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The n^{th} Power Signed Graphs-II

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. In this paper, we present solutions of some signed graph switching equations involving the line signed graph, complement and n^{th} power signed graph operations.

Keywords: Smarandachely k-signed graphs, Smarandachely k-marked graphs, signed graphs, marked graphs, balance, switching, line signed graph, complementary signed graph, n^{th} power signed graph.

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§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [6]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (See [7]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

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A *marking* of S is a function $\mu : V(G) \to \{+,-\}$; A signed graph S together with a marking μ by S_μ.

The following characterization of balanced signed graphs is well known.

Proposition 1.1(E. Sampathkumar [8]) *A signed graph $S = (G,\sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Given a marking μ of S, by *switching* S with respect to μ we mean reversing the sign of every edge of S whenever the end vertices have opposite signs in $\mathcal{S}_\mu$ [1]. We denote the signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and this signed graph is called the *μ-switched signed graph* or just *switched signed graph*. A signed graph S_1 switches to a signed graph S_2 (that is, they are *switching equivalent* to each other), written $S_1 \sim S_2$, whenever there exists a marking μ such that $\mathcal{S}_\mu(S_1) \cong S_2$.

Two signed graphs $S_1 = (G,\sigma)$ and $S_2 = (G',\sigma')$ are said to be *weakly isomorphic* (see [13]) or *cycle isomorphic* (see [14]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [14]):

Proposition 1.2(T. Zaslavsky [14]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

Behzad and Chartrand [4] introduced the notion of line signed graph $L(S)$ of a given signed graph S as follows: Given a signed graph $S = (G,\sigma)$ its *line signed graph* $L(S) = (L(G),\sigma')$ is the signed graph whose underlying graph is $L(G)$, the line graph of G, where for any edge e_ie_j in $L(S)$, $\sigma'(e_ie_j)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [5], is as follows:

The *line signed graph* of a signed graph $S = (G,\sigma)$ is a signed graph $L(S) = (L(G),\sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [5] (See also E. Sampathkumar et al. [9]).

Proposition 1.3(**M. Acharya** [2]) *For any signed graph $S = (G,\sigma)$, its line signed graph $L(S) = (L(G),\sigma')$ is balanced.*

For any positive integer k, the k^{th} iterated line signed graph, $L^k(S)$ of S is defined as follows:

$$L^0(S) = S,\ L^k(S) = L(L^{k-1}(S)).$$

Corollary 1.4 *For any signed graph $S = (G,\sigma)$ and for any positive integer k, $L^k(S)$ is balanced.*

Let $S = (G,\sigma)$ be a signed graph. Consider the marking μ on vertices of S defined as follows: for each vertex $v \in V$, $\mu(v)$ is the product of the signs on the edges incident with v. *The complement* of S is a signed graph $\overline{S} = (\overline{G},\sigma^c)$, where for any edge $e = uv \in \overline{G}$,

$\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S}$ as defined here is a balanced signed graph due to Proposition 1.1.

§2. n^{th} Power signed graph

The n^{th} *power graph* G^n of G is defined in [3] as follows:

The n^{th} power has same vertex set as G, and has two vertices u and v adjacent if their distance in G is n or less.

In [12], we introduced a natural extension of the notion of n^{th} power graphs to the realm of signed graphs: Consider the marking μ on vertices of S defined as follows: for each vertex $v \in V$, $\mu(v)$ is the product of the signs on the edges incident at v. The n^{th} *power signed graph* of S is a signed graph $S^n = (G^n, \sigma')$, where G^n is the underlying graph of S^n, where for any edge $e = uv \in G^n$, $\sigma'(uv) = \mu(u)\mu(v)$.

The following result indicates the limitations of the notion of n^{th} power signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to n^{th} power signed graphs.

proposition 2.1(P. Siva Kota Reddy et al.[12]) *For any signed graph $S = (G, \sigma)$, its n^{th} power signed graph S^n is balanced.*

For any positive integer k, the k^{th} iterated n^{th} power signed graph, $(S^n)^k$ of S is defined as follows:

$$(S^n)^0 = S,\ (S^n)^k = S^n((S^n)^{k-1}).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and any positive integer k, $(S^n)^k$ is balanced.*

The *degree* of a signed graph switching equation is then the maximum number of operations on either side of an equation in standard form. For example, the degree of the equation $S \sim \overline{L(S)}$ is one, since in standard form it is $L(S) \sim \overline{S}$, and there is one operation on each side of the equation. In [12], the following signed graph switching equations are solved:

- $\overline{S} \sim (L(S))^n$ (1)
- $L(\overline{S}) \sim (L(S))^n$ (2)
- $\overline{L(S)} \sim \overline{S}^n$, where $n \geq 2$ (3)
- $L^2(S) \sim S^n$, where $n \geq 2$ (4)
- $L^2(S) \sim \overline{S^n}$, where $n \geq 2$, and (5)
- $L^2(S) \sim \overline{S}^n$, where $n \geq 2$. (6)

Recall that $L^2(S)$ is the second iterated line signed graph S.

