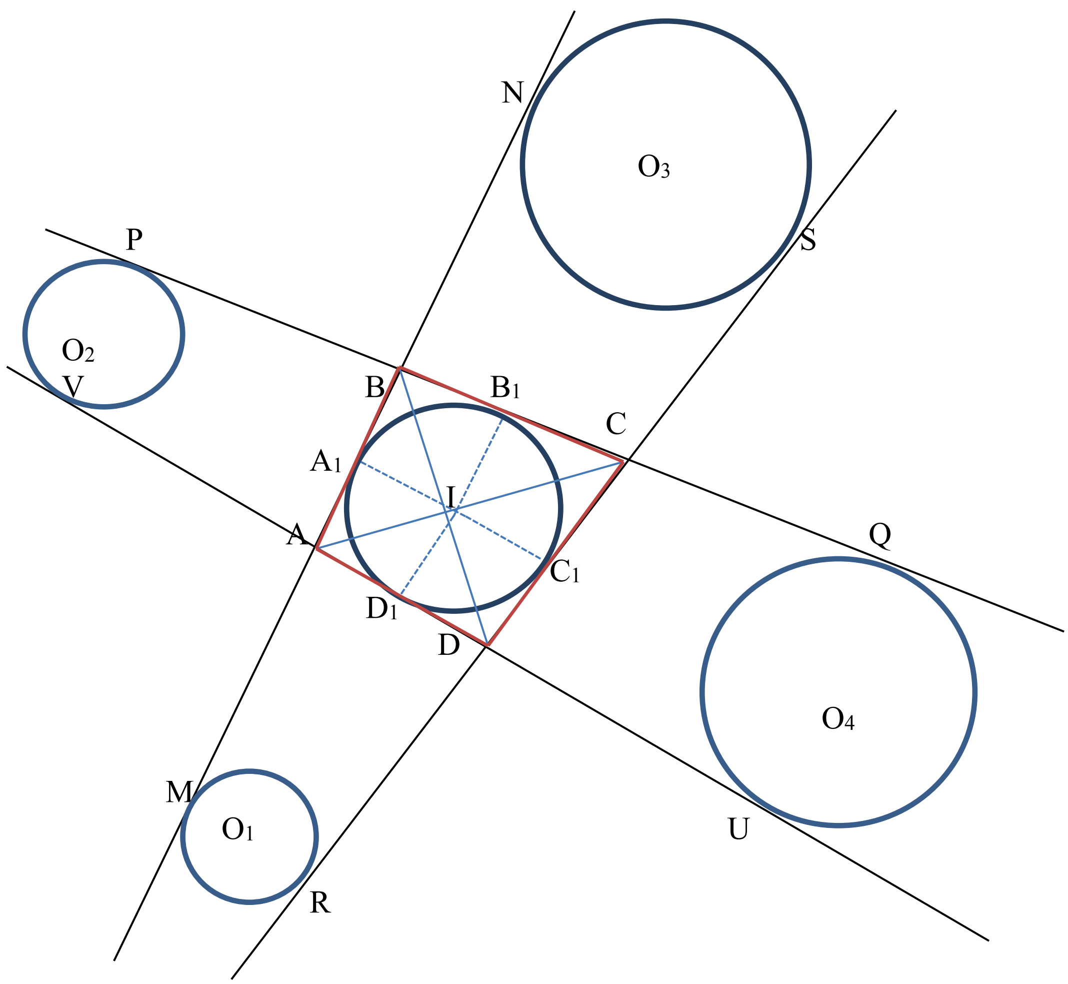

Fig. 1

We constructed the circles O_1, O_2, O_3, O_4 tangent to the extensions of the quadrilateral $ABCD$ such that

$$A_1M = A_1N = B_1P = B_1Q = C_1R = C_1S = D_1U = D_1V$$

See Fig. 1.

From $A_1M = A_1N = C_1R = C_1S$ it results that the points A_1 and C_1 have equal powers in relation to the circles O_1 and O_3, therefore A_1C_1 is the radical axis of these circles. Similarly B_1D_1 is the radical axis of the circles O_2 and O_4.

