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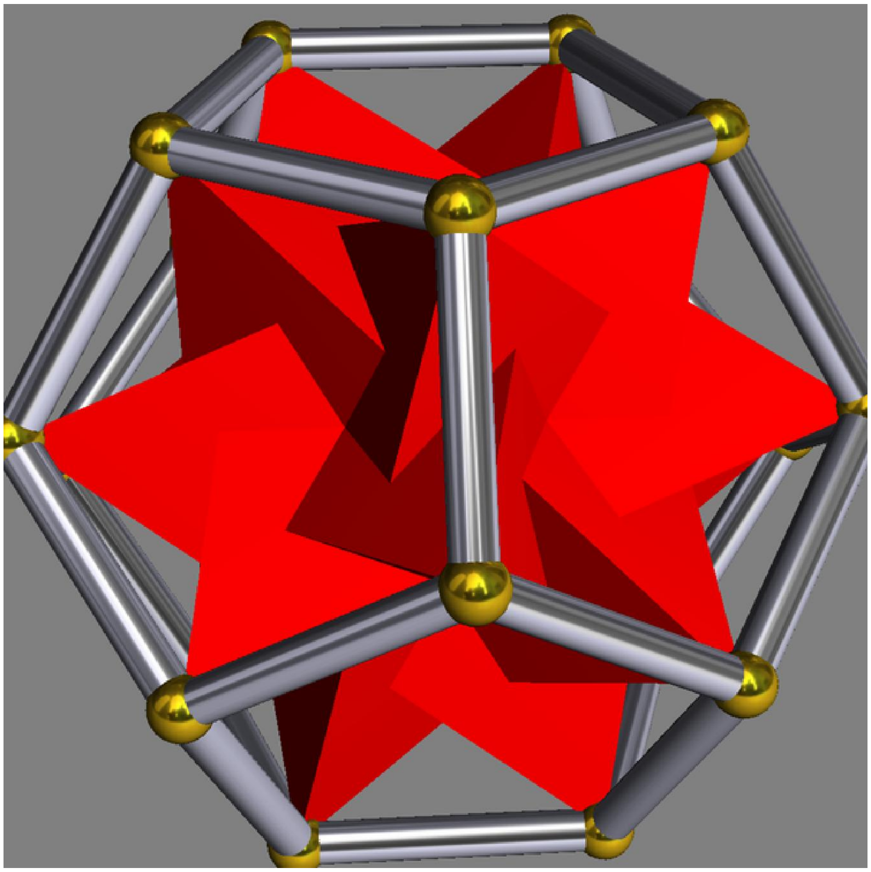
Poincare Dodecahedral Space and the Qi Men Dun Jia Model



By John Frederick Sweeney

Abstract

In Vedic Nuclear Physics, the number 28 plays a key role, and this will be discussed in a future paper. The 28 aspects must be extruded or dispersed along structures which contain factors of 12 or 6. The Poincare Dodecahedral Space contains these factors and relates to the 120-element binary icosahedral group, which double covers the simple 60-element icosahedral group. This, in turn, enjoys isometric relationships to the 60 Jia Zi and the 60 Na Yin of Chinese metaphysics, function to add Five Element and temporal qualities to matter as it becomes visible, in the Qi Men Dun Jia Model.

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Introduction

In earlier papers published on Vixra, the author has begun to delineate the outline of the Qi Men Dun Jia Model, which is a mathematical version of the Qi Men Dun Jia Cosmic Board. Qi Men Dun Jia is an ancient Chinese form which allows multiple types of divination, including military, medical, business, homicidal, criminal, weather, etc. Qi Men Dun Jia was held as a state secret by Chinese emperors, and is said to have helped the Ming attain the throne, through the Qi Men Dun Jia abilities of Liu Bo Wen.

The Qi Men Dun Jia Model begins in the substratum of "black hole" material with the natural logarithm of e, then ascends through the substratum as matter forms itself into cylindrical and then spherical shapes. In the substratum, or the invisible spectrum of "black hole" matter, matter begins the process of formation at the level of the natural e (Euler) logarithm (2.718) and ascends towards a sphere shape before emerging as visible matter.

The Qi Men Dun Jia Model to the present articulation includes the icosahedron, or the twin icosahedron and the 60 stellated permutations of the icosahedron, which correspond to the 60 Jia Zi and the 60 Na Yin, in the visible spectrum of matter.

To this structure the author now adds the Poincare Dodecahedral Space with its properties of the icosahedron and the dodecahedron. Vedic Nuclear Physics requires a structure composed of 12 sections, which help to accommodate things of order 28, and this reason will be explained in a future paper.

Five tetrahedra fit inside the dodecahedron, as Frank "Tony" Smith explains, and the tetrahedra correspond to the Sedenions. The Qi Men Dun Jia Model must accommodate things of order 28, as mentioned above, and so five copies of the Sedenions per dodecahedron help to accomplish this goal.

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As 15 copies of the Fano Plane, the Octonion multiplication table, comprise the Sedenions, then 15 copies of the Octonions would prove necessary for each set of Sedenions. So much for Sir Roger Penrose and his dismissal of the Octonions as a "lost cause" of physics.

This paper includes two sections by other authors, Frank "Tony" Smith and John Baez to describe the Poincare Dodecahedral Space (as well as an appendix with a third description from Wikipedia). By including separate takes on the Poincare Dodecahedral Space, we hope to elucidate all of its features, whether currently regarded as major or minor, for God lies in the details.

Next, an interesting twist: the "twist" in the Poincare Dodecahedral Space, described by Gert Vegter and Rien van de Weigaert in their discussion of Poincare Dodecahedral Space and its tessellations. We note here that the Qi Men Dun Jia Model follows the basic outline of Plato, his solids and his notion of triangles, as described in the Timaeus.

Finally, in the present discussion of the shape of the universe, we find that the Poincare Dodecahedral Space offers a leading model. Since the ancient Hindu saying, "As above, so below" implies that the minor matches the major, the micro the macro, then the Qi Men Dun Jia Model lends support to the notion of a universe in the shape of and given the qualities of the Poincare Dodecahedral Space.

At this point, however, the goal of the author is merely to articulate the model of the Qi Men Dun Jia Cosmic Board in terms of mathematical physics, not to engage in debate about the size and shape of the universe. Even so, the logical articulation of the Qi Men Dun Jia Model does lend critical support to the notion of a universe similar to the Poincare Dodecahedral Space.

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Poincaré Dodecahedral Space
(by Frank "Tony" Smith)

WHAT ABOUT the 3-sphere S3, of dimension 2^k + 1 for k=1?

It is accurate to say that there is no exotic S3,
in the sense that anything homeomorphic to normal S3
is also diffeomorphic to normal S3.

HOWEVER, the 3-dimensional analogue S3# of the exotic Milnor sphere
DOES exist, but it is not only not diffeomorphic to normal S3,
it is NOT even homeomorphic to normal S3.
That is because S3# is NOT SIMPLY CONNECTED.

WHAT IS S3# ?

S3# is called the Poincare Dodecahedral Space.

Poincare, because it would be a counterexample
to the Poincare conjecture in 3 dimensions if it were simply connected.

Dodecahedral, because it has dodecahedral/icosahedral symmetry
of the 120-element binary icosahedral group,
which double covers the simple 60-element icosahedral group.

A reference is:

Topology and Geometry, by Glen Bredon, Springer (1993).

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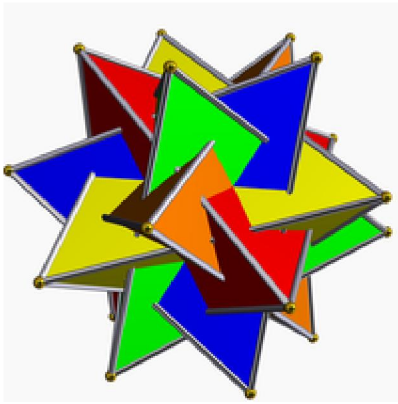

The Time Star is one of my favorite Archetypes.
Start with a dodedecahedron.
Five tetrahedra fit inside the dodecahedron:

The alternating permutation group of the 5 tetrahedra is
the 60-element icosahedral group.

Now, to see things clearly, look at just one tetrahedron.
You can symmetries more clearly when you put
an octhedron inside the tetrahedron and
a cuboctahedron inside the octahedron:

Take the one tetrahedron and put it inside a cube,
with one edge of the tetrahedron in each face of the cube.
Now rotate the cube around inside the dodecahedron,
while you also rotate each of the 6 edges of the tetrahedron
each of the 6 faces of the cube.

The tetrahedra edges now are parallel to the cube edges.
36 more degrees, after 72 degrees total rotation,
the edges will have again formed a tetrahedron.
Keep rotating.

After 360 degrees, you have made 5 tetrahedra
(one each 72 degrees), and this is what you have:

The cube is back like it was,

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BUT THE TETRAHEDRON IS ORIENTED OPPOSITELY
with respect to the cube from its original position.

YOU HAVE TO ROTATE 720 degrees TO GET BACK LIKE YOU STARTED.

That means that, to make S3#,
instead of taking the quotient of SO(3)
by the 60-element icosahedral group,
you should
take the quotient of S3 = Spin(3) = SU(2),
the double cover of SO(3),
by the 120-element binary icosahedral group.

Therefore, S3# is a natural spinor space,
and
5-fold Golden Ratio Icosahedral Symmetry
is
a manifestation in 3 and 4 dimensions
of
the Milnor sphere structure of 7 and 8 dimensions.

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Poincaré Dodecahedral Space (John Baez)

John Baez, in a 2003-01-23 post to the sci.physics.research thread "The magic of 8", said:

"... "even unimodular lattices" and "invertible symmetric integer matrices with even entries on the diagonal" ...[are]... secretly the same thing as long as your matrix is positive definite ... a "lattice" is a subgroup of R^n that's isomorphic to Z^n. If we give R^n its usual inner product, an "even" lattice is one such that the inner product of any two vectors in the lattice is an even integer. A lattice is "unimodular" if the volume of each cell of the lattice is 1. To get from an even unimodular lattice to a matrix, pick a basis of vectors in the lattice and form the matrix of their inner products. This matrix will then be symmetric, have determinant +-1, and have even entries down the diagonal. ...[a famous 8 x 8 invertible integer matrix with even entries on the diagonal and signature +8 is]...

```
 2 -1  0  0  0  0  0  0
-1  2 -1  0  0  0  0  0
 0 -1  2 -1  0  0  0  0
 0  0 -1  2 -1  0  0  0
 0  0  0 -1  2 -1  0 -1
 0  0  0  0 -1  2 -1  0
 0  0  0  0  0 -1  2  0
 0  0  0  0 -1  0  0  2
```

It's called E8, and it leads to huge wads of amazing mathematics. For example, suppose we take 8 dots and connect the ith and jth dots with an edge if there is a "-1" in the ij entry of the above matrix. We get this:

```
o----o---o----o----o----o----o
                       |
                       |
                       o
```

Now, make a model with one ring for each dot in the above picture, where the rings link if the corresponding dots have an edge connecting them. We get this:

