ON A EQUATION OF SMARANDACHE AND ITS INTEGER SOLUTIONS*

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ABSTRACT. Let Q denotes the set of all rational numbers, $a \in Q\backslash\{-1,0,1\}$. The main purpose of this paper is to prove that the equation

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^{x} = 2a$$

has one and only one integer solution $x = 1$. This solved a problem of Smarandache in book [1].

1. INTRODUCTION

Let Q denotes the set of all rational numbers, $a \in Q\backslash\{-1,0,1\}$. In problem 50 of book [1], Professor F.Smarandache ask us to solve the equation

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^{x} = 2a. \tag{1}$$

About this problem, it appears that no one had studied it yet, at least, we have not seen such a result before. The problem is interesting because it can help us to understand some new indefinite equations. In this paper, we use elementary method and analysis method to study the equation (1), and prove the following conclusion:

Theorem. *For all $a \in Q\backslash\{-1,0,1\}$, the equation*

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^{x} = 2a$$

has one and only one integer solution $x = 1$.

Key words and phrases. F.Smarandache equation; Integer solution; One and only one solution.
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2. Proof of the Theorem

In this section, we use elementary methods and the Rolle's Theorem in mathematical analysis to complete the proof of the Theorem. First we prove that the Theorem holds for $a > 1$. In fact in this case, let x is an integer solution of (1), we must have $x > 0$. Then using the inequality $|u| + |v| \geq 2\sqrt{|u| \cdot |v|}$ we have

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^x \geq 2 \cdot \sqrt{x \cdot a^{\frac{1}{x}} \cdot \frac{1}{x} \cdot a^x} = 2 \cdot a^{\frac{x + \frac{1}{x}}{2}} \geq 2 \cdot a,$$

and the equality holds if and only if $x = 1$. This proved that for $a > 1$, the equation (1) has one and only one integer solution $x = 1$.

Now we consider $0 < a < 1$. Let x_0 is any integer solution of (1), then from equation (1) we know that $x_0 > 0$. To prove $x_0 = 1$, we suppose $x_0 \neq 1$, let $0 < x_0 < 1$ (the proof for case $x_0 > 1$ is the same as for $0 < x_0 < 1$), then $\frac{1}{x_0} > 1$, we define the function $f(x)$ as follows:

$$f(x) = x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^x - 2a$$

It is clear that $f(x)$ is a continuous function in the closed interval $\left[x_0, \frac{1}{x_0}\right]$, and a derivable function in the open interval $\left(x_0, \frac{1}{x_0}\right)$, and more $f(x_0) = f(\frac{1}{x_0}) = f(1) = 0$. So from the Rolle's Theorem in mathematical analysis we know that $f'(x)$ must have two zero points in the open interval $\left(x_0, \frac{1}{x_0}\right)$, and $f''(x)$ must have one zero point in the same open interval. But from the definition of $f(x)$ we have

$$f'(x) = a^{\frac{1}{x}} - \frac{1}{x} \cdot a^{\frac{1}{x}} \cdot \ln a - \frac{1}{x^2} \cdot a^x + \frac{1}{x} \cdot a^x \cdot \ln a$$

and

$$\begin{aligned} f''(x) &= \frac{1}{x^3} \cdot a^{\frac{1}{x}} \cdot \ln^2 a + \frac{2}{x^3} \cdot a^x - \frac{1}{x^2} \cdot a^x \cdot \ln a - \frac{1}{x^2} \cdot a^x \cdot \ln a + \frac{1}{x} \cdot a^x \cdot \ln^2 a \\ &= \frac{1}{x^3} \cdot a^{\frac{1}{x}} \cdot \ln^2 a + \frac{2}{x^3} \cdot a^x + \frac{2}{x^2} \cdot a^x \cdot \ln \frac{1}{a} + \frac{1}{x} \cdot a^x \cdot \ln^2 a \\ &> 0, \quad x \in \left(x_0, \frac{1}{x_0}\right), \end{aligned}$$

where we have used $0 < a < 1$ and $\ln \frac{1}{a} > 0$. This contradiction with that $f''(x)$ must have one zero point in the open interval $\left(x_0, \frac{1}{x_0}\right)$. This proved that the Theorem holds for $0 < a < 1$.

If $a < 0$ and $a \neq -1$, and equation (1) has an integer solution x, then $|x|$ must be an odd number, because negative number has no real square root. So in this case, the equation (1) become the following equation:

$$2|a| = -2a = -x \cdot a^{\frac{1}{x}} - \frac{1}{x} \cdot a^x = -x \cdot (-1)^{\frac{1}{x}} \cdot |a|^{\frac{1}{x}} - \frac{1}{x} \cdot (-1)^x \cdot |a|^x$$
$$= x \cdot |a|^{\frac{1}{x}} + \frac{1}{x} \cdot |a|^x.$$

Then from the above conclusion we know that the Theorem is also holds. This completes the proof of the Theorem.

Note. In fact from the process of the proof of the Theorem we can easily find that we have proved a more general conclusion:

Theorem. *Let R denotes the set of all real numbers. For any $a \in R \setminus \{-1, 0, 1\}$, the equation*

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^x = 2a$$

has one and only one integer solution $x = 1$; It has one and only one real number solution $x = 1$, if $a > 0$.

References

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