SMARANDACHE RECIPROCAL FUNCTION AND AN ELEMENTARY INEQUALITY

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The Smarandache Function is defined as $S(n) = k$. Where k is the smallest integer such that n divides $k!$

Let us define $S_c(n)$ **Smarandache Reciprocal Function** as follows:

$S_c(n) = x$ where $x + 1$ does not divide $n!$ and for every $y \le x$, $y \mid n!$

THEOREM-I.

If $S_c(n) = x$,and $n \ne 3$, then $x + 1$ is the smallest prime greater than n.

PROOF: It is obvious that n! is divisible by 1 , 2, . . . up to n. We have to prove that n! is also divisible by $n + 1$, $n + 2$, . . . $n + m$ (= x) , where $n + m + 1$ is the smallest prime greater than n.. Let r be any of these composite numbers. Then r must be factorable into two factors each of which is ≥ 2. Let $r = p.q$, where $p , q \ge 2$. If one of the factors (say q) is $\ge n$ then $r = p.q \ge 2n$. But according to the Bertrand's postulate there must be a prime between n and 2n, there is a contradiction here since all the numbers from $n + 1$ to $n + m$ ($n + 1 \le r < n + m$) are assumed to be composite. Hence neither of the two factors p , q can be $\ge n$. So each must be $< n$. Now there are two possibilities:

Case-I $p \neq q$.

In this case as each is $< n$ so $p.q = r$ divides $n!$

Case-II $p = q$ = prime

In this case $r = p^2$ where p is a prime. There are again three possibilities:

(a) $p \geq 5$. Then $r = p^2 > 4p \Rightarrow 4p < r < 2n \Rightarrow 2p < n$. Therefore both p and 2p are less than n so p^2 divides $n!$

(b) $p = 3$, Then $r = p^2 = 9$ Therefore n must be 7 or 8 . and 9 divides 7! and 8!.

(c) $p = 2$, then $r = p^2 = 4$. Therefore n must be 3 . But 4 does not divide 3! , Hence the theorem holds for all integral values of n except $n = 3$. This completes the proof.

Remarks: Readers can note that n! is divisible by all the composite numbers between n and 2n.

Note: We have the well known inequality $S(n) \leq n$. ---------(2)

From the above theorem one can deduce the following inequality.

If p_r is the r^{th} prime and $p_r \leq n < p_{r+1}$ then $S(n) \leq p_r$. (A slight improvement on (2)).

i.e. $S(p_r) = p_r$, $S(p_r + 1) < p_r$, $S(p_r + 2) < p_r$, . . . $S(p_{r+1} - 1) < p_r$, $S(p_{r+1}) = p_{r+1}$

Summing up for all the numbers $p_r \le n < p_{r+1}$ one gets

$$\sum_{t=0}^{p_{r+1} - p_r - 1} S(p_r + t) \le (p_{r+1} - p_r)\, p_r$$

Summing up for all the numbers up to the s^{th} prime, defining $p_0 = 1$, we get

$$\sum_{k=1}^{p_s} S(k) \le \sum_{r=0}^{s} (p_{r+1} - p_r)\, p_r \quad \text{--------(3)}$$

More generally from Ref. [1] following inequality on the nth partial sum of the Smarandache (Inferior) Prime Part Sequence directly follows.

Smarandache (Inferior) Prime Part Sequence

For any positive real number n one defines $p_p(n)$ as the largest prime number less than or equal to n. In [1] Prof. Krassimir T. Atanassov proves that the value of the n^{th} partial sum of this sequence $X_n = \sum_{k=1}^{n} p_p(k)$ is given by

$$X_n = \sum_{k=2}^{\pi(n)} (p_k - p_{k-1}).\, p_{k-1} + (n - p_{\pi(n)} + 1).\, p_{\pi(n)} \quad \text{-----(4)}$$

From (3) and (4) we get

$$\sum_{k=1}^{n} S(k) \le X_n$$

REFERENCES:

[1] "Krassimir T. Atanassov" , ' ON SOME OF THE SMARANDACHE'S PROBLEMS' AMERICAN RESEARCH PRESS Lupton, AZ USA. 1999. (22-23)

[2] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

[3] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA