The curvature tensor of the stationary accelerated frame in the gravity field

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ABSTRACT
In the general relativity theory, we define the accelerated frame that moves in \( \hat{r} \)-axis in the curved time-space. And we calculate the curvature tensor of the stationary accelerated frame in the gravity field. In this time, the curvature tensor divide the observational curvature tensor of the people and the curvature tensor of the people’s self on the planet in the gravity field.

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             The curved time-space,
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1. Introduction

This theory’s object is that defines the accelerated frame that moves in $\hat{r}$-axis in the curved space-time. The Schwarzschild solution is

$$d\tau^2 = (1 - \frac{2GM}{rc^2})dt^2 - \frac{1}{c^2} \left[ \frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$  \hspace{1cm} (1)

In this time, a moving matter’s acceleration is $\mathbf{a}$ in the Schwarzschild space-time.

$$a = a_{\text{inertial}} - g = \frac{d}{dt} \left( \frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right), u = \sqrt{1 - \frac{2GM}{rc^2}}$$  \hspace{1cm} (2)

$a_{\text{inertial}}$ is the inertial acceleration, $g$ is the pure gravity acceleration.

If $a_0 = a / \sqrt{1 - \frac{2GM}{rc^2}} = -g / \sqrt{1 - \frac{2GM}{rc^2}}$, $a_{\text{inertial}} = 0$ is

$$a_0 = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{d}{dt} \left( \frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right) = \frac{d}{dt} \left( \frac{V}{\sqrt{1 - \frac{V^2}{c^2}}} \right),$$

$$V = \frac{df}{dt} = \frac{dr}{dt} \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}}, \quad df = dt \sqrt{1 - \frac{2GM}{rc^2}}, \quad \dot{f} = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}}$$

$$a_0 \hat{f} = \frac{V}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad V = \sqrt{1 + a_0^2 \hat{f}^2}, \quad V \text{ is the } \hat{f} \text{-axis’s velocity}$$  \hspace{1cm} (3)

If $\frac{d\theta}{dt} = \frac{d\phi}{dt} = 0$, the solution is

$$d\tau^2 = (1 - \frac{2GM}{rc^2})dt^2 - \frac{1}{c^2} \frac{dr^2}{1 - \frac{2GM}{rc^2}} = \dot{f}^2 - \frac{1}{c^2} \dot{\theta}^2 = \dot{f}^2 \left( 1 - \frac{V^2}{c^2} \right)$$

$$= \frac{df^2}{\sqrt{1 + a_0^2 \hat{f}^2}}$$  \hspace{1cm} (4)

In this time,
\[ \tau = \int d\tau = \int \frac{df}{\sqrt{1 + \frac{a_0^2 \dot{t}^2}{c^2}}} = \frac{c}{a_0} \sinh^{-1}(\frac{a_0 \dot{t}}{c}). \]

\[ \dot{t} = \frac{c}{a_0} \sinh(\frac{a_0 \tau}{c}), \quad \ddot{t} = \int Vd\dot{t} = \int \frac{a_0 \dot{t}d\dot{t}}{\sqrt{1 + \frac{a_0^2 \dot{t}^2}{c^2}}} = \frac{c^2}{a_0} \int \frac{a_0^2 \dot{t}^2}{c^2} \quad \frac{c^2}{a_0} \cosh(\frac{a_0 \tau}{c}) \]

\[ \frac{df}{d\tau} = \cosh(\frac{a_0 \tau}{c}). \quad \frac{d\ddot{t}}{d\tau} = \sinh(\frac{a_0 \tau}{c}) \quad \tag{5} \]

2. The tetrad in the curved space-time

The tetrad \( \mathbf{e}_\mu^\hat{\alpha} \) is the unit vector defined by the following formula.

\[ \eta_{\hat{a}\hat{b}} \mathbf{e}_\mu^\hat{a} \mathbf{e}_\nu^\hat{b} = g_{\mu\nu} \quad \tag{6} \]

In this time, if a matter moves in \( \hat{t} \)-axis in the curved space-time,

\[ \eta_{\hat{a}\hat{b}} \mathbf{e}_\mu^\hat{a}(\tau) \mathbf{e}_\nu^\hat{b}(\tau) = g_{\mu\nu} \quad \eta_{\hat{a}\hat{b}} = \eta_{\hat{b}\hat{a}} \cap g_{\hat{a}\hat{b}} = g_{\hat{b}\hat{a}} \quad \tag{7} \]