Let $I \in A_1C_1 \cap B_1D_1$. The point I has equal powers in rapport to circles O_1, O_2, O_3, O_4. Because $BA_1 = BB_1$ from $B_1P = A_1N$ it results that $BP = BN$, similarly, from $DD_1 = DC_1$ and $D_1V = C_1S$ it results that $DV = DS$, therefore B and D have equal powers in rapport with the circles O_3 and O_4, which shows that BD is the radical axis of these circles. Consequently, $I \in BD$, similarly it results that $I \in AC$, and the proof is complete.

Theorem 2.

In an inscribed quadrilateral in which the opposite sides intersect, the intersection points of the tangents constructed to the circumscribed circle with the opposite vertexes and the points of intersection of the opposite sides are collinear.

Proof

We'll prove this theorem applying the configuration from the Newton theorem, o transformation through duality in rapport with the circle inscribed in the quadrilateral. Through this transformation to the lines AB, BC, CD, DA will correspond, respectively, the points A_1, B_1, C_1, D_1 their pols. Also to the lines $A_1B_1, B_1C_1, C_1D_1, D_1A_1$ correspond, respectively, the points B, C, D, A. We note $X \in AB \cap CD$ and $Y \in AD \cap BC$, these points correspond, through the considered duality, to the lines A_1C_1 respectively B_1D_1. If $I \in A_1C_1 \cap B_1D_1$ then to the point I corresponds line XY, its polar.

To line BD corresponds the point $Z \in A_1D_1 \cap C_1B_1$.

To line AC corresponds the point $T \in A_1D_1 \cap C_1B_1$.

To point $\{I\} = BD \cap AC$ corresponds its polar ZT.

We noticed that to the point I corresponds the line XY, consequently the points X, Y, Z, T are collinear.

We obtained that the quadrilateral $A_1B_1C_1D_1$ inscribed in a circle has the property that if $A_1D_1 \cap C_1B_1 = \{Z\}$, $A_1D_1 \cap C_1B_1 = \{T\}$, the tangent in A_1 and the tangent in C_1 intersect in the point X; the tangent in B_1 and the tangent in D_1 intersect in Y, then X, Y, Z, T are collinear (see Fig. 2).

Theorem 3.

In a circumscribed octagon, the four cords, determined by the octagon's contact points with the circle of the octagon opposite sides, are concurrent.

Proof

We'll transform through reciprocal polar the configuration in figure 3.

To point Z corresponds through this transformation the line determined by the tangency points with the circle of the tangents constructed from Z - its polar; to the point Y it corresponds the line determined by the contact points of the tangents constructed from T at the circle; to the point X corresponds its polar A_1C_1.

To point A_1 corresponds through duality the tangent A_1X, also to the points B_1, C_1, D_1 correspond the tangents B_1Y, C_1T, D_1Z.