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```
    /\    /\    /\    /\    /\    /\    /\
   /  \/  \/  \/  \/  \/  \/  \
  /    \     \     \     \     \     \     \
 /     /\   /\   /\   /\   /\   /\     \
 \     \/   \/   \/   \/   \/   \/     /
  \     \     \     \     \/\ \     \    /
   \  /\  /\  /\  /\  \\  /\  /
    \/    \/    \/    \//\/\\/    \/
                      /       \
                      \       /
                       \     /
                        \   /
                         \/
```

Next imagine this model as living in the 3-sphere. Hollow out all these rings: actually delete the portion of space that lies inside them! We now have a 3-manifold M whose boundary dM consists of 8 connected components, each a torus. Of course, a solid torus also has a torus as its boundary. So attach solid tori to each of these 8 components of dM, but do it via this attaching map: (x,y) -> (y,-x+2y) where x and y are the obvious coordinates on the torus, numbers between 0 and 2pi, and we do the arithmetic mod 2pi.

We now have a new 3-manifold without boundary. This manifold is called the "Poincare homology sphere". Poincare invented it as a counterexample to his own conjecture that any 3-manifold with the same homology groups as a 3-sphere must *be* the 3-sphere. But he didn't invent it this way. Instead, he got it by taking a regular dodecahedron and identifying its opposite faces in the simplest possible way, namely by a 1/10th turn. So, we've gone from E8 to the dodecahedron!

... The fundamental group of the Poincare homology sphere has 120 elements. In fact, we can describe it as follows. The rotational symmetry group of the dodecahedron has 60 elements. Take the "double cover" of this 60-element group, namely the 120-element subgroup of SU(2) consisting of elements that map to rotational symmetries of the dodecahedron under the double cover p: SU(2) -> SO(3).

This is the fundamental group of the Poincare homology ...[sphere]... Now,

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this 120-element group has finitely many irreducible representations. One of them just comes from restricting the 2-dimensional representation of SU(2) to this subgroup: call that R. There are 8 others: call them R(i) for i = 1,...,8. Draw a dot for each one, and draw a line from the ith dot to the jth dot if the tensor product of R and R(i) contains R(j) as a sub-representation. We get this picture:

```
o----o---o----o----o----o----o
                      |
                      |
                      o
```

Voila! Back to E8. ...".

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Poincaré Homology Sphere Wikipedia

The Poincaré homology sphere (also known as Poincaré dodecahedral space) is a particular example of a homology sphere. Being a spherical 3-manifold, it is the only homology 3-sphere (besides the 3-sphere itself) with a finite fundamental group. Its fundamental group is known as the binary icosahedral group and has order 120. This shows the Poincaré conjecture cannot be stated in homology terms alone.

Construction

A simple construction of this space begins with a dodecahedron. Each face of the dodecahedron is identified with its opposite face, using the minimal clockwise twist to line up the faces. Gluing each pair of opposite faces together using this identification yields a closed 3-manifold. (See Seifert–Weber space for a similar construction, using more "twist", that results in a hyperbolic 3-manifold.)
Alternatively, the Poincaré homology sphere can be constructed as the quotient space SO(3)/I where I is the icosahedral group (i.e. the rotational symmetry group of the regular icosahedron and dodecahedron, isomorphic to the alternating group A_5). More intuitively, this means that the Poincaré homology sphere is the space of all geometrically distinguishable positions of an icosahedron (with fixed center and diameter) in Euclidean 3-space. One can also pass instead to the universal cover of SO(3) which can be realized as the group of unit quaternions and is homeomorphic to the 3-sphere. In this case, the Poincaré homology sphere is isomorphic to $S^3/\tilde{I}$ where $\tilde{I}$ is the binary icosahedral group, the perfect double cover of I embedded in S^3.
Another approach is by Dehn surgery. The Poincaré homology sphere results from +1 surgery on the right-handed trefoil knot.

Cosmology

In 2003, lack of structure on the largest scales (above 60 degrees) in the cosmic microwave background as observed for one year by the WMAP spacecraft led to the suggestion, by Jean-Pierre Luminet of the Observatoire de Paris and colleagues, that the shape of the Universe is a Poincaré sphere.[1][2] In 2008, astronomers found the best orientation on the sky for the model and confirmed some of the predictions of the model, using three years of observations by the WMAP spacecraft.[3] However, there is no strong support for the correctness of the model, as yet.

Constructions and examples

- Surgery on a knot in the 3-sphere S^3 with framing +1 or − 1 gives a homology sphere.
- More generally, surgery on a link gives a homology sphere whenever the matrix given by intersection numbers (off the diagonal) and framings (on the diagonal) has determinant +1 or −1.
- If p, q, and r are pairwise relatively prime positive integers then the link of the singularity $x^p + y^q + z^r = 0$ (in other words, the intersection of a small 5-sphere around 0 with this complex surface) is a homology 3-sphere, called a Brieskorn 3-sphere $\Sigma(p, q, r)$. It is homeomorphic to the standard 3-sphere if one of p, q, and r is 1, and $\Sigma(2, 3, 5)$ is the Poincaré sphere.
- The connected sum of two oriented homology 3-spheres is a homology 3-sphere. A homology 3-sphere that cannot be written as a connected sum of two homology 3-spheres is called **irreducible** or **prime**, and every homology 3-sphere can be written as a connected sum of prime homology 3-spheres in an essentially unique way. (See Prime decomposition (3-manifold).)

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- Suppose that a_1, ..., a_r are integers all at least 2 such that any two are coprime. Then the Seifert fiber space

$$\{b, (o_1, 0); (a_1, b_1), \ldots, (a_r, b_r)\}$$

over the sphere with exceptional fibers of degrees a_1, ..., a_r is a homology sphere, where the b's are chosen so that

$$b + b_1/a_1 + \cdots + b_r/a_r = 1/(a_1 \cdots a_r).$$

(There is always a way to choose the b's, and the homology sphere does not depend (up to isomorphism) on the choice of b's.) If r is at most 2 this is just the usual 3-sphere; otherwise they are distinct non-trivial homology spheres. If the a's are 2, 3, and 5 this gives the Poincaré sphere. If there are at least 3 a's, not 2, 3, 5, then this is an acyclic homology 3-sphere with infinite fundamental group that has a Thurston geometry modeled on the universal cover of $SL_2(\mathbf{R})$.

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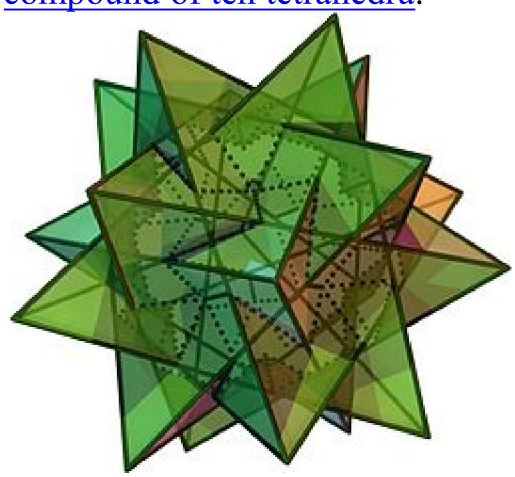
Compound of Five Tetrahedra

Wikipedia explains icosahedral symmetry

This compound polyhedron is also a stellation of the regular icosahedron. It was first described by Edmund Hess in 1876.

It can be constructed by arranging five tetrahedra in rotational icosahedral symmetry (**I**), as colored in the upper right model. It is one of five regular compounds which can be constructed from identical Platonic solids.
It shares the same vertex arrangement as a regular dodecahedron.
There are two enantiomorphous forms (the same figure but having opposite chirality) of this compound polyhedron. Both forms together create the reflection symmetric compound of ten tetrahedra.



Transparent Models (Animation)

As a stellation

It can also be obtained by stellating the icosahedron, and is given as Wenninger model index 24

Group theory

The compound of five tetrahedra is a geometric illustration of the notion of orbits and stabilizers, as follows.
The symmetry group of the compound is the (rotational) icosahedral group I of order 60, while the stabilizer of a single chosen tetrahedron is the (rotational) tetrahedral group T of order 12, and the orbit space I/T (of order $60/12 = 5$) is naturally identified with the 5 tetrahedra – the coset gT

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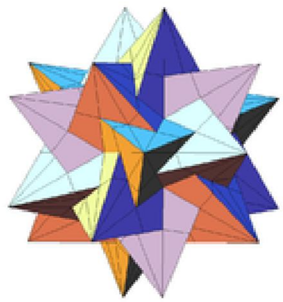
corresponds to which tetrahedron *g* sends the chosen tetrahedron to.

An unusual dual property



Compound of five tetrahedra

This compound is unusual, in that the dual figure is the enantiomorph of the original. This property seems to have led to a widespread idea that the dual of any chiral figure has the opposite chirality. The idea is generally quite false: a chiral dual nearly always has the same chirality as its twin. For example if a polyhedron has a right hand twist, then its dual will also have a right hand twist.
In the case of the compound of five tetrahedra, if the faces are twisted to the right then the vertices are twisted to the left. When we dualise, the faces dualise to right-twisted vertices and the vertices dualise to left-twisted faces, giving the chiral twin. Figures with this property are extremely rare.

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The Twist

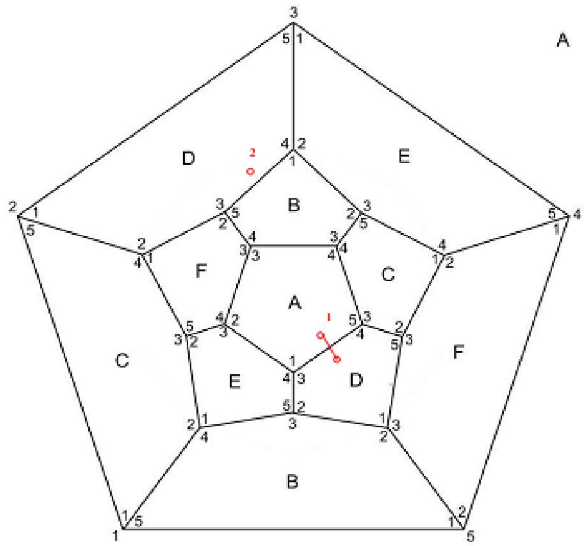

Figure: Schlegel diagram of dodecahedron. Opposite faces identified with minimal twist $\pi/5$

The process of forming visible matter in the universe involves a sudden twisting. A book on Vedic Nuclear Physics describes the process in this way:

While radiation of energy needs 7 Ne's to be accelerated at the same instant, below that level, any number of Ne’s may transmigrate, as a result of changing stresses, without giving any clue to the underlying interactive process.

Such tunneling activity creates phase changes in synchronized states that alter the potential. Dynamic and flexible molecular, cellular and genetic structures undergo a twisting or unwinding of stresses (without observable movements) through small angles, that upset balanced states. Living organisms sense the change and react.

This subtle twisting in the nature of matter requires compensation in the Qi Men Dun Jia Model, which one finds in the Poincare Dodecahedral Space.

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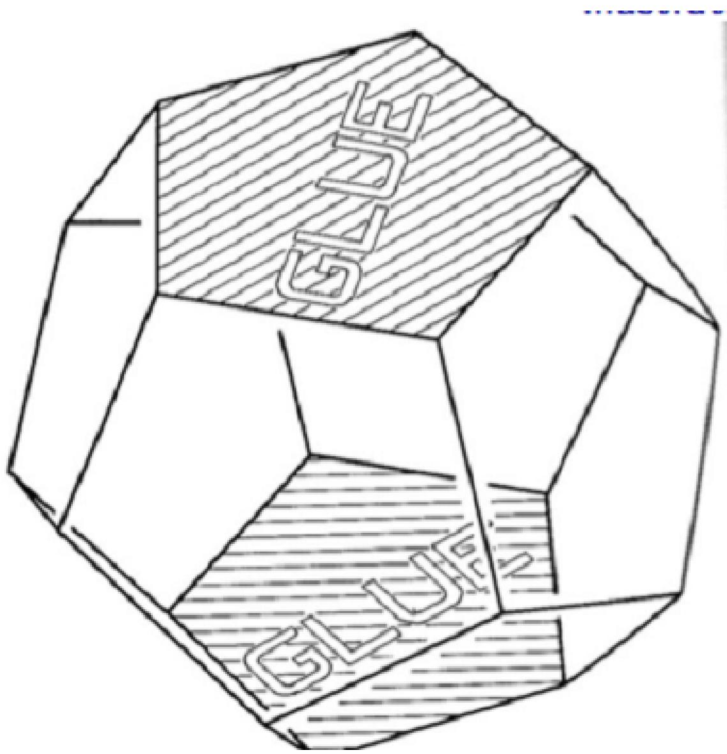
PDS: Identify opposite faces with twist



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The Shape of the Universe

The **shape of the universe** is a matter of debate in physical cosmology over the local and global geometry of the universe which considers both curvature and topology, though, strictly speaking, it goes beyond both. In practice, more formally, theorists seek a 3-manifold model that corresponds to the spatial section (in comoving coordinates) of the 4-dimensional space-time of the universe. The recent Wilkinson Microwave Anisotropy Probe (WMAP) measurements have led NASA to state, "We now know that the universe is flat with only a 0.4% margin of error."[1] Within the Friedmann–Lemaître–Robertson–Walker (FLRW) model, the presently most popular shape of the Universe found to fit observational data according to cosmologists is the infinite flat model,[2] while other FLRW models that fit the data include the Poincaré dodecahedral space[3][4] and the Picard horn.[5]

Proposed models

Various models have been proposed for the global geometry of the universe. In addition to the primitive geometries, these proposals include the:

- Poincaré dodecahedral space, a positively curved space, colloquially described as "soccerball-shaped", as it is the quotient of the 3-sphere by the binary icosahedral group, which is very close to icosahedral symmetry, the symmetry of a soccer ball. This was proposed by Jean-Pierre Luminet and colleagues in 2003[3][11] and an optimal orientation on the sky for the model was estimated in 2008.[4]

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Conclusion