Hence, Eq(6), Eq(7) is

\[ \eta_{\hat{a}\hat{b}} \mathbf{e}_\mu^\hat{a}(\tau) \mathbf{e}_\nu^\hat{b}(\tau) = \eta_{\hat{0}\hat{0}} = -1 \quad \tag{8} \]

\[ d\tau^2 = -\frac{1}{c^2} \eta_{\hat{a}\hat{b}} d\mathbf{x}^\hat{a} d\mathbf{x}^\hat{b} \]

\[ \rightarrow -1 = \eta_{\hat{a}\hat{b}} \left( \frac{1}{c} \frac{d\mathbf{x}^\hat{a}}{d\tau} \right) \left( \frac{1}{c} \frac{d\mathbf{x}^\hat{b}}{d\tau} \right) = \eta_{\hat{a}\hat{b}} \mathbf{e}_\mu^\hat{a}(\tau) \mathbf{e}_\nu^\hat{b}(\tau) \]

\[ \mathbf{x}^\hat{a} = (c\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}) \quad \tag{9} \]

According to Eq(5), Eq(9)

\[ \mathbf{e}_\mu^\hat{0}(\tau) = \frac{1}{c} \frac{d\mathbf{x}_\mu}{d\tau} = (\cosh(\frac{a_0 \tau}{c}), \sinh(\frac{a_0 \tau}{c}), 0, 0) \quad \tag{10} \]

About \( \hat{\theta} \)-axis’s and \( \hat{\phi} \)-axis’s orientation

\[ \eta_{\hat{2}\hat{2}} \mathbf{e}_\mu^{\hat{2}}(\tau) \mathbf{e}_\nu^{\hat{2}}(\tau) = \eta_{\hat{2}\hat{2}} = 1, \quad \mathbf{e}_\mu^{\hat{2}}(\tau) = (0, 0, 1) \]
\[ \eta_{33} e^{\hat{3}}(r) e^{\hat{3}}(r) = \eta_{33} = 1, \ e^{\hat{3}}(r) = (0, 0, 0, 1) \] (11)

And the other vector \( e^{\hat{1}}(r) \) has to satisfy the tetrad condition, Eq (6), Eq (7)

\[ e^{\hat{1}}(r) = (\sinh(\frac{\mathbf{a}_0 r}{c}), \cosh(\frac{\mathbf{a}_0 r}{c}), 0, 0) \] (12)

In this time,

\[ \overline{\sigma}^\rho_i = (1/\sqrt{1 - \frac{2GM}{r c^2}}, 0, 0, 0), \overline{\sigma}^\rho_j = (0, \sqrt{1 - \frac{2GM}{r c^2}}, 0, 0) \]

\[ \overline{\sigma}^\rho_\theta = (0, 0, 1/r, 0), \overline{\sigma}^\rho_\phi = (0, 0, 0, 1/r \sin \theta) \]

\[ g_{\rho \sigma} \overline{\sigma}^\rho_i \overline{\sigma}^\sigma_j = \eta_{i j} \] (13)

\[ \frac{\mathbf{a}_0}{c} \hat{t} = \sinh(\frac{\mathbf{a}_0}{c} r) \sqrt{1 - \frac{v}{c}} \sqrt{1 + \frac{\mathbf{a}_0^2 t^2}{c^2}} = \cosh(\frac{\mathbf{a}_0}{c} r) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \] (14)

Therefore, the Lorentz transformation \( \mathbf{B}^{\hat{\nu}}_{\hat{\mu}}(\nu) \) is

\[ \mathbf{B}^{\hat{\nu}}_{\hat{\mu}}(\nu) = \begin{pmatrix}
\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} & \frac{v}{c} & 0 & 0 \\
\frac{\mathbf{v}}{c} & \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} & 0 & 0 \\
0 & 0 & \sqrt{1 - \frac{v^2}{c^2}} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \]

\[ = e^{\hat{\nu}}_{\hat{\mu}}(r) = \begin{pmatrix}
\cosh(\frac{\mathbf{a}_0}{c} r) & \sinh(\frac{\mathbf{a}_0}{c} r) & 0 & 0 \\
\sinh(\frac{\mathbf{a}_0}{c} r) & \cosh(\frac{\mathbf{a}_0}{c} r) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \] (15)

\[ \overline{\sigma}^\rho_{\hat{\nu}} = \mathbf{B}^{\hat{\nu}}_{\hat{\mu}}(\nu) \overline{\sigma}^\rho_{\hat{\mu}} = e^{\hat{\nu}}_{\hat{\mu}}(r) \overline{\sigma}^\rho_j \] (16)

Hence,

\[ g_{\rho \sigma} \overline{\sigma}^\rho_i \overline{\sigma}^\sigma_j = \eta_{i j} \]

\[ g_{\rho \sigma} \mathbf{B}^{\hat{\nu}}_{\hat{\mu}}(\nu) \overline{\sigma}^\rho_i \overline{\sigma}^\sigma_j = g_{\rho \sigma} \overline{\sigma}^\rho_{\hat{\nu}} \overline{\sigma}^\sigma_{\hat{\mu}} = \eta_{i j} e^{\hat{\nu}}_{\hat{\mu}}(r) e^{\hat{\mu}}_{\hat{\nu}}(r) = \eta_{i j} \] (17)
3. The accelerated frame in the curved space-time.

About the accelerated frame $\xi$ in the curved space-time,

$$d\tau^2 = -\frac{1}{c^2} \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = -\frac{1}{c^2} \eta_{\alpha\beta} \frac{\partial \hat{\xi}^\alpha}{\partial \xi^\mu} \frac{\partial \hat{\xi}^\beta}{\partial \xi^v} d\xi^\mu d\xi^v$$

$$= -\frac{1}{c^2} \eta_{\alpha\beta} e^{\hat{\alpha} \mu} e^{\hat{\beta} \nu} d\xi^\mu d\xi^v$$

$$= -\frac{1}{c^2} g_{\mu\nu} d\xi^\mu d\xi^v$$

$$e^{\hat{\alpha} \mu} = \frac{\partial \hat{\xi}^\alpha}{\partial \xi^\mu}, \quad \frac{\partial e^{\hat{\alpha} \beta}}{\partial \xi^1} = \frac{\partial^2 \hat{\xi}^\alpha}{\partial \xi^0 \partial \xi^1} = \frac{\partial e^{\hat{\alpha} \beta}}{\partial \xi^0}$$

(18)

$$e^{\hat{\alpha} \beta}(\xi^0) = \frac{1}{c} \frac{\partial \hat{\xi}^\alpha}{\partial \xi^0} = (1 + \frac{a_{0,21}}{c^2}) \cosh(\frac{a_{0,20}}{c}), (1 + \frac{a_{0,21}}{c^2}) \sinh(\frac{a_{0,20}}{c}), 0, 0)$$

(20)

$$e^{\hat{\alpha} \beta}(\xi^0) = \frac{\partial \hat{\xi}^\alpha}{\partial \xi^1} = \sinh(\frac{a_{0,20}}{c}), \cosh(\frac{a_{0,20}}{c}), 0, 0)$$

(21)

$$e^{\hat{\alpha} \beta}(\xi^0) = \frac{\partial \hat{\xi}^\alpha}{\partial \xi^2} = (0, 0, 1, 0), e^{\hat{\alpha} \beta}(\xi^0) = (0, 0, 1, 0)$$

(22)

$$d\xi^\alpha \frac{\partial \hat{\xi}^\alpha}{\partial \xi^\mu} d\xi^\mu = e^{\hat{\alpha} \beta}(\xi^0) d\xi^\beta + e^{\hat{\alpha} \beta}(\xi^0) d\xi^\beta + e^{\hat{\alpha} \beta}(\xi^0) d\xi^\beta$$

(23)

Hence,

$$c d\bar{t} = c dt \sqrt{1 - \frac{2GM}{rc^2}} = (1 + \frac{a_{0,21}}{c^2}) \cosh(\frac{a_{0,20}}{c}) d\xi^\beta + \sinh(\frac{a_{0,20}}{c}) d\xi^1$$

$$d\bar{t} = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} = (1 + \frac{a_{0,21}}{c^2}) \sinh(\frac{a_{0,20}}{c}) d\xi^\beta + \cosh(\frac{a_{0,20}}{c}) d\xi^1$$

$$d\theta = d\xi^2, \quad d\phi = d\xi^3$$

(24)
\[ d\tau^2 = (1 - \frac{2GM}{rc^2})dt^2 - \frac{1}{c^2} \left[ \frac{dr^2}{c^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]

\[ = dt^2 - \frac{1}{c^2} \left[ dr^2 + d\theta^2 + d\phi^2 \right] \]

\[ = (1 + \frac{a_0 \xi^1}{c^2})^2 (d\xi^0)^2 - \frac{1}{c^2} [(d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2] \]  \hspace{1cm} (25)

The coordinate transformation is

\[ c\hat{t} = (\frac{c^2}{a_0} + \xi^1) \sinh(\frac{a_0 \xi^0}{c}), \hat{r} = (\frac{c^2}{a_0} + \xi^1) \cosh(\frac{a_0 \xi^0}{c}) - \frac{c^2}{a_0} \]

\[ \hat{\theta} = \xi^2, \hat{\phi} = \xi^3 \]  \hspace{1cm} (26)

The inverse-transformation is

\[ \xi^0 = \frac{c}{a_0} \text{tanh}^{-1}\left(\frac{c\hat{t}}{\hat{r} + \frac{c^2}{a_0}}\right), \xi^1 = \sqrt{(\hat{r} + \frac{c^2}{a_0})^2 - c^2 \hat{t}^2} - \frac{c^2}{a_0} \]

\[ \xi^2 = \hat{\theta}, \xi^3 = \hat{\phi} \]  \hspace{1cm} (27)

If we calculate the curvature tensor \( R_{\mu\nu\lambda\delta}(\xi) \),

\[ R_{\mu\nu\lambda\delta}(\xi) = \frac{\partial^\lambda}{\partial \xi^\alpha} \frac{\partial^\mu}{\partial \xi^\beta} \frac{\partial^\nu}{\partial \xi^\gamma} \frac{\partial^\delta}{\partial \xi^\lambda} R_{\alpha\beta\gamma\delta}(\hat{X}) \]

\[ = e^{\hat{\mu}}(\xi^0)e^{\hat{\nu}}(\xi^0)e^{\hat{\rho}}(\xi^0)e^{\hat{\lambda}}(\xi^0)R_{\alpha\beta\gamma\delta}(\hat{X}) \]  \hspace{1cm} (28)

\[ R_{\hat{t}\hat{t}\hat{t}\hat{t}} = -R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = \frac{2GM}{r^3 c^2}. \]

\[ R_{\hat{t}\hat{t}\hat{t}\hat{t}} = -R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = -\frac{GM}{r^3 c^2} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = -R_{\hat{t}\hat{t}\hat{t}\hat{t}} \]

\[ R_{\hat{t}\hat{t}\hat{t}\hat{t}} = -R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = \frac{2GM}{r^3 c^2} \]

\[ R_{\hat{t}\hat{t}\hat{t}\hat{t}} = -R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = \frac{GM}{r^3 c^2} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{t}\hat{t}} = -R_{\hat{t}\hat{t}\hat{t}\hat{t}} \]  \hspace{1cm} (29)
Therefore,
\[ e^{\xi_0}(\xi^0) = \left(1 + \frac{a_0 \xi^0}{c^2}\right) \cosh\left(\frac{a_0 \xi^0}{c}\right), \left(1 + \frac{a_0 \xi^0}{c^2}\right) \sinh\left(\frac{a_0 \xi^0}{c}\right), 0, 0 \]
\[ e^{\xi_1}(\xi^0) = \left(\sinh\left(\frac{a_0 \xi^0}{c}\right), \cosh\left(\frac{a_0 \xi^0}{c^2}\right), 0, 0 \right) \]
\[ e^{\xi_2}(\xi^0) = (0, 0, 1, 0), e^{\xi_3}(\xi^0) = (0, 0, 1, 0) \]

(30)

\[ R_{0i0j}(\xi) = \frac{2GM}{r^2c^2} (1 + \frac{a_0 \xi^1}{c^2}), R_{0i0j}(\xi) = R_{0j0i}(\xi) = -\frac{GM}{r^3c^2} (1 + \frac{a_0 \xi^1}{c^2}) \]
\[ R_{2323}(\xi) = -\frac{2GM}{r^3c^2}, R_{1212}(\xi) = R_{1313}(\xi) = \frac{GM}{r^3c^2} \]

(31)

Specially, if \( t = 0 \),
\[ u = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{dr}{dt} = 0 \rightarrow V = \frac{dr}{dt} = \frac{a_0 \xi^0}{\sqrt{1 + \frac{a_0^2 \xi^2}{c^2}}} \frac{dr}{dt} = \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)} = 0 \]

(32)

Therefore, if \( t = \xi^0 = 0 \), the theory treats the real situation.

