ON RADU'S PROBLEM

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For a positive integer n, the Smarandache function S(n) is defined as the smallest positive integer such that S(n)! is divisible by n. Radu [1] noticed that for nearly all values of n up to 4800 there is always at least one prime number between S(n) and S(n+1) including possibly S(n) and S(n+1). The exceptions are n=224 for which S(n)=8 and S(n+1)=10 and n=2057 for which S(n)=22 and S(n+1)=21. Radu conjectured that, except for a finite set of numbers, there exists at least one prime number between S(n) and S(n+1). The conjecture does not hold if there are infinitely many solutions to the following problem.

Find consecutive integers n and n+1 for which two consecutive primes p_k and p_{k+1} exist so that $p_k < Min(S(n),S(n+1))$ and $p_{k+1} > Max(S(n),S(n+1))$.

Consider

$$n+1 = xp_r^s \tag{1}$$

and

$$n = yp_{r+1}^s \tag{2}$$

where p_r and p_{r+1} are consecutive prime numbers. Subtract (2) form (1).

$$xp_r^s - yp_{r+1}^s = 1 \tag{3}$$

The greatest common divisor $(p_r^s, p_{r+1}^s) = 1$ divides the right hand side of (3) which is the condition for this diophantine equation to have infinitely many integer solutions. We are interested in positive integer solutions (x,y) such that the following conditions are met.

$$S(n+1) = sp_r, \text{ i.e } S(x) < sp_r \tag{4}$$

$$S(n) = sp_{r+1}, \text{ i.e } S(y) < sp_{r+1} \tag{5}$$

In addition we require that the interval

$$sp_r^s < q < sp_{r+1}^s \text{ is prime free, i.e. q is not a prime.} \tag{6}$$

Euclid's algorithm is used to obtain principal solutions (x_0,y_0) to (3). The general set of solutions to (3) are then given by

$$x = x_0 + p_{r+1}^s t, \qquad y = y_0 - p_r^s t \tag{7}$$

with t an integer.

These algorithms were implemented for different values of the parameters $d=p_{r+1}-p_r$, s and t resulted in a very large number of solutions. Table 1 shows the 20 smallest (in respect of n) solutions found. There is no indication that the set would be finite. A pair of primes may produce several solutions.

Table 1. The 20 smallest solutions which ocurred for s=2 and d=2.

#	n	S(n)	S(n+1)	P1	P2	t
1	265225	206	202	199	211	0
2	843637	302	298	293	307	0
3	6530355	122	118	113	127	-1
4	24652435	926	922	919	929	0
5	35558770	1046	1042	1039	1049	0
6	40201975	142	146	139	149	1
7	45388758	122	118	113	127	-4
8	46297822	1142	1138	1129	1151	0
9	67697937	214	218	211	223	0
10	138852445	1646	1642	1637	1657	0
11	157906534	1718	1714	1709	1721	0
12	171531580	1766	1762	1759	1777	0
13	299441785	2126	2122	2113	2129	0
14	551787925	2606	2602	2593	2609	0
15	1223918824	3398	3394	3391	3407	0
16	1276553470	3446	3442	3433	3449	0
17	1655870629	3758	3754	3739	3761	0
18	1853717287	3902	3898	3889	3907	0
19	1994004499	3998	3994	3989	4001	0
20	2256222280	4166	4162	4159	4177	0

Within the limits set by the design of the program the largest prime difference for which a solution was found is d=42 and the largest exponent which produced solutions is s=4. Some numerically large examples illustrating the these facts are given in table 2.

Table 2.

n/n+1	S(n)/ S(n+1)	d	s	t	p_r/p_{r+1}
11822936664715339578483018	3225562	42	2	-2	1612781
11822936664715339578483017	3225646				1612823
111579064978581002637386836 34	165999	4	3	0	55333
11157906497858100263738683635	166011				55337
17549865213221162413502236227	16599	4	3	-1	55333
17549865213221162413502236226	166011				55337
270329975921205253634707051822848570391314	669764	2	4	0	167441
270329975921205253634707051822848570391313	669772				167443

To see the relation between these large numbers and the corresponding values of the Smarandache function in table 2 the factorisations of these large numbers are given below:

$118229366647153395784830 18 = 2 \cdot 3 \cdot 89 \cdot 193 \cdot 431 \cdot 1612781^2$
$11822936664715339578483017 = 509 \cdot 3253 \cdot 1612823^2$

$1115790649785810026373868 3634 = 2 \cdot 7 \cdot 37^2 \cdot 56671 \cdot 55333^3$
$11157906497858100263738683635 = 3 \cdot 5 \cdot 11 \cdot 19^2 \cdot 16433 \cdot 55337^3$

$17549865213221162413502236227 = 3 \cdot 11^2 \cdot 307 \cdot 12671 \cdot 55333^3$
$17549865213221162413502236226 = 2 \cdot 23 \cdot 37 \cdot 71 \cdot 419 \cdot 743 \cdot 55337^3$

$270329975921205253634707051822848570391314 = 2 \cdot 3^3 \cdot 47 \cdot 1289 \cdot 2017 \cdot 119983 \cdot 167441^4$
$270329975921205253634707051822848570391313 = 37 \cdot 23117 \cdot 24517 \cdot 38303 \cdot 167443^4$

It is also interesting to see which are the nearast smaller P_k and nearast bigger P_{k+1} primes to S_1 = Min(S(n),S(n+1)) and S_2 = Max(S(n),S(n+1)) respectively. This is shown in table 3 for the above examples.

Table 3.

P_k	S_1	S_2	P_{k+1}	$G=P_{k+1}-P_k$
3225539	3225562	3225646	3225647	108
165983	165999	166011	166013	30
669763	669764	669772	669787	24

Conclusion: There are infintely many intervals {Min(S(n),S(n-1)),Max(S(n),S(n-1))} which are prime free.

References:
I. M. Radu, *Mathematical Spectrum, Sheffield Univeristy, UK*, Vol. 27, No.2, 1994/5, p. 43.