The combination of the icosahedron and the dodecahedron in the Poincare Dodecahedral Space makes this an integral part of the Qi Men Dun Jia Model, since it incorporates Base 60 math to include the 60 Jia Zi and the 60 Na Yin, as well as the 12 sectors of the dodecahedron. Moreover, the 60 stellations of the icosahedron form an isomorphic relationship to the Poincare Dodecahedral Space.

This structure is critical to the formation of visible matter via the substratum, since matter gets stamped with Five Element, frequency and temporal aspects as soon as matter emerges from the substratum of "black hole" material. Since black hole material absorbs Time, then we may say that time begins in the visible material universe only when matter or events emerge from the substratum.

Upon emerging from the substratum, matter is instantly branded with Five Elements, frequency and temporal aspects through the agency of the 60 Na Yin and the 60 Na Jia, as well as Yin and Yang or binary aspects. Since all material and events are thus branded, then it becomes possible through Chinese metaphysics, and especially Qi Men Dun Jia, to monitor and to predict appearances and events.

In the process of writing this paper, the author has discovered yet another isometric relationship of the Qi Men Dun Jia Model: the 27 +1 Exotic Milnor Spheres share the isometric relationship with the 28 Nakshastra of the Atharaveda of Hindu tradition, which are known as astrological houses. In fact, those astrological houses and their given characteristics share an isometric relationship with 28 qualities of Vedic Nuclear Physics, which the author will discuss in a future paper. Suffice it to say that the Qi Men Dun Jia Model continues to from isometric

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relationships with well - formed concepts of mathematical physics, as the Exotic Milner Spheres were discovered during the decade of the 1950's.

This is why Krsanna Duran and Richard Hawkins devised the Maya Time Star, which perhaps functions along the same lines.

In this series of papers, all published on Vixra, the author has been trying to articulate the outline of the Qi Men Dun Jia Model in terms of mathematical physics. While a foundation has been laid, much remains to articulate, yet the Poincare Dodecahedral Space goes far in terms of explanation and articulation.

"As above, so below," goes the Hermestian dictum, which originated in the Hindu tradition. In this sense, if the Qi Men Dun Jia Model proves capable of making predictions about the world, then we might conclude that this model comprises a true scientific model which is capable of making predictions, which are capable of replication by other scientists.

If the Qi Men Dun Jia Model proves adequate in our world, then one might expect something similar on a universal scale, and thus the Qi Men Dun Jia Model helps to support the Poincare Dodecahedral Space as a model for the shape of the universe. That is to say that if the Qi Men Dun Jia Model functions on a micro level in the lives of humans, then we may have every reason to assume that the universe takes on a similar shape, that is, of the Poincare Dodecahedral Space.

Yet the evidence is not all yet in, the model is still in the stages of articulation, and other possibilities remain, such as a 3 - torus.

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Baez, John, www.**tony5m17h.net**/PDS3.html

Frank "Tony" Smith, www.**tony5m17h.net**/PDS3.html

Wikipedia

Wolfram

Platonic solids Polychorons (4D) Tiling the 3-sphere The **Poincaré dodecahedral space** Gert Vegter and Rien van de Weijgaert (joint work with Guido Senden)

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Appendix I

Tetrahedron 5-Compound / Wolfram

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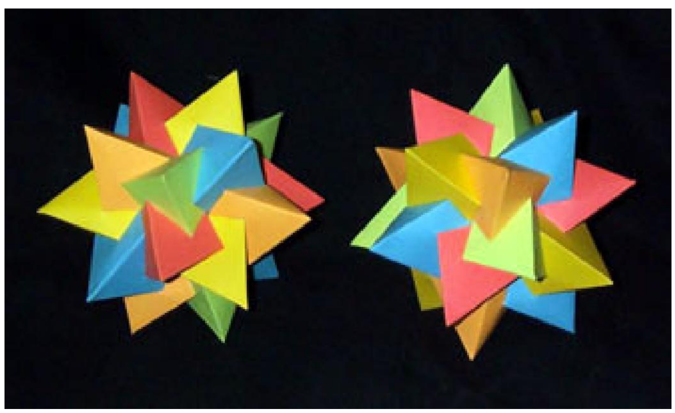
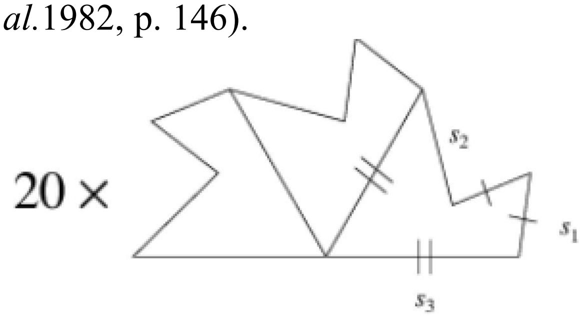
A polyhedron compound composed of five tetrahedra which is also one of the icosahedron stellations. The 5×4 vertices of the tetrahedron are then 20 vertices of the dodecahedron. Two tetrahedron 5-compounds of opposite chirality combine to make a tetrahedron 10-compound (Cundy and Rollett 1989).

It is implemented in *Mathematica* as PolyhedronData["TetrahedronFiveCompound"].



The illustration above shows paper sculptures of both handednesses of the tetrahedron 5-compound.

Escher built his own model of the tetrahedron 5-compound as a study for his woodcuts (Bool *et al.*1982, p. 146).



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There are two techniques for constructing the compounds. The first, advocated by Wenninger (1989, p. 44) uses 20 identical pieces as illustrated above, each assembled into a small three-piece pyramid. The 20 pyramids are then assembled in rings of five into the solid. The side lengths illustrated in the net are given by