\[ \xi^1 = \sqrt{(\hat{r} + \frac{c^2}{a_0})^2 - c^2 \hat{t}^2 - \frac{c^2}{a_0}} = \hat{r} \]
\[ d\hat{r} = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} \rightarrow \hat{r} = \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{r^2c^2} \ln |\sqrt{r} + \sqrt{r - \frac{2GM}{c^2}}| - \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{r^2c^2} \ln |\sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}}| \]

\[ a_0 = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - u^2}}\right) = \frac{a}{\sqrt{1 - \frac{2GM}{rc^2}}} \]
\[ a = -g = \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - u^2}}\right) \]

\[ g \] is the pure gravity acceleration.
\[ r_0 \] is the location of the stationary accelerated frame

(33)
In this time, in the curved space-time, the curvature tensor $R_{\mu\nu\rho\lambda}(\hat{\xi})$ of the stationary accelerated frame is

$$R_{0101}(\hat{\xi}) = \frac{2GM}{r^3c^2} (1 + \frac{a_0\hat{x}_1}{c^2})^2 = \frac{2GM}{r^3c^2} (1 + \frac{a_0\hat{r}}{c^2})^2$$

$$= \frac{2GM}{r^3c^2} \left[ 1 + \frac{a_0}{c^2} \sqrt{r} \right. \left. \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln |\sqrt{r} + \sqrt{r - \frac{2GM}{c^2}}| \right.$$  

$$\left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln |\sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}}| \right] \}^2$$

$$= \frac{2GM}{r^3c^2} \left[ 1 - \frac{1}{c^2} \left\{ \sqrt{r} \right\} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln |\sqrt{r} + \sqrt{r - \frac{2GM}{c^2}}| \right.$$  

$$\left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln |\sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}}| \right] \}^2$$

$$R_{0202}(\hat{\xi}) = R_{0303}(\hat{\xi}) = -\frac{GM}{r^3c^2} (1 + \frac{a_0\hat{x}_1}{c^2})^2 = -\frac{GM}{r^3c^2} (1 + \frac{a_0\hat{r}}{c^2})^2$$

$$= -\frac{GM}{r^3c^2} \left[ 1 + \frac{a_0}{c^2} \{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln |\sqrt{r} + \sqrt{r - \frac{2GM}{c^2}}| \right.$$  

$$\left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln |\sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}}| \right] \}^2$$

$$= -\frac{GM}{r^3c^2} \left[ 1 - \frac{1}{c^2} \left\{ \sqrt{r} \right\} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln |\sqrt{r} + \sqrt{r - \frac{2GM}{c^2}}| \right.$$  

$$\left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln |\sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}}| \right] \}^2$$

$$R_{2323}(\hat{\xi}) = -\frac{2GM}{r^3c^2}, \quad R_{1212}(\hat{\xi}) = R_{1313}(\hat{\xi}) = \frac{GM}{r^3c^2}$$

$g$ is the pure gravity acceleration.
In this time, in Eq(10), Eq(11), Eq(12), if uses $\xi^0$ instead of $\tau$ and multiply $\exp(\frac{a_0}{c^2} \xi^1)$

$$e^{a_0}(\xi^0) = \frac{1}{c} \frac{\partial \hat{x}^a}{\partial \xi^0} = (\exp(\frac{a_0}{c^2} \xi^0), \exp(\frac{a_0}{c^2} \xi^1), \exp(\frac{a_0}{c^2} \xi^2), \exp(\frac{a_0}{c^2} \xi^3), \exp(\frac{a_0}{c^2} \xi^0), 0, 0)$$ (35)

$$e^{a_1}(\xi^0) = \frac{\partial \hat{x}^a}{\partial \xi^1} = (\exp(\frac{a_0}{c^2} \xi^0), \exp(\frac{a_0}{c^2} \xi^1), \exp(\frac{a_0}{c^2} \xi^2), \exp(\frac{a_0}{c^2} \xi^3), \exp(\frac{a_0}{c^2} \xi^0), 0, 0)$$ (36)

$$e^{a_2}(\xi^0) = \frac{\partial \hat{x}^a}{\partial \xi^2} = (0, 0, 1, 0), e^{a_3}(\xi^0) = \frac{\partial \hat{x}^a}{\partial \xi^3} = (0, 0, 1, 0)$$ (37)

$$d\hat{x}^a = \frac{\partial \hat{x}^a}{\partial \xi^\mu} d\xi^\mu = e^{a_0}(\xi^0) \frac{dc}{dt} + e^{a_1}(\xi^0) \frac{dr}{dt} + e^{a_2}(\xi^0) \frac{d\theta}{dt} + e^{a_3}(\xi^0) \frac{d\phi}{dt}$$ (38)

Hence,

$$cd\hat{t} = cd\tau \sqrt{1 - \frac{2GM}{rc^2}} = \exp(\frac{a_0}{c^2} \xi^1) \frac{dc}{dt} + \exp(\frac{a_0}{c^2} \xi^0) \frac{dr}{dt} + \exp(\frac{a_0}{c^2} \xi^1) \frac{d\theta}{dt} + \exp(\frac{a_0}{c^2} \xi^3) \frac{d\phi}{dt}$$

$$d\tau = \sqrt{1 - \frac{2GM}{rc^2}} c d\hat{t} = \exp(\frac{a_0}{c^2} \xi^1) \frac{dc}{dt} + \exp(\frac{a_0}{c^2} \xi^0) \frac{dr}{dt} + \exp(\frac{a_0}{c^2} \xi^1) \frac{d\theta}{dt} + \exp(\frac{a_0}{c^2} \xi^3) \frac{d\phi}{dt}$$

$$d\hat{\theta} = d\xi^2, \quad d\hat{\phi} = d\xi^3$$ (39)

$$d\tau^2 = (1 - \frac{2GM}{rc^2}) c d\hat{t}^2 - \frac{1}{c^2} \left[ \frac{dc}{d\hat{t}} d\hat{t}^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

$$= d\hat{t}^2 - \frac{1}{c^2} \left[ d\hat{t}^2 + d\hat{\theta}^2 + d\hat{\phi}^2 \right]$$

$$= \exp(2 \frac{a_0}{c^2} \xi^1) (d\xi^0)^2 - \frac{1}{c^2} \left[ \exp(2 \frac{a_0}{c^2} \xi^1) (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 \right]$$ (40)

The coordinate transformation is...
\[ c^\ell = \frac{c^2}{a_0} \exp\left(\frac{a_0}{c^2} \xi^1\right) \sinh\left(\frac{a_0}{c^2} \xi^0\right), \quad \hat{r} = \frac{c^2}{a_0} \exp\left(\frac{a_0}{c^2} \xi^1\right) \cosh\left(\frac{a_0}{c^2} \xi^0\right) - \frac{c^2}{a_0} \]

\[ \hat{\theta} = \hat{\xi}^2, \quad \hat{\phi} = \hat{\xi}^3 \]  

(41)

The inverse-transformation is

\[ \hat{\xi}^0 = \frac{c}{a_0} \tanh^{-1}\left(\frac{c^\ell}{\hat{r} + \frac{c^2}{a_0}}\right), \quad \hat{\xi}^1 = \frac{c^2}{a_0} \ln \left| \frac{a_0}{c^2} \sqrt{\left(\hat{r} + \frac{c^2}{a_0}\right)^2 - c^2 \hat{\xi}^2} \right| \]

\[ \hat{\xi}^2 = \hat{\theta}, \quad \hat{\xi}^3 = \hat{\phi} \]  

(42)

If we calculate the curvature tensor \( R_{\mu\nu\rho\lambda}(\xi) \).

\[ R_{\mu\nu\rho\lambda}(\xi) = \frac{\partial \hat{x}^\mu}{\partial \xi^\alpha} \frac{\partial \hat{x}^\rho}{\partial \xi^\beta} \frac{\partial \hat{x}^\gamma}{\partial \xi^\gamma} \frac{\partial \hat{x}^\delta}{\partial \xi^\lambda} R_{\alpha\beta\gamma\delta}(\hat{X}) \]

\[ = e^\hat{\alpha}_{\hat{\mu}}(\hat{\xi}^0) e^\hat{\beta}_{\hat{\nu}}(\hat{\xi}^0) e^\hat{\gamma}_{\hat{\rho}}(\hat{\xi}^0) e^\hat{\delta}_{\hat{\lambda}}(\hat{\xi}^0) R_{\alpha\beta\gamma\delta}(\hat{X}) \]

(43)

\[ R_{\hat{t}\hat{t}\hat{t}\hat{t}} = -R_{\hat{t}\hat{r}\hat{r}\hat{r}} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -R_{\hat{r}\hat{t}\hat{r}\hat{r}} = -\frac{2GM}{r^3 c^2}, \]

\[ R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -R_{\hat{t}\hat{t}\hat{r}\hat{r}} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -R_{\hat{r}\hat{t}\hat{t}\hat{r}} = -\frac{GM}{r^3 c^2} = R_{\hat{t}\hat{r}\hat{t}\hat{r}} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -R_{\hat{r}\hat{t}\hat{r}\hat{r}} = -R_{\hat{r}\hat{t}\hat{t}\hat{r}} \]

(44)

Hence,

\[ e^{\hat{\alpha}_{\hat{0}}(\hat{\xi}^0)} = (\exp(\frac{a_0}{c^2} \xi^1) \cosh(\frac{a_0}{c^2} \xi^0), \exp(\frac{a_0}{c^2} \xi^1) \sinh(\frac{a_0}{c^2} \xi^0)), 0, 0) \]

\[ e^{\hat{\alpha}_{\hat{1}}(\hat{\xi}^0)} = (\exp(\frac{a_0}{c^2} \xi^1) \sinh(\frac{a_0}{c^2} \xi^0), \exp(\frac{a_0}{c^2} \xi^1) \cosh(\frac{a_0}{c^2} \xi^0)), 0, 0) \]

\[ e^{\hat{\alpha}_{\hat{2}}(\hat{\xi}^0)} = (0, 0, 1, 0), e^{\hat{\alpha}_{\hat{3}}(\hat{\xi}^0)} = (0, 0, 1, 0) \]  

(45)
\[ R_{0101}(\xi) = \frac{2GM}{r^3c^2} \exp\left(4\frac{a_0\xi}{c^2}\right), \quad R_{0000}(\xi) = R_{0000}(\xi) = -\frac{GM}{r^3c^2} \exp\left(2\frac{a_0\xi}{c^2}\right) \]

\[ R_{2222}(\xi) = -\frac{2GM}{r^3c^2}, \quad R_{1212}(\xi) = R_{1313}(\xi) = \frac{GM}{r^2c^2} \exp\left(2\frac{a_0\xi}{c^2}\right) \]

(46)

Specially, if \( t = 0 \),

\[
\frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{dr}{dt} = 0 \rightarrow V = \frac{d\hat{r}}{dt} = \frac{a_0\hat{r}}{\sqrt{1 + \frac{a_0^2\hat{r}^2}{c^2}}} = \frac{dr}{dt} \left(1 - \frac{2GM}{rc^2}\right) = 0
\]

(47)

Hence, if \( t = \hat{t} = \xi^0 = 0 \), the theory treats the real situation.

\[
\xi^1 = \frac{c^2}{a_0} \ln \left| \frac{a_0^2}{c^2} \left(\hat{r} + \frac{c^2}{a_0^2}\right)^2 - c^2\hat{r}^2 \right| = \frac{c^2}{a_0} \ln \left|1 + \frac{a_0^2\hat{r}}{c^2}\right|
\]

\[
\exp\left(\frac{a_0\xi^1}{c^2}\right) = 1 + \frac{a_0\hat{r}}{c^2}
\]

\[
d\hat{r} = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} \rightarrow \hat{r} = \sqrt{r} \left[ \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \\
\left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right]
\]

\[
a_0 = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right) = \frac{a}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \\
\]

\[
a = -g = \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right)
\]

\( g \) is the pure gravity acceleration.

\( r_0 \) is the location of the stationary accelerated frame

(48)

In this time, in the curved space-time, the curvature tensor \( R_{\mu\nu\rho\sigma}(\xi) \) of the stationary accelerated frame is

\[
R_{0101}(\xi) = \frac{2GM}{r^3c^2} \exp\left(4\frac{a_0\xi}{c^2}\right) = \frac{2GM}{r^3c^2} \left(1 + \frac{a_0\hat{r}}{c^2}\right)^4
\]
\[
\frac{2GM}{r^3c^2} \left[ 1 + \frac{a_0}{c^2} \left\{ \sqrt{r} \left( \frac{1}{ \sqrt{r} - \frac{2GM}{c^2} } + \frac{2GM}{c^2} \ln | \sqrt{r} + \frac{r^2 - 2GM}{c^2} | \right) - \sqrt{r_0} \left( \frac{1}{ \sqrt{r_0} - \frac{2GM}{c^2} } - \frac{2GM}{c^2} \ln | \sqrt{r_0} + \frac{r_0^2 - 2GM}{c^2} | \right) \right\}^4 \right]
\]

