NOTE ON THE DIOPHANTINE EQUATION $2x^2 - 3y^2 = p$

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The solving of the Diophantine equation

$$2x^2 - 3y^2 = 5 \tag{1}$$

i.e.,

$$2x^2 - 3y^2 - 5 = 0$$

was put as an open Problem 78 by F. Smarandache in [1]. Below this problem is solved completely. Also, we consider here the Diophantive equation

$$l^2 - 6m^2 = -5, \tag{2}$$

i.e.,

$$l^2 - 6m^2 + 5 = 0$$

and the Pellian equation

$$u^2 - 6v^2 = 1, \tag{3}$$

i.e.,

$$u^2 - 6v^2 - 1 = 0.$$

Here we use variables x and y only for equation (1) and l, m for equation (2).

We will need the following denotations and definitions:

$$\mathcal{N} = \{1, 2, 3, ...\};$$

if

$$F(t, w) = 0$$

is an Diophantive equation, then:

(a_1) we use the denotation $< t, w >$ if and only if (or briefly: iff) t and w are integers which satisfy this equation.

(a_2) we use the denotation $< t, w > \in \mathcal{N}^2$ iff t and w are positive integers;
$K(t, w)$ denotes the set of all $< t, w >$;
$K^\circ(t, w)$ denotes the set of all $< t, w > \in \mathcal{N}^2$;
$K'(t, w) = K^\circ(t, w) - \{< 2, 1 >\}$.

LEMMA 1: If $< t, w > \in \mathcal{N}^2$ and $< x, y > \neq < 2, 1 >$, then there exists $< l, m >$, such that $< l, m > \in \mathcal{N}^2$ and the equalities

$$x = l + 3m \text{ and } y = l + 2m \tag{4}$$

hold.

LEMMA 2: Let $< l, m > \in \mathcal{N}^2$. If x and y are given by (1), then x and y satisfy (4) and $< x, y > \in \mathcal{N}^2$.

We shall note that lemmas 1 and 2 show that the map $\varphi : K^0(l,m) \to K'(x,y)$ given by (4) is a bijection.

Proof of Lemma 1: Let $< x, y > \in \mathcal{N}^2$ be chosen arbitrarily, but $< x, y > \neq < 2, 1 >$. Then $y \geq 2$ and $x > y$. Therefore,

$$x = y + m \tag{5}$$

and m is a positive integer. Subtracting (5) into (1), we obtain

$$y^2 - 4my + 5 - 2m^2 = 0. \tag{6}$$

Hence

$$y = y_{1,2} = 2m \pm \sqrt{6m^2 - 5}. \tag{7}$$

For $m = 1$ (7) yields only

$$y = y_1 = 3.$$

indeed

$$1 = y = y_2 < 2$$

contradicts to $y \geq 2$.

Let $m > 1$. Then

$$2m - \sqrt{6m^2 - 5} < 0.$$

Therefore $y = y_2$ is impossible again. Thus we always have

$$y = y_1 = 2m + \sqrt{6m^2 - 5}. \tag{8}$$

Hence

$$y - 2m = \sqrt{6m^2 - 5}. \tag{9}$$

The left-hand side of (9) is a positive integer. Therefore, there exists a positive integer l such that

$$6m^2 - 5 = l^2.$$

Hence l and m satisfy (2) and $< l, m > \in \mathcal{N}^2$.

The equalities (4) hold because of (5) and (8). ◇

Proof of Lemma 2: Let $< l, m > \in \mathcal{N}^2$. Then we check the equality

$$2(l + 3m)^2 - 3(l + 2m)^2 = 5,$$

under the assumption of validity of (2) and the lemma is proved. ◇

Theorem 108 a, Theorem 109 and Theorem 110 from [2] imply the following

THEOREM 1: There exist sets $K_i(l,m)$ such that

$$K_i(l,m) \subset K(l,m) \quad (i = 1,2),$$

$$K_1(l,m) \cap K_2(l,m) = \emptyset,$$

and $K(l,m)$ admits the representation

$$K(l,m) = K_1(l,m) \cup K_2(l,m).$$

The fundamental solution of $K_1(l,m)$ is $< -1, 1 >$ and the fundamental solution of $K_2(l,m)$ is $< 1, 1 >$.

Moreover, if $< u, v >$ runs $K(u,v)$, then:

(b$_1$) $< l, m >$ runs $K_1(l,m)$ iff the equality

$$l + m\sqrt{6} = (-1 + \sqrt{6})(u + v\sqrt{6}) \tag{10}$$

holds;

(b$_2$) $< l, m >$ runs $K_2(l,m)$ iff the equality

$$l + m\sqrt{6} = (1 + \sqrt{6})(u + v\sqrt{6}) \tag{11}$$

holds.

