Spin as a Manifestation of a Nonlinear Constitutive Tensor and a Non-Riemannian Geometry

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Abstract
The electromagnetic constitutive tensor can be used to introduce a classical form of spin. For charged particles at rest, with no external forces, the spin does not affect the energy density of the particle. The particle solutions satisfy a non-Riemannian form of the Einstein-Maxwell equations.

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1. Introduction
The 4-dimensional electromagnetic constitutive tensor can be combined with a generalized stress-energy tensor to describe electric monopole particles with a classical form of spin. We will begin by looking at the problem in a 3-dimensional notation in order to develop a physical understanding of the role of the constitutive tensor. In particular, there are terms which do not affect the energy density. We will construct particular solutions which have some of the properties required for the elementary particles. We will then proceed to a 4-dimensional notation and will show that the solutions satisfy a generalized form of the Einstein-Maxwell equations. The spin portion of the electromagnetic stress-energy tensor will be equal to non-Riemannian curvature terms arising from a general symmetric connection. There are also electromagnetic wave solutions that couple only to the non-Riemannian part of the curvature tensor.

2. Maxwell’s Equations in 3-Dimensions
Maxwell’s equations can be written in 3-dimensions, using SI units, as:

\[ E_i = -\partial_t A_i \]
\[ D_i = \varepsilon_{ij} E^j - \gamma_{ji} B^j \]
\[ \rho = D^i \]
\[ B^i = \varepsilon^{ijk} A_{k,j} \]
\[ H_i = \alpha_{ij} B^j + \gamma_{ij} E^j \]
\[ j^i = \varepsilon^{ijk} H_{k,j} - \partial_t D^j \]

where \( \varepsilon^{ijk} \) is the Levi-Civita tensor and \( \alpha_{ij} \) is the inverse permeability. In free space, with metric \( g_{ij} \),

\[ \varepsilon_{ij} = \varepsilon_0 g_{ij} \]
\[ \alpha_{ij} = \mu_0^{-1} g_{ij} \]
\[ c^2 \varepsilon_0 \mu_0 = 1 \]

(2.2)

The following vector-dyadic notation will also be useful:

\[ D = \varepsilon \cdot E - B \cdot \gamma \]
\[ H = \alpha \cdot B + \gamma \cdot E \]

(2.3)

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\[ D = \varepsilon \cdot E