MEASURING COMPLEXITY BY USING REDUCTION TO
SOLVE P VS NP AND NC & PH

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1. Abstract

This article describes about that NC and PH is proper (especially P is not NP) by using problem reduction. If L is not P, we can prove P is not NP by using difference between logarithm space reduction and polynomial time reduction. Like this, we can also prove that NC is proper by using AL0 is not NC1. This means L is not P. Therefore P is not NP. And we can also prove that PH is proper by using P is not NP.

2. P is not NP if L is not P

Definition 1. We will use the term “L”, “P”, “P – Complete”, “NP”, “NP – Complete”, “FL”, “FP” as each complexity classes. These complexity classes also use Turing Machine (TM) set that compute target complexity classes problems. “f ◦ g” as composite TM that accepting configurations of g are starting configurations of f.

Theorem 2. L ⊆ P → P ⊆ NP

Proof. To prove it by using contraposition P = NP → L = P.

As we all know that if P = NP then all NP can reduce P – Complete under FL.

P = NP → ∀A ∈ P – Complete, B ∈ NP∀C ∈ FL (A ◦ C = B)

This is correct even if NP reduce by any FP.

P = NP → ∀D ∈ P – Complete, E ∈ NP, F ∈ FP∀G ∈ FL (D ◦ G = E ◦ F)

If P = NP, all NP can reduce {1} under some FP.

P = NP → ∀D ∈ P – CompleteG ∈ FL (D ◦ G = {1})

This means L = P. Therefore, this theorem was shown.

3. NC is proper

We use circuit problem as follows;

Definition 3. We will use the term “AC”, “NC” as each complexity decision problems classes. “FAC” as function problems class of “AC”. These complexity classes also use uniform circuits family set that compute target complexity classes problems. “f ◦ g” as composite circuit that output of g are input of f. In this case, we also use complexity classes to show target circuit. For example, A ◦ BB when A is circuits family and BB is circuits family set mean that a ◦ b | a ∈ A, b ∈ B ∈ BB. Circuits family uniformity is that these circuits can compute FAC0.

Theorem 4. NL \leq_{AC} NC^2
**Theorem 5.** $AC^0$ has Universal Circuits Family that can emulate all $AC^1$ circuits family. That is, every $AC^1$ has $AC^0$ – Complete.

**Proof.** To prove this theorem by making universal circuit family $A^i \in AC^1$ that emulate circuit family $\{C_j\} \in AC^0$ by using “depth circuit tableau”. Universal circuit $U_j \in A^i$ have partial circuit $u_{k,d}$ that emulate all $C_j$ gates $g_k \in n$ (include input value) and connected wires $w_p,q$ from $g_p$ output to $g_q$ input in every depth $d$. ($w_{p,q}$ always exist)

$u_{v \in n,d}$ have inputs from all $u_{w \in n,d-1}$ and $g_u$ information that mean

a) validity of $u_{w,d-1}$
b) $u_{w,d-1}$ output (true if $g_u$ output true)
c) existence of $u_{w,v}$ (true if $u_{w,u}$ exists)
d) negation of $u_{w,v}$ (true if $u_{w,v}$ include not gate)
e) gate type of $g_v$ (Or gate or And gate)

and outputs to $u_{w \in n,d+1}$ that mean

A) validity of $u_{v,d}$
B) $u_{v,d}$ output

These $u_{v,d}$ compute output like this:

If $u_{w,d-1}$ a) or c) input false then $u_{v,d}$ ignore $u_{w,d-1}$.
If $u_{w,d-1}$ a) and c) input true then $u_{v,d}$ A) output true and $u_{v,d}$ B) output $g_k$ value that compute from e), b), d), b), d) include another $u_{w \in n,d-1}$ b), d).
If all a) input false then $u_{k,d}$ A) output false.
If all c) input false then $u_{k,d}$ A) output false.

And depth 0 circuit compute additional condition;

If $u_{k,0}$ is $C_j$ input then $u_{k,0}$ A) output true and $u_{i,d}$ B) output $C_j$ input value, else $u_{k,0}$ A) output false.