We must note that the fundamental solution of (3) is $< 5, 2 >$. Let u_n and v_n be given by

$$u_n + v_n\sqrt{6} = (5 + 2\sqrt{6})^n \quad (n \in \mathcal{N}. \tag{12}$$

Then u_n and v_n satisfy (11) and $< u_n, v_n > \in \mathcal{N}^2$. Moreover, if n runs $\mathcal{N}$, then $< u_n, v_n >$ runs $K^o(u,v)$.

Let the sets $K_i^o(l,m)$ $(i = 1, 2)$ are introduced by

$$K_i^o(l,m) = K_i(l,m) \cap \mathcal{N}^2. \tag{13}$$

As a corollary from the above remark and Theorem 1 we obtain

THEOREM 2: The set $K^o(l,m)$ may be represented as

$$K^o(l,m) = K_1^o(l,m) \cup K_2^o(l,m), \tag{14}$$

where

$$K_1^o(l,m) \cap K_2^o(l,m) = \emptyset. \tag{15}$$

Moreover:

(c$_1$) If n runs $\mathcal{N}$ and the integers l_n and m_n are defined by

$$l_n + m_n\sqrt{6} = (-1 + \sqrt{6})(5 + 2\sqrt{6})^n, \tag{16}$$

then l_n and m_n satisfy (2) and $< l_n, m_n >$ runs $K_1^o(l,m)$;

(c$_2$) If n runs $\mathcal{N} \cup \{0\}$ and the integers l_n and m_n are defined by

$$l_n + m_n\sqrt{6} = (1 + \sqrt{6})(5 + 2\sqrt{6})^n, \tag{17}$$

then l_n and m_n satisfy (2) and $< l_n, m_n >$ runs $K_2^o(l,m)$.

Let φ be the above mentioned bijection. The sets $K_i'^o(x,y)$ $(i = 1,2)$ are introduced by

$$K_i'^o(x,y) = \varphi(K_i^o(l,m)). \tag{18}$$

From Theorem 2, and especially from (14), (15), and (18) we obtain

THEOREM 3: The set $K'^o(x,y)$ may have the representation

$$K'^o(x,y) = K_1^o(x,y) \cup K_2^o(x,y), \tag{19}$$

where

$$K_1^o(x,y) \cap K_2^o(x,y) = \emptyset. \tag{20}$$

Moreover:

(d_1) If n runs $\mathcal{N}$ and the integers x_n and y_n are defined by

$$x_n = l_n + 3m_n \text{ and } y_n = l_n + 2m_n, \tag{21}$$

where l_n and m_n are introduced by (16), then x_n and y_n satisfy (1) and $< x_n, y_n >$ runs $K_1^o(x,y)$;

(d_2) If n runs $\mathcal{N} \cup \{0\}$ and the integers x_n and y_n are defined again by (21), but l_n and m_n now are introduced by (17), then x_n and y_n satisfy (1) and $< x_n, y_n >$ runs $K_2^o(x,y)$.

Theorem 3 completely solves F. Smarandache's Problem 78 from [1], because l_n and m_n could be expressed in explicit form using (16) or (17) as well.

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Below we shall introduce a generalization of Smarandache's problem 87 from [1].

If we have to consider the Diophantine equation

$$2x^2 - 3y^2 = p, \tag{22}$$

where $p \neq 2$ is a prime number, then using [2, Ch. VII, exercize 2] and the same method as in the case of (1), we obtain the following result.

THEOREM 4: (1) The necessary and sufficient condition for the solvability of (22) is:

$$p \equiv 5 (mod 24) \quad \text{or} \quad p \equiv 23 (mod 24) \qquad (23);$$

(2) If (23) is valid, then there exists exactly one solution $< x, y > \in \mathcal{N}^2$ of (22) such that the inequalities $x < \sqrt{\frac{3}{2}.p}; y < \sqrt{\frac{2}{3}.p}$ hold. Every other solution $< x, y > \in \mathcal{N}^2$ of (22) has the form:

$$x = l + 3m$$

$$y = l + 2m,$$

where $< l, m > \in \mathcal{N}^2$ is a solution of the Diophantine equation

$$l^2 - 6m^2 = -p.$$

The question how to solve the Diophantine equation, a special case of which is the above one, is considered in Theorem 110 from [2].

REFERENCES:

[1] F. Smarandache, Proposed Problems of Mathematics. Vol. 2, U.S.M., Chişinău, 1997.
[2] T. Nagell, Introduction to Number Theory. John Wiley & Sons, Inc., New York, 1950.

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