Unsolved Problems

Edited by

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Welcome to the first installment of what is to be a regular feature in **Smarandache Notions**! In this column, we will present problems where the solution is either unknown or incomplete. This is meant to be an interactive endeavor, so input from readers is strongly encouraged. Always feel free to contact the editor at any of the addresses given above. It is hoped that we can together advance the flow of mathematics in some small way. There will be no deadlines here, and even if a problem is completely solved, new insights or more elegant proofs are always welcome. All correspondents who are the first to resolve any issue appearing here will have their efforts acknowledged in a future column.

While there will almost certainly be an emphasis on problems related to Smarandache notions, it will not be exclusive. Our goal here is to be interesting, challenging and maybe at times even profound. In modern times computers are an integral part of mathematics and this column is no exception. Feel free to include computer programs with your submissions, but please make sure that adequate documentation is included. If you are someone with significant computer resources and would like to be part of a collective effort to resolve outstanding problems, please contact the editor. If such a group can be formed, then sections of a problem can be parceled out and all those who participated will be given credit for the solution.

And now, it is time to stop chatting and get to work!

Definition of the Smarandache function S(n):

S(n) = m, smallest positive integer such that m! is evenly divisible by n.

In [1], T. Yau posed the following question:

For what triplets n, n+1, and n+2 does the Smarandache function satisfy the Fibonacci relationship

$$S(n) + S(n+1) = S(n+2)?$$

And two solutions

$$S(9) + S(10) = S(11); \quad S(119) + S(120) = S(121)$$

were given.

In [2], C. Ashbacher listed the additional solutions

$$S(4900) + S(4901) = S(4902); \; S(26243) + S(26244) = S(26245);$$
$$S(32110) + S(32111) = S(32112) ; \; S(64008) + S(64009) = S(64010);$$
$$S(368138) + S(368139) = S(368140) ; \; S(415662) + S(415663) = S(415664)$$

discovered in a computer search up through n = 1,000,000. He then presented arguments to support the conjecture that the number of solutions is in fact infinite.

Recently, Henry Ibstedt from Sweden sent a letter in response to this same problem appearing in the October issue of **Personal Computer World**. He has conducted a more extensive computer search, finding many other solutions. His conclusion was, "This study strongly indicates that the set of solutions is infinite." The complete report has been submitted to **PCW** for publication.

Another problem dealing with the Smarandache function has been given the name Radu's problem, having been first proposed by I.M. Radu[3].

Show that, except for a finite set of numbers, there exist at least one prime number between S(n) and S(n+1).

Ashbacher also dealt with this problem in [2] and conducted another computer search up through n = 1,000,000. Four instances where there are no primes between S(n) and S(n+1) were found.

n = 224 = 2*2*2*2*2*7 S(n) = 8 n+1 = 225 = 3*3*5*5 S(225) = 10
n = 2057 = 11*11*17 S(n) = 22 n+1 = 2058 = 2*3*7*7*7 S(2058) = 21
n = 265225 = 5*5*103*103 S(n) = 206 n+1 = 265226 = 2*13*101*101
S(265226) = 202
n = 843637 = 37*151*151 S(n) = 302 n+1 = 843638 = 2*19*149*149
S(843638) = 298

The fact that the last two solutions involve the pairs of twin primes (101,103) and (149,151) was one point used to justify the conjecture that there is an infinite set of numbers such that there is no prime between S(n) and S(n+1).

Ibstedt also extended the computer search for solutions and found many other cases where there is no prime between S(n) and S(n+1). His conclusion is quoted below.

"A very large set of solutions was obtained. There is no indication that the set would be finite."

This conclusion is also due to appear in a future issue of **Personal Computer World**.

The following statement appears in [4].

$$1141^6 = 74^6 + 234^6 + 402^6 + 474^6 + 702^6 + 894^6 + 1077^6$$

This is the smallest known solution for 6th power as the sum of 7 other 6th powers.

Is this indeed the smallest such solution? No one seems to know. The editor would be interested in any information about this problem. Clearly, given enough computer time, it can be resolved. This simple problem is also a prime candidate for a group effort at resolution.