$$s_1 = \tfrac{1}{2}\left(3-\sqrt{5}\right)$$

$$= \phi^{-2}$$

$$s_2 = \sqrt{7-3\sqrt{5}}$$

$$= \sqrt{2}\,\phi^{-2}$$

$$s_3 = \sqrt{3-\sqrt{5}}$$

$$= \sqrt{2}\,\phi^{-1},$$

where ϕ is the golden ratio, for a compound produced starting from a dodecahedron with unit edge lengths.

A fancier construction is advocated by (Cundy and Rollett 1989). While allegedly simpler than the first construction, its differently shaped pieces actually prove more difficult to assemble correctly in practice. Cundy and Rollett's method involves constructing a base tetrahedron, placing a "cap" around one of the apexes (thus giving an attractive tetrahedron 2-compound as an intermediate step), and then affixing a triangular pyramid to the opposite face. Twelve pyramids of the same type as above are then constructed and attached edge-to-edge in chains of three. The four chains of pyramids are then arranged about the eight vertices of the original two tetrahedra, with the points of coincidence of the three pyramids in each chain attached such that they coincide with intersections of the original two tetrahedra such that five pyramids touch at a single point.

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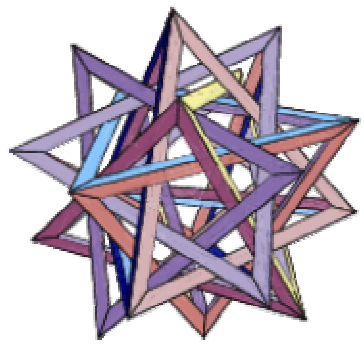
The convex hull of the compound's vertices is a regular dodecahedron.



By replacing the solid tetrahedra with beveled struts along its edges, the attractive structure above is obtained.

A regular icosahedron has 60 rotational (or orientation-preserving) symmetries, and a symmetry order of 120 including transformations that combine a reflection and a rotation. A regular dodecahedron has the same set of symmetries, since it is the dual of the icosahedron. The set of orientation-preserving symmetries forms a group referred to as A_5 (the alternating group on 5 letters), and the full symmetry group (including reflections) is the product $A_5 \times Z_2$. The latter group is also known as the Coxeter group H_3, and is also represented by Coxeter notation, [5,3] and Coxeter diagram .

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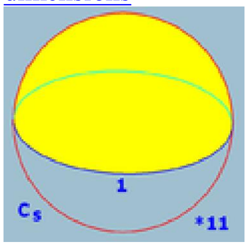
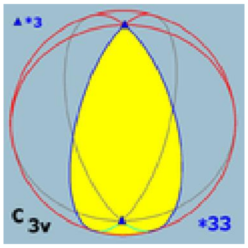
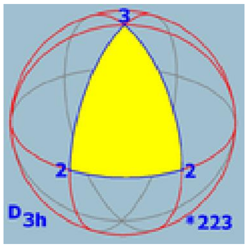
Point groups in three dimensions



Involutional symmetry
C_s, [1], (*)

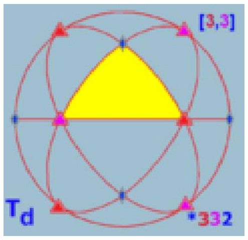


Cyclic symmetry
C_{nv}, [n], (*nn)

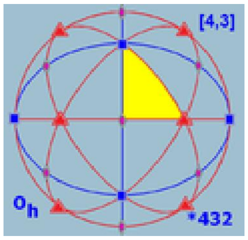


Dihedral symmetry
D_{nh}, [n,2], (*n22)

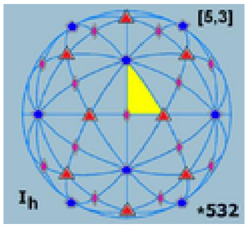
Polyhedral group, [n,3], (*n32)



Tetrahedral symmetry
T_d, [3,3], (*332)



Octahedral symmetry
O_h, [4,3], (*432)



Icosahedral symmetry
I_h, [5,3], (*532)

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Apart from the two infinite series of prismatic and antiprismatic symmetry, **rotational icosahedral symmetry** or **chiral icosahedral symmetry** of chiral objects and **full icosahedral symmetry** or **achiral icosahedral symmetry** are the discrete point symmetries (or equivalently, symmetries on the sphere) with the largest symmetry groups.
Icosahedral symmetry is *not* compatible with translational symmetry, so there are no associated crystallographic point groups or space groups.

Schönflies crystallographic notation	Coxeter notation	Orbifold notation	Order
I	$[3,5]^+$	532	60
I_h	[3,5]	*532	120

Presentations corresponding to the above are:

$$I : \langle s,t \mid s^2, t^3, (st)^5 \rangle$$
$$I_h : \langle s,t \mid s^3(st)^{-2}, t^5(st)^{-2} \rangle.$$

These correspond to the icosahedral groups (rotational and full) being the (2,3,5) triangle groups.
The first presentation was given by William Rowan Hamilton in 1856, in his paper on icosian calculus.[1]
Note that other presentations are possible, for instance as an alternating group (for I).

Group Structure

The **icosahedral rotation group** $\boldsymbol{I}$ is of order 60. The group I is isomorphic to A_5, the alternating group of even permutations of five objects. This isomorphism can be realized by I acting on various compounds, notably the compound of five cubes (which inscribe in the dodecahedron), the compound of five octahedra, or either of the two compounds of five tetrahedra (which are enantiomorphs, and inscribe in the dodecahedron).
The group contains 5 versions of T_h with 20 versions of D_3 (10 axes, 2 per axis), and 6 versions of D_5.
The **full icosahedral group** $\boldsymbol{I_h}$ has order 120. It has I as normal subgroup of index 2. The group I_h is isomorphic to $I \times Z_2$, or $A_5 \times Z_2$, with the inversion in the center corresponding to element (identity,-1), where Z_2 is written multiplicatively.