This $U_j$ that consists of $u$ emulate $C_j$. We can make every $u$ in $FAC^0$, so that $A^i$ in $AC^1$.

Therefore, this theorem was shown. □

**Theorem 6.** $NC^1 \subseteq NC^{i+1}$

**Proof.** To prove it using reduction to absurdity. We assume that $NC^i = NC^{i+1}$. It is trivial that $NC^i = AC^0 = \cdots = NC^{2i}$.

Mentioned above 5, all $AC^i$ can reduce $AC^i – Complete$ under $AC^0$. Therefore if $NC^i = NC^{i+1}$ then all $NC^{2i}$ can reduce $AC^i – Complete$ under $AC^0$.

$NC^i = NC^{i+1} \Rightarrow \forall A \in AC^i – Complete, B \in NC^{2i} \exists C \in AC^0 \left(A \circ C = B\right)$

All $NC^i \circ NC^i$ is in $NC^{2i}$. Therefore above is correct even if $NC^i$ is $NC^i \circ NC^i$.

$NC^i = NC^{i+1} \Rightarrow \forall D \in AC^0 – Complete, E, F \in NC^i \exists G \in AC^0 \left(D \circ G = E \circ F\right)$

All $NC^0$ can reduce $\{1\}$ under some $NC^i$.

$NC^i = NC^{i+1} \Rightarrow \forall D \in AC^0 – Complete \exists G \in AC^0 \left(D \circ G = \{1\}\right)$

This means $AC^0 = AC^i$. But this contradict $AC^0 \subseteq NC^1 \subset AC^i$.

Therefore, this theorem was shown than reduction to absurdity. □

**Theorem 7.** $P \neq NP$
**Proof.** Mentioned above 2, \( L \subseteq P \rightarrow P \subseteq NP \). And mentioned above 6, \( L \subseteq NC^0 \subseteq NC^{i+1} \subseteq P \). Therefore \( P \not\subseteq NP \). \( \square \)

5. PH is proper

**Theorem 8.** \( \Pi_k \subseteq \Pi_{k+2} \)

**Proof.** To prove it using reduction to absurdity. We assume that \( \Pi_k = \Pi_{k+2} \). It is trivial that \( \Pi_k = \Pi_{k+2} = \cdots = \Pi_{2k} \).

Mentioned [2] Theorem 6.26, \( QSAT \subseteq \Pi_k \) - Complete under polynomial time reduction. All \( \Pi_k \) can reduce \( \Pi_k \) - Complete under FP. Therefore if \( \Pi_k = \Pi_{k+2} \) then all \( \Pi_{2k} \) can reduce \( \Pi_k \) - Complete under FP.

\( \Pi_k = \Pi_{k+2} \rightarrow \forall A \in \Pi_k - \text{Complete}, B \in \Pi_{2k} \exists C \in FP (A \circ C = B) \)

All \( \Pi_k \) is in \( \Pi_{2k} \). Therefore, if \( \Pi_k = \Pi_{k+2} \) then above is correct even if \( \Pi_k \) is \( \Pi_k \circ \Pi_k \).

\( \Pi_k = \Pi_{k+2} \rightarrow \forall D \in \Pi_k - \text{Complete}, E, F \in \Pi_{2k} \exists G \in FP (D \circ G = E \circ F) \)

All \( \Pi_k \) can reduce \( \{1\} \) under some \( \Pi_k \).

\( \Pi_k = \Pi_{k+2} \rightarrow \forall D \in \Pi_k - \text{Complete} \exists G \in FP (D \circ G = \{1\}) \)

This means \( FP \equiv \Pi_k \). But this contradict \( FP \not\subseteq NP \subset \Pi_k \) mentioned above.7

Therefore, this theorem was shown than reduction to absurdity. \( \square \)

**Theorem 9.** \( \Delta_k \subseteq \Sigma_k, \Sigma_k \not\subseteq \Pi_k \)

**Proof.** Mentioned [2] Theorem 6.12,

\( \Sigma_k = \Pi_k \rightarrow \Sigma_k = \Pi_k = PH \)