Another related problem that would be also be a prime candidate for a group effort at computer resolution appeared as problem 1223 in **Journal of Recreational Mathematics**.

Find the smallest integer that is the sum of two unequal fifth powers in two different ways, or prove that there is none.

The case of third powers is well known as a result of the famous story concerning the number of a taxicab

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

as related by Hardy[4].

It was once conjectured that there might be a solution for the fifth power case where the sum had about 25 decimal digits, but a computer search for a solution with sum $< 1.02 \times 10^{26}$ yielded no solutions[5].

Problem (24) in [6] involves the Smarandache Pierced Chain(SPC) sequence.

{ 101, 1010101, 10101010101, 101010101010101, . . . }

or

$$\mathrm{SPC}(n) = 101 * 1\ \underbrace{0001}_{|----|}\ 0001 \ldots 0001$$

where the section in | ---- | appears n-1 times.

And the question is, how many of the numbers

SPC(n) / 101 are prime?

It is easy to verify that if n is evenly divisible by 3, then the number of 1's in SPC(n) is evenly divisible by 3. Therefore, so is SPC(n). And since 101 is not divisible by 3, it follows that

SPC(n) / 101

must be divisible by 3.

A simple induction proof verifies that SPC(2k)/101 is evenly divisible by 73 for k = 1,2,3, . . .

Basis step:

SPC(2)/101 = 73*137

Inductive step:

Assume that SPC(2k)/101 is evenly divisible by 73. From this, it is obvious that 73 divides SPC(2k). Following the rules of the sequence, SPC(2(k+1)) is formed by appending 01010101 to the end of SPC(2k). Since

01010101 / 73 = 13837

it follows that SPC(2(k+1)) must also be divisible by 73.

Therefore, SPC(2k) is divisible by 73 for all k > 0. Since 73 does not divide 101, it follows that SPC(2k) / 101 is also divisible by 73.

Similar reasoning can be used to obtain the companion result.

SPC(3 + 4k) is evenly divisible by 37 for all k > 0.

With these restrictions, the first element in the sequence that can possibly be prime when divided by 101 is

SPC(5) = 1010101010101010101.

However, this does not yield a prime as

SPC(5) = 41 * 101 * 271 * 3541 * 9091 * 27961.

Furthermore, since the elements of the sequence SPC(5k), k > 0 are made by appending the string

$$0101010101010101010101 = 41 * 101 * 271 * 3541 * 9091 * 27961$$

to the previous element, it is also clear that every number SPC(5k) is evenly divisible by 271 and therefore so is SPC(5k)/101.

Using these results to reduce the field of search, the first one that can possibly be prime is SPC(13)/101. However,

$$SPC(13)/101 = 53 * 79 * 521 * 859 * \ldots\ldots$$

SPC(17)/101 is the next not yet been filtered out. But it is also not prime as

$$SPC(17)/101 = 103 * 4013 * \ldots..$$

The next one to check is SPC(29)/101, which is also not prime as

$$SPC(29)/101 = 59 * 349 * 3191 * 16763 * 38861 * 43037 * 62003 * \ldots.$$

SPC(31)/101 is also not prime as

$$SPC(31)/101 = 2791 * \ldots.$$

At this point we can stop and argue that the numerical evidence strongly indicates that there are no primes in this sequence. The problem is now passed on to the readership to perform additional testing or perhaps come up with a proof that there are no primes in this sequence.

References

1. T. Yau: `A Problem Concerning the Fibonacci Series', Smarandache Function Journal, V. 4-5, No. 1, (1994), page 42.

2. C. Ashbacher, **An Introduction To the Smarandache Function**, Erhus University Press, 1995.

3. I. M. Radu, Letter to the Editor, Math. Spectrum, Vol. 27, No. 2, (1994/1995), page 44.

4. D. Wells, **The Penguin Dictionary of Curious and Interesting Numbers**, Penguin Books, 1987.

5. R. Guy, **Unsolved Problems in Number Theory 2nd. Ed.**, Springer Verlag, 1994.

6. F. Smarandache, **Only Problems, Not Solutions!**, Xiquan Publishing House, 1993.