I_h acts on the compound of five cubes and the compound of five octahedra, but -1 acts as the identity (as cubes and octahedra are centrally symmetric).

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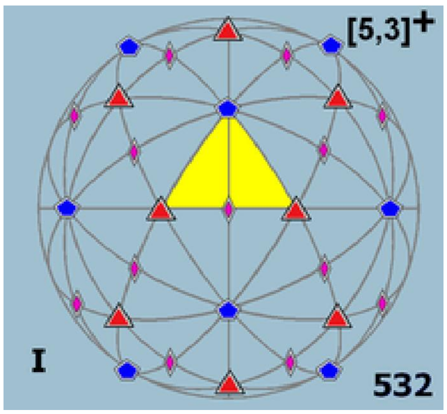
It acts on the compound of ten tetrahedra: I acts on the two chiral halves (compounds of five tetrahedra), and -1 interchanges the two halves. Notably, it does *not* act as S_5, and these groups are not isomorphic; see below for details.
The group contains 10 versions of D_{3d} and 6 versions of D_{5d} (symmetries like antiprisms).
I is also isomorphic to $PSL_2(5)$, but I_h is not isomorphic to $SL_2(5)$.



The icosahedral rotation group I with fundamental domain

Conjugacy classes

The conjugacy classes of I are:

- identity
- 12 × rotation by 72°, order 5
- 12 × rotation by 144°, order 5
- 20 × rotation by 120°, order 3
- 15 × rotation by 180°, order 2

Those of I_h include also each with inversion:

- inversion
- 12 × rotoreflection by 108°, order 10
- 12 × rotoreflection by 36°, order 10
- 20 × rotoreflection by 60°, order 6
- 15 × reflection, order 2

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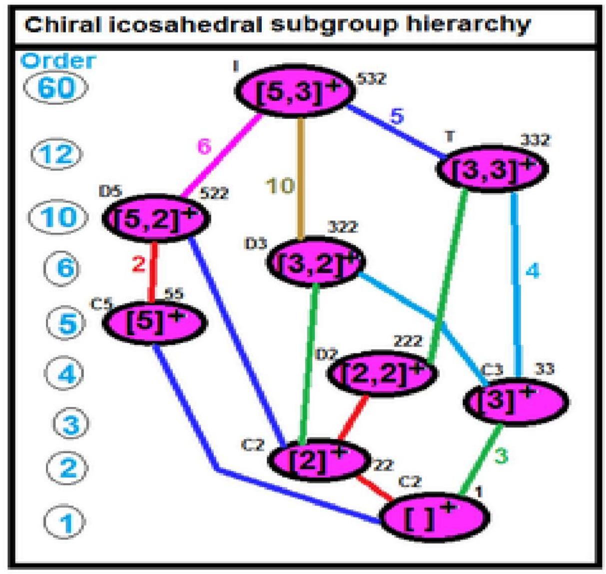
Subgroups of chiral icosahedral symmetry

Schoenflies notation	Coxeter notation	Orbifold notation	Hermann Mauguin notation	Order	Index
I	$[5,3]^+$	532	532	60	1
T	$[3,3]^+$	332	332	12	5
D_5	$[5,2]^+$	522	522	10	6
D_3	$[3,2]^+$	322	322	6	10
C_5	$[5]^+$	55	5	5	12
D_2	$[2,2]^+$	222	222	4	15
C_3	$[3]^+$	33	3	3	20
C_2	$[2]^+$	22	2	2	30
C_1	$[\,]^+$	11	1	1	60

Subgroup relations



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Subgroups of full icosahedral symmetry

Schoenflies notation	Coxeter notation	Orbifold notation	Hermann Mauguin notation	Order	Index
I_h	[5,3]	*532	532/m	120	1
I	$[5,3]^+$	532	532	60	2
D_{5d}	$[2^+,10]$	2*5	10m2	20	6
T_h	$[4,3^+]$	3*2	m3	12	10
T	$[3,3]^+$	332	332	12	10
D_{3d}	$[2^+,6]$	2*3	3m	12	10
D_5	$[5,2]^+$	522	522	10	12
C_{5v}	[5]	*55	5m	10	12
S_{10}	$[2^+,10^+]$	5×	5	10	12
D_{2h}	[2,2]	*222	mmm	8	15
D_3	$[3,2]^+$	322	322	6	20
C_{3v}	[3]	*33	3m	6	20
S_6	$[2^+,6^+]$	3×	3	6	20
C_5	$[5]^+$	55	5	5	24
D_2	$[2,2]^+$	222	222	4	30
C_{2h}	$[2,2^+]$	2*	2/m	4	30
C_{2v}	[2]	*22	mm2	4	30
C_3	$[3]^+$	33	3	3	40
C_s	[ ]	*	2or m	2	60
C_2	$[2]^+$	22	2	2	60
S_2	$[2^+,2^+]$	×	1	2	60
C_1	$[\ ]^+$	11	1	1	120

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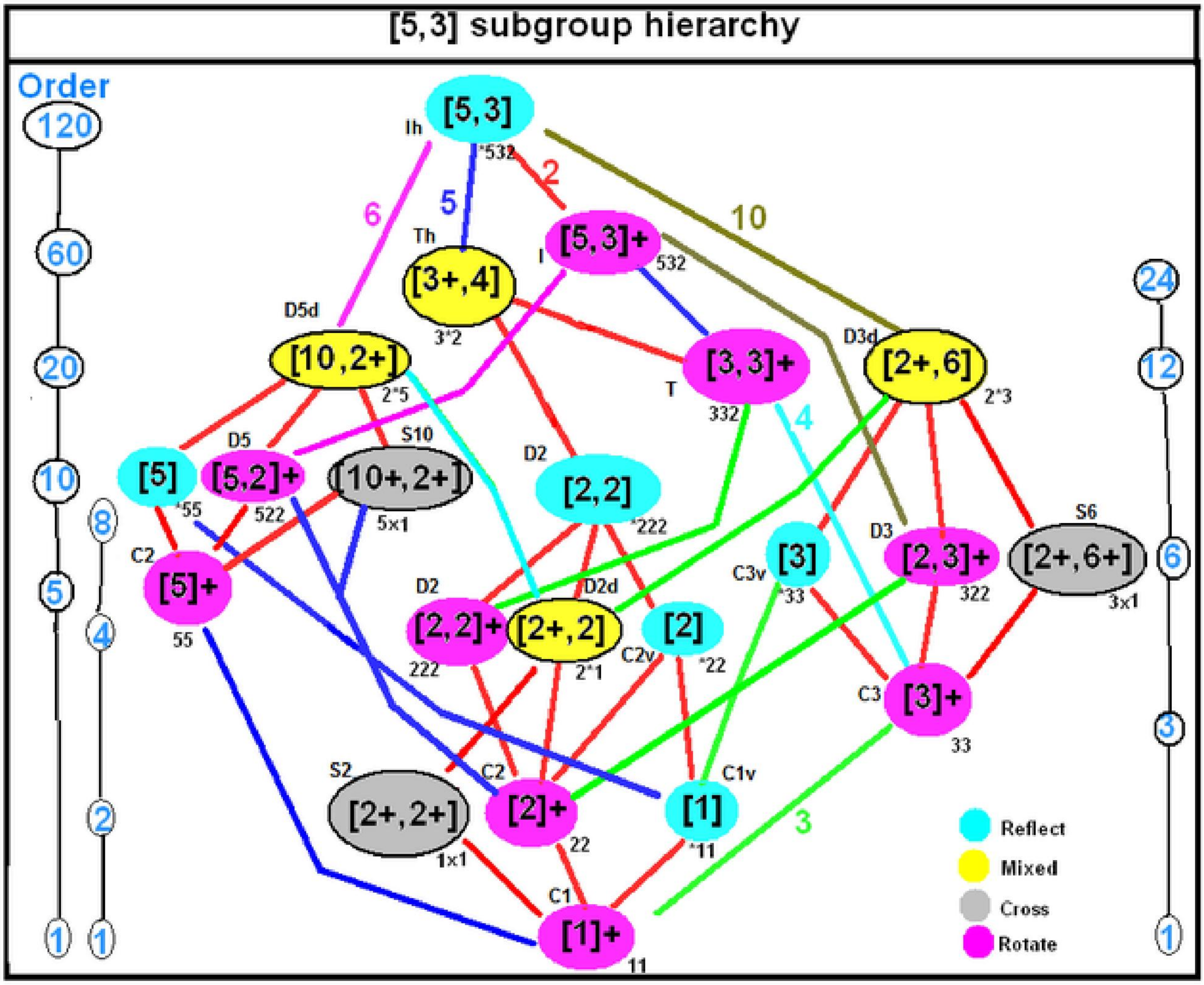

All of these classes of subgroups are conjugate (i.e., all vertex stabilizers are conjugate), and admit geometric interpretations.
Note that the stabilizer of a vertex/edge/face/polyhedron and its opposite are equal, since -1 is central.

Vertex stabilizers

Stabilizers of an opposite pair of vertices can be interpreted as stabilizers of the axis they generate.

- vertex stabilizers in I give cyclic groups C_3
- vertex stabilizers in I_h give dihedral groups D_3
- stabilizers of an opposite pair of vertices in I give dihedral groups D_3
- stabilizers of an opposite pair of vertices in I_h give $D_3 \times \pm 1$

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Edge stabilizers

Stabilizers of an opposite pair of edges can be interpreted as stabilizers of the rectangle they generate.

- edges stabilizers in I give cyclic groups Z_2
- edges stabilizers in I_h give Klein four-groups $Z_2 \times Z_2$
- stabilizers of a pair of edges in I give Klein four-groups $Z_2 \times Z_2$; there are 5 of these, given by rotation by 180° in 3 perpendicular axes.
- stabilizers of a pair of edges in I_h give $Z_2 \times Z_2 \times Z_2$; there are 5 of these, given by reflections in 3 perpendicular axes.

Face stabilizers

Stabilizers of an opposite pair of faces can be interpreted as stabilizers of the anti-prism they generate.

- face stabilizers in I give cyclic groups C_5
- face stabilizers in I_h give dihedral groups D_5
- stabilizers of an opposite pair of faces in I give dihedral groups D_5
- stabilizers of an opposite pair of faces in I_h give $D_5 \times \pm 1$

Polyhedron stabilizers

For each of these, there are 5 conjugate copies, and the conjugation action gives a map, indeed an isomorphism, $I \tilde{\to} A_5 < S_5$.

- stabilizers of the inscribed tetrahedra in I are a copy of T
- stabilizers of the inscribed tetrahedra in I_h are a copy of T_h
- stabilizers of the inscribed cubes (or opposite pair of tetrahedra, or octahedrons) in I are a copy of O
- stabilizers of the inscribed cubes (or opposite pair of tetrahedra, or octahedrons) in I_h are a copy of O_h

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'Other people, he said, see things and say why? But I dream things that never were and I say, why not?"

Robert F. Kennedy, after George Bernard Shaw