\( \Delta_k = \Sigma_k \rightarrow \Delta_k = \Sigma_k = \Pi_k = PH \)

This contraposition is,

\( (\Sigma_k \not\subseteq PH) \vee (\Pi_k \not\subseteq PH) \rightarrow \Sigma_k \not= \Pi_k \)

\( (\Delta_k \not\subseteq PH) \vee (\Sigma_k \not\subseteq PH) \vee (\Pi_k \not\subseteq PH) \rightarrow \Delta_k \not= \Sigma_k \)

From mentioned above 8,

\( \Sigma_k \subseteq \Pi_{k+1} \subseteq PH \)

Therefore, \( \Delta_k \not= \Sigma_k, \Sigma_k \not= \Pi_k \).

**Theorem 10.** \( \Pi_k \not\subseteq \Sigma_k, \Sigma_k \not\subseteq \Pi_k \)

**Proof.** To prove it using reduction to absurdity. We assume that \( \Pi_k \subset \Sigma_k \). This means that all \( \Sigma_k = \Pi_k \) is also \( \Sigma_k \).

\( \Pi_k \subset \Sigma_k \rightarrow \forall A \in \Sigma_k (A \in \Pi_k \subset \Sigma_k) \)

Mentioned [2] Theorem 6.10,

\( \Sigma_k \subseteq \Sigma_{k+1}, \Pi_k \subset \Pi_{k+1}, \forall k \geq 1 (\Delta_k \subset (\Sigma_k \cap \Pi_k) \subset (\Sigma_k \cup \Pi_k) \subset \Delta_{k+1}) \)

Therefore, \( \Delta_k \not\subseteq \Sigma_k, \Sigma_k \not\subseteq \Pi_k \).

Therefore \( \Pi_k \not\subseteq \Sigma_k, \Sigma_k \not\subseteq \Pi_k \).
We can prove $\Sigma_k \not\subseteq \Pi_k$ like this.
Therefore, this theorem was shown than reduction to absurdity.

**Theorem 11.** $\Delta_k \subseteq \Pi_k$

**Proof.** To prove it using reduction to absurdity. We assume that $\Delta_k = \Pi_k$.
$\Sigma_k \subseteq \Sigma_{k+1}$, $\Pi_k \subseteq \Pi_{k+1}$, $\forall k \geq 1$ (\(\Delta_k \subseteq (\Sigma_k \cap \Pi_k) \subseteq (\Sigma_k \cup \Pi_k) \subseteq \Delta_{k+1}\))
Therefore
\[\Delta_k = \Pi_k\]
\[\Rightarrow \Delta_k = \Pi_k \subseteq (\Sigma_k \cap \Pi_k) \subseteq \Sigma_k \subseteq (\Sigma_k \cup \Pi_k) \subseteq \Delta_{k+1}\]
\[\Rightarrow \Pi_k \subseteq \Sigma_k\]
But this result contradict mentioned above 10.
Therefore, this theorem was shown than reduction to absurdity.

**Theorem 12.** $\Sigma_k \not\subseteq \Delta_{k+1}$, $\Pi_k \not\subseteq \Delta_{k+1}$

**Proof.** To prove it using reduction to absurdity. We assume that $\Sigma_k = \Delta_{k+1}$.
$\forall k \geq 1$ (\(\Delta_k \subseteq (\Sigma_k \cap \Pi_k) \subseteq (\Sigma_k \cup \Pi_k) \subseteq \Delta_{k+1}\))
Therefore
\[\Sigma_k = \Delta_{k+1}\]
\[\Rightarrow \Delta_k \subseteq (\Sigma_k \cap \Pi_k) \subseteq \Pi_k \subseteq (\Sigma_k \cup \Pi_k) \subseteq \Sigma_k = \Delta_{k+1}\]
\[\Rightarrow \Pi_k \subseteq \Sigma_k\]
But this result contradict mentioned above 10. Therefore $\Sigma_k \subseteq \Delta_{k+1}$.
We can prove $\Pi_k \subseteq \Delta_{k+1}$ like this.
Therefore, this theorem was shown than reduction to absurdity.

**References**