\[
= \frac{2GM}{r^3c^2} \left[ 1 - \frac{1}{c^2} \frac{\sqrt{r}}{\sqrt{1 - \frac{2GM}{rc^2}}} \left\{ \sqrt{r} \left( \frac{1}{ \sqrt{r} - \frac{2GM}{c^2} } + \frac{2GM}{c^2} \ln | \sqrt{r} + \frac{r^2 - 2GM}{c^2} | \right) - \sqrt{r_0} \left( \frac{1}{ \sqrt{r_0} - \frac{2GM}{c^2} } - \frac{2GM}{c^2} \ln | \sqrt{r_0} + \frac{r_0^2 - 2GM}{c^2} | \right) \right\}^4 \right]
\]

\[R_{0202}(\xi) = R_{0303}(\xi) = -\frac{GM}{r^3c^2} \exp\left(2\frac{a_0}{c^2} \xi \right) = \frac{GM}{r^3c^2} \left(1 + \frac{a_0}{c^2} r \right)^2
\]

\[-\frac{GM}{r^3c^2} \left[ 1 + \frac{a_0}{c^2} \left\{ \sqrt{r} \left( \frac{1}{ \sqrt{r} - \frac{2GM}{c^2} } + \frac{2GM}{c^2} \ln | \sqrt{r} + \frac{r^2 - 2GM}{c^2} | \right) - \sqrt{r_0} \left( \frac{1}{ \sqrt{r_0} - \frac{2GM}{c^2} } - \frac{2GM}{c^2} \ln | \sqrt{r_0} + \frac{r_0^2 - 2GM}{c^2} | \right) \right\}^2 \right]
\]

\[-\frac{GM}{r^3c^2} \left[ 1 - \frac{1}{c^2} \frac{\sqrt{r}}{\sqrt{1 - \frac{2GM}{rc^2}}} \left\{ \sqrt{r} \left( \frac{1}{ \sqrt{r} - \frac{2GM}{c^2} } + \frac{2GM}{c^2} \ln | \sqrt{r} + \frac{r^2 - 2GM}{c^2} | \right) - \sqrt{r_0} \left( \frac{1}{ \sqrt{r_0} - \frac{2GM}{c^2} } - \frac{2GM}{c^2} \ln | \sqrt{r_0} + \frac{r_0^2 - 2GM}{c^2} | \right) \right\}^2 \right]
\]

\[R_{2323}(\xi) = -\frac{2GM}{r^3c^2}.
\]

\[R_{1212}(\xi) = R_{1313}(\xi) = \frac{GM}{r^3c^2} \exp\left(2\frac{a_0}{c^2} \xi \right) = \frac{GM}{r^3c^2} \left(1 + \frac{a_0}{c^2} r \right)^2
\]

\[= \frac{GM}{r^3c^2} \left[ 1 + \frac{a_0}{c^2} \left\{ \sqrt{r} \left( \frac{1}{ \sqrt{r} - \frac{2GM}{c^2} } + \frac{2GM}{c^2} \ln | \sqrt{r} + \frac{r^2 - 2GM}{c^2} | \right) - \sqrt{r_0} \left( \frac{1}{ \sqrt{r_0} - \frac{2GM}{c^2} } - \frac{2GM}{c^2} \ln | \sqrt{r_0} + \frac{r_0^2 - 2GM}{c^2} | \right) \right\} \right]
\]
\[-\sqrt{r_0} \left( \frac{2GM}{c^2} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right)^2 \]

\[= \frac{GM}{r^3 c^2} \left[ 1 - \frac{1}{c^2} \sqrt{1 - \frac{2GM}{rc^2}} \right] \left( \sqrt{r} - \frac{2GM}{c^2} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right) \]

\[-\sqrt{r_0} \left( \frac{2GM}{c^2} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right)^2 \]

\[g \text{ is the pure gravity acceleration.} \]

\[r_0 \text{ is the location of the stationary accelerated frame} \quad (49)\]

4. Conclusion

In the general relativity theory, we define the accelerated frame that moves in \( \hat{t} \)-axis in the curved space-time. Specially, if \( t = \hat{t} = \xi^0 = 0 \), this theory treats the curvature tensor of the stationary accelerated frame in the curved space-time in two-cases. In this time, \( R_{\hat{t} \hat{t} \hat{x} \hat{x}}(\hat{\xi}) \) is the observational curvature tensor of the people on the planet in the gravity field but \( R_{\hat{a} \hat{b} \hat{c} \hat{d}}(\hat{X}) \) is the curvature tensor of the people’s self on the planet in the gravity field.

Reference

relativity":Arxiv:gr-qc/0006095(2000)