Several of these signed graph switching equations can be viewed as generalized of earlier work [11]. For example, equation (1) is a generalization of $L(S) \sim \overline{S}$, which was solved by Siva Kota Reddy and Subramanya [11]. When $n = 1$ in equations (3) and (4), we get $L(S) \sim S$ and $L^2(S) \sim S^2$, which was solved in [11]. If $n = 1$ in (5) and (6), the resulting signed graph switching equation was solved by Siva Kota Reddy and Subramanya [11].

Further, in this paper we shall solve the following three signed graph switching equations:

- $L(S) \sim S^n$ (7)
- $\overline{L(S)} \sim S^n \quad (or L(S) \sim \overline{S^n})$ (8)
- $L(S) \sim (\overline{S})^n$ (9)

In the above expressions, the equivalence (i.e, $\sim$) means the switching equivalent between corresponding graphs.

Note that for $n = 1$, the equation (7) is reduced to the following result of E. Sampathkumar et al. [10].

Proposition 2.3(E. Sampathkumar et al. [10]) *For any signed graph $S = (G, \sigma)$, $L(S) \sim S$ if, and only if, S is a balanced signed graph and G is 2-regular.*

Note that for $n = 1$, the equations (8) and (9) are reduced to the signed graph switching equation which is solved by Siva Kota Reddy and Subramanya [11].

Proposition 2.4 (P. Siva Kota Reddy and M. S. Subramanya [11]) *For any signed graph $S = (G, \sigma)$, $L(S) \sim \overline{S}$ if, and only if, G is either C_5 or $K_3 \circ K_1$.*

§3. The Solution of $L(S) \sim S^n$

We now characterize signed graphs whose line signed graphs and its n^{th} power line signed graphs are switching equivalent. In the case of graphs the following result is due to J. Akiyama et. al [3].

Proposition 3.1(J. Akiyama et al. [3]) *For any $n \geq 2$, the solutions to the equation $L(G) \cong G^n$ are graphs $G = mK_3$, where m is an arbitrary integer.*

Proposition 3.2 *For any signed graph $S = (G, \sigma)$, $L(S) \sim S^n$, where $n \geq 2$ if, and only if, G is mK_3, where m is an arbitrary integer.*

Proof Suppose $L(S) \sim S^n$. This implies, $L(G) \cong G^n$ and hence by Proposition 3.1, we see that the graph G must be isomorphic to mK_3.

Conversely, suppose that G is mK_3. Then $L(G) \cong G^n$ by Proposition 3.1. Now, if S is a signed graph with underlying graph as mK_3, by Propositions 1.3 and 2.1, $L(S)$ and S^n are balanced and hence, the result follows from Proposition 1.2. $\square$

§4. Solutions of $\overline{L(S)} \sim S^n$

In the case of graphs the following result is due to J. Akiyama et al. [3].

Proposition 4.1(J. Akiyama et al. [3]) *For any $n \geq 2$, $G = C_{2n+3}$ is the only solution to the equation $\overline{L(G)} \cong G^n$.*

Proposition 4.2 *For any signed graph $S = (G, \sigma)$, $\overline{L(S)} \sim S^n$, where $n \geq 2$ if, and only if, G is C_{2n+3}.*

Proof Suppose $\overline{L(S)} \sim S^n$. This implies, $\overline{L(G)} \cong G^n$ and hence by Proposition 4.1, we see that the graph G must be isomorphic to C_{2n+3}.

Conversely, suppose that G is C_{2n+3}. Then $\overline{L(G)} \cong G^n$ by Proposition 4.1. Now, if S is a signed graph with underlying graph as C_{2n+3}, by definition of complementary signed graph and Proposition 2.1, $\overline{L(S)}$ and S^n are balanced and hence, the result follows from Proposition 1.2. □

In [3], the authors proved there are no solutions to the equation $L(G) \cong (\overline{G})^n, n \geq 2$. So its very difficult, in fact, impossible to construct switching equivalence relation of $L(S) \sim (\overline{S})^n$.

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