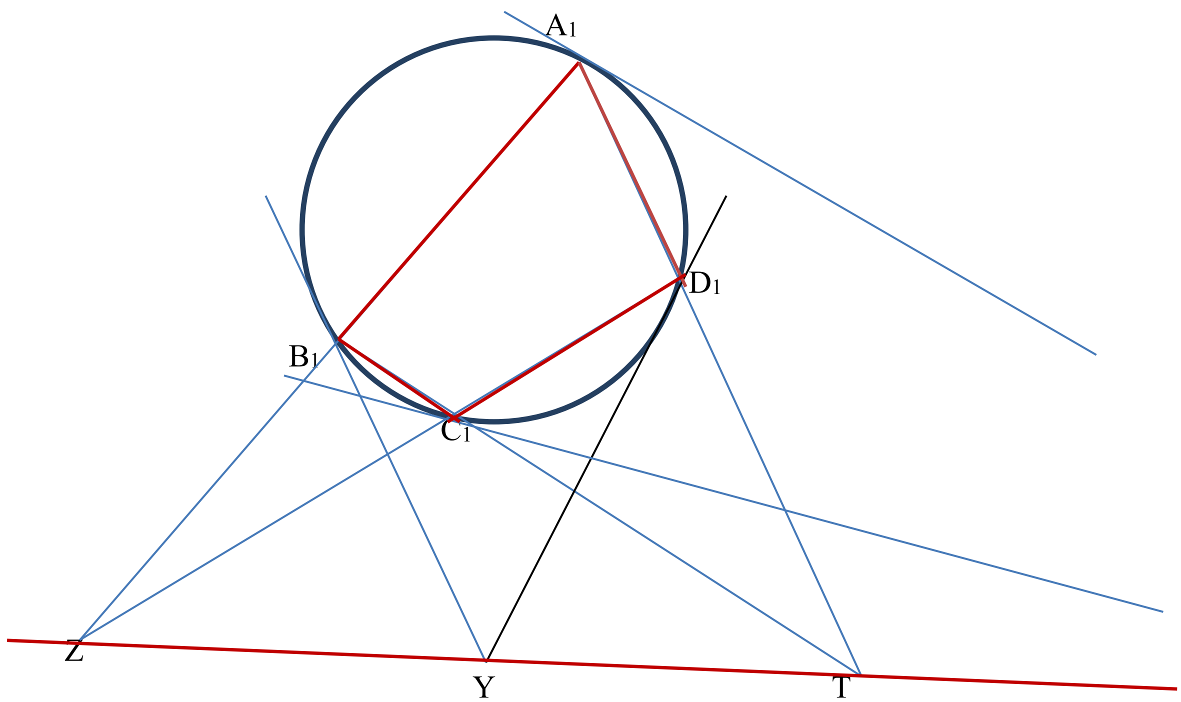

Fig. 2

These four tangents together with the tangents constructed from X and Y (also four) will contain the sides of an octagon circumscribed to the given circle.

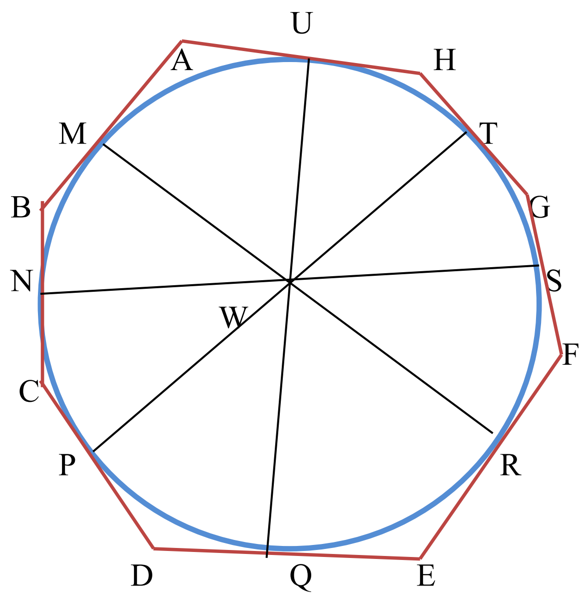

Fig .3

In this octagon A_1C_1 and B_1D_1 will connect the contact points of two pairs of sides opposed to the circle, the other two cords determined by the contact points of the opposite sides of the octagon with the circle will be the polar of the points Z and T.

Because the transformation through reciprocal polar will make that to collinear points will correspond concurrent lines, these lines are the cords from our initial statement.

Observation

In figure 3 we represented an octagon $ABCDEFGH$ circumscribed to a circle.

As it can be observed the cords MR, NS, PT, QU are concurrent in a point notated W

References

[1] Coxeter H. S. M, Greitzer S. L. –Geometry revisited – Toronto – New York, 1957 (translation in Russian, 1978)

[2] Smarandache F., Pătraşcu I. – The Geometry of Homological Triangles – The Education Publisher, Inc. Columbus, Ohio, U.S.A. -2012.