MEASURING COMPLEXITY BY USING REDUCTION TO
SOLVE P VS NP AND NC & PH

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1. Abstract

This article describes about that NC and PH is proper (especially P is not NP) by using problem reduction. If L is not P, we can prove P is not NP by using difference between logarithm space reduction and polynomial time reduction. Like this, we can also prove that NC is proper by using difference between ALO and NC! This means L is not P. Therefore P is not NP. And we can also prove that PH is proper by using P is not NP.

2. P IS NOT NP IF L IS NOT P

Definition 1. We will use the term “L”, “P”, “NP”, “FL”, “FP” as each complexity classes. These complexity classes also use Turing Machine (TM) set that compute target complexity classes problems. “f ∘ g” as composite TM that accepting configurations of g are starting configurations of f. In this case, we also use complexity classes to show target TM. For example, a ∘ bb when a is TM and bb is complexity class mean that a ∘ b | b ∈ bb.

Theorem 2. L ⊆ P → P ⊆ NP

Proof. To prove it by using contraposition P = NP → L = P. As we all know NP ∩ FP ∈ NP. From assumption P = NP, all NP ∩ FP correspond to P. Therefore

P = NP → ∀C ∈ NP ∩ FP ∃ D ∈ FP ∃ E ∈ P(C ∩ D = E)

Mentioned [1] Theorem 10.43, CIRCUIT – VALUE are closed under logarithm space reduction FL. That is,

∀ H ∈ P ∃ G ∈ FL(CIRCUIT – VALUE ∩ G = H)

Therefore

P = NP

→ ∀ C ∈ NP ∩ D ∈ FP ∃ G ∈ FL(C ∩ D = CIRCUIT – VALUE ∩ G)
→ ∀ D ∈ FP ∃ G ∈ FL(CIRCUIT – VALUE ∩ D = CIRCUIT – VALUE ∩ G)
→ ∀ D ∈ FP ∃ G ∈ FL(D = G)

This means L = P. Therefore, this theorem was shown. □

3. NC IS PROPER

And we use circuit problem as follows:

Definition 3. We will use the term “AC”, “NC” as each complexity decision problems classes. “FA” as function problems class of “AC”. These complexity classes also use uniform circuits family set that compute target complexity classes problems. “f ∘ g” as composite circuit that output of g are input of f. In this case,
we also use complexity classes to show target circuit. For example, \( A \circ BB \) when \( A \) is circuits family and \( BB \) is circuits family set mean that \( a \circ b \mid a \in A, b \in B \in BB \). Circuits family uniformity is that these circuits can compute \( FAC^0 \).

**Theorem 4.** \( NL \subseteq_{AC^0} NC^2 \)

**Proof.** Mentioned [1] Theorem 10.40, all \( NC^2 \) are closed by FL reduction. This reduction is validity of \((e_1, e_2)\) transition function. Transition function change \( O(1) \) memory and keep another memory. Therefore this validity can compute \( AC^0 \) and we can replace FL to \( FAC^0 \). \( \square \)

**Theorem 5.** \( AC^i \) has Universal Circuits Family that can emulate all \( AC^i \) circuits family.

**Proof.** To prove this theorem by making universal circuit family \( A^i \in AC^i \) that emulate circuit family \( \{C_j\} \in AC^i \) by using “depth circuit tableau”. Universal circuit \( U_j \in A^i \) have partial circuit \( u_{k,d} \) that emulate all \( C_j \) gates \( g_{k \in n} \) (include input value) and connected wires \( w_{p,q} \) from \( g_p \) output to \( g_q \) input in every depth \( d \).

\( (w_{p,p} \text{} always \text{} exist) \)

\( u_{v \in n, d} \) have inputs from all \( u_{u \in n, d-1} \) and \( g_u \) information that mean

\( a) \) validity of \( u_{u,d-1} \)
\( b) \) \( u_{u,d-1} \) output (true if \( g_u \) output true)
\( c) \) existence of \( w_{u,v} \) (true if \( w_{u,v} \) exists)
\( d) \) negation of \( w_{u,v} \) (true if \( w_{u,v} \) include not gate)
\( e) \) gate type of \( g_v \) (Or gate or And gate)

and outputs to \( u_{w \in n, d+1} \) that mean

\( A) \) validity of \( u_{w,d} \)
\( B) \) \( w_{v,d} \) output

These \( w_{v,d} \) compute output like this;

If \( u_{u,d-1} \) a) or c) input false then \( u_{v,d} \) ignore \( u_{u,d-1} \).
If \( u_{u,d-1} \) a) and c) input true then \( u_{v,d} \) A) output true and \( w_{v,d} \) B) output \( g_k \) value that compute from c), b), d), e) include another \( u_{w \in n, d-1} \) b), d).
If all a) input false then \( u_{k,d} \) A) output false.
If all c) input false then \( u_{k,d} \) A) output false.
And depth 0 circuit compute additional condition;
If \( u_{k,0} \) is \( C_j \) input then \( u_{k,0} \) A) output true and \( u_{i,d} \) B) output \( C_j \) input value, else \( u_{k,0} \) A) output false.

This \( U_j \) that consists of \( u \) emulate \( C_j \). We can make every \( u \) in \( FAC^0 \), so that \( A^i \) in \( AC^i \).

Therefore, this theorem was shown. \( \square \)

**Definition 6.** We will use the term “\( A^m \)” as universal circuits family that compute \( AC^i \) problem, “\( N^m \)” as universal circuits family that compute \( NC^i \) problem.

**Theorem 7.** \( FAC^0 \) can reduce all \( AC^i \) to \( A^i \). That is, \( A^i \) is closed under \( FAC^0 \) reduction.

**Proof.** Mentioned above 35, we can make all \( AC^i \) by using \( AC^0 \) and we can connect these \( AC^i \) to \( A^i \). That is, we can emulate all \( AC^i \) circuit by using \( A^i \circ AC^0 \). From the view of \( A^i \), \( AC^0 \) is input reduction from \( AC^i \) to \( A^i \). Therefore, this theorem was shown. \( \square \)
As we all know, all $NC^i$ decision problems can embed $NC^1$ function problems. To simplify, we define “Padding” that embed decision problems in function problems.

**Definition 8.** We will use the term “Padding function” and “Pad$_{N^i}$ (NC$^1$)” as function that change decision problem $NC^i$ to function problems Pad$_{N^i}$ (NC$^1$) that outputs fit to $N^i$ inputs. This Pad$_{N^i}$ (NC$^1$) output must include $NC^1$ output in head. (Other output make additional $AC^0$ circuit that input is some $NC^1$ gate output.)

**Theorem 9.** $NC^i \subseteq NC^{i+1}$

*Proof.* We can prove this theorem like mentioned above 2.

To prove it using reduction to absurdity. We assume that $NC^i = AC^i = NC^{i+1}$.

From assumption $NC^i = AC^i$, there is $N^i$ that equal $A^i$.

$NC^i = AC^i \rightarrow \forall A^i \in AC^i \exists N^i \in NC^i (A^i = N^i)$

From view of circuit structure, it is trivial that $N^i \circ Pad_{N^i} (NC^1) \in NC^{i+1}$.

From assumption $NC^i = AC^i = NC^{i+1}$, all $N^i \circ Pad_{N^i} (NC^1)$ correspond to $NC^i$.

Therefore

$NC^i = AC^i = NC^{i+1} \rightarrow \forall C \in NC^i \exists D \in NC^i (N^i \circ Pad_{N^i} (C) = D)$

Mentioned above 7, all $AC^i$ are closed by $FAC^0$ reduction to universal circuit $A^i$. That is,

\[
\forall H \in AC^i \exists G \in FAC^0 (A^i \circ G = H)
\]

Therefore

\[
NC^i = AC^i = NC^{i+1} \rightarrow \forall C \in NC^i \exists D \in NC^i \forall H \in AC^i \exists G \in FAC^0 (A^i \circ G = H) \land (N^i \circ Pad_{N^i} (C) = D)
\]

\[
\rightarrow \forall C \in NC^i \exists G \in FAC^0 (A^i \circ G = N^i \circ Pad_{N^i} (C))
\]

\[
\rightarrow \forall C \in NC^i \exists G \in FAC^0 (G = Pad_{N^i} (C))
\]

But this means $AC^0 = NC^1$ because head of $Pad_{N^i} (C)$ output is $C$ output. It is contradict $AC^0 \not\subseteq NC^1$.

Therefore, this theorem was shown than reduction to absurdity. $\square$

**Theorem 10.** $P \neq NP$

*Proof.* Mentioned above 2, $L \subseteq P \rightarrow P \subseteq NP$. And mentioned above 9, $L \subseteq NC^i \subseteq NC^{i+1} \subset P$. Therefore $P \subseteq NP$. $\square$

**5. PH is proper**

**Theorem 11.** $\Pi_k \not\subseteq \Sigma_{k+1}$

*Proof.* We can prove this theorem like mentioned above 9.

To prove it using reduction to absurdity. We assume that $\Pi_k = \Sigma_{k+1}$. As we all know $\Pi_k \circ \Sigma_1 \subseteq \Sigma_{k+1}$. From assumption, all $\Pi_k \circ \Sigma_1$ correspond to $\Pi_k$. Therefore

\[
\Pi_k = \Sigma_{k+1} \rightarrow \forall C \in \Pi_k \forall D \in \Sigma_1 \exists E \in \Pi_k (C \circ D = E)
\]

Mentioned [2] Theorem 6.26, $QSAT'_k$ are $\Pi_k – Complete$ under polynomial time reduction. That is,

\[
\forall H \in \Pi_k \exists G \in FP (QSAT'_k \circ G = H)
\]

Therefore

\[
\Pi_k = \Sigma_{k+1} \rightarrow \forall C \in \Pi_k \forall D \in \Sigma_1 \exists G \in P (C \circ D = QSAT'_k \circ G)
\]
\[ \rightarrow \forall D \in \Sigma_1 \exists G \in FP (QSAT_k^P \circ D = QSAT_k^P \circ G) \]
\[ \rightarrow \forall D \in \Sigma_1 \exists G \in FP (D = G) \]

But this means \( P = NP \) and contradict \( P \neq NP \). Therefore \( \Pi_k \subseteq \Sigma_{k+1} \).

Therefore, this theorem was shown than reduction to absurdity.

**Theorem 12.** \( \Delta_k \subseteq \Sigma_k, \Sigma_k \neq \Pi_k \)

\[ \Sigma_k = \Pi_k \rightarrow \Sigma_k = \Pi_k = PH \]
\[ \Delta_k = \Sigma_k \rightarrow \Delta_k = \Sigma_k = \Pi_k = PH \]

This contraposition is,
\[ (\Sigma_k \subseteq PH) \lor (\Pi_k \subseteq PH) \rightarrow \Sigma_k \neq \Pi_k \]
\[ (\Delta_k \subseteq PH) \lor (\Sigma_k \subseteq PH) \lor (\Pi_k \subseteq PH) \rightarrow \Delta_k \neq \Sigma_k \]

From mentioned above 11,
\[ \Sigma_k \subseteq \Pi_{k+1} \in PH \]

Therefore, \( \Delta_k \neq \Sigma_k, \Sigma_k \neq \Pi_k \).

Mentioned [2] Theorem 6.10,
\[ \Sigma_k \in \Sigma_{k+1}, \Pi_k \in \Pi_{k+1}, \forall k \geq 1 (\Delta_k \in (\Sigma_k \cap \Pi_k) \in (\Sigma_k \cup \Pi_k) \in \Delta_{k+1}) \]

Therefore, \( \Delta_k \subseteq \Sigma_k, \Sigma_k \neq \Pi_k \).

**Theorem 13.** \( \Pi_k \not\subseteq \Sigma_k, \Sigma_k \not\subseteq \Pi_k \)

**Proof.** To prove it using reduction to absurdity. We assume that \( \Pi_k \subseteq \Sigma_k \). This means that all \( \Sigma_k = \Pi_k \) is also \( \Sigma_k \).
\[ \Pi_k \in \Sigma_k \rightarrow \forall A \in \Sigma_k (A \not\in \Pi_k \not\in \Sigma_k) \]

Mentioned [2] Theorem 6.21, all \( \Sigma_k \) are closed under polynomial time conjunctive reduction. We can emulate these reduction by using \( \Pi_1 \). That is,
\[ \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 (B \circ D = C) \]

Therefore,
\[ \Pi_1 \subseteq \Sigma_k \]
\[ \rightarrow \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 \forall A \in \Sigma_k (B \circ D = C) \land (A \not\in \Pi_k \not\in \Sigma_k) \]
\[ \rightarrow \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 (B \circ D = C) \land (B \in \Sigma_k) \]
\[ \rightarrow \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 (B \circ D = C) \land (B \in \Pi_1) \]

Therefore \( \Sigma_k \subseteq \Pi_k \) because \( B \circ D \in \Pi_k \). But this means \( \Sigma_k = \Pi_k \) and contradict mentioned above 12 \( \Sigma_k \neq \Pi_k \). Therefore \( \Pi_k \not\subseteq \Sigma_k \).

We can prove \( \Sigma_k \not\subseteq \Pi_k \) like this.

Therefore, this theorem was shown than reduction to absurdity.

**Theorem 14.** \( \Delta_k \subseteq \Pi_k \)

**Proof.** To prove it using reduction to absurdity. We assume that \( \Delta_k = \Pi_k \).

Mentioned [2] Theorem 6.10,
\[ \Sigma_k \in \Sigma_{k+1}, \Pi_k \in \Pi_{k+1}, \forall k \geq 1 (\Delta_k \in (\Sigma_k \cap \Pi_k) \in (\Sigma_k \cup \Pi_k) \in \Delta_{k+1}) \]

Therefore
\[ \Delta_k = \Pi_k \]
\[ \rightarrow \Delta_k = \Pi_k \in (\Sigma_k \cap \Pi_k) \in (\Sigma_k \cup \Pi_k) \in \Delta_{k+1} \]
\[ \rightarrow \Pi_k \in (\Sigma_k \cap \Pi_k) \]

But this result contradict mentioned above 13.

Therefore, this theorem was shown than reduction to absurdity.

**Theorem 15.** \( \Sigma_k \subseteq \Delta_{k+1}, \Pi_k \subseteq \Delta_{k+1} \)
Proof. To prove it using reduction to absurdity. We assume that \( \Sigma_k = \Delta_{k+1} \).

Mentioned [2] Theorem 6.10,
\[
\forall k \geq 1 (\Delta_k \subset (\Sigma_k \cap \Pi_k) \subset (\Sigma_k \cup \Pi_k) \subset \Delta_{k+1})
\]
Therefore
\[
\Sigma_k = \Delta_{k+1}
\]
\[
\to \Delta_k \subset (\Sigma_k \cap \Pi_k) \subset \Pi_k \subset (\Sigma_k \cup \Pi_k) \subset \Sigma_k = \Delta_{k+1}
\]
\[
\to \Pi_k \subset \Sigma_k
\]
But this result contradict mentioned above 13. Therefore \( \Sigma_k \subset \Delta_{k+1} \).
We can prove \( \Pi_k \subset \Delta_{k+1} \) like this.
Therefore, this theorem was shown than reduction to absurdity. \( \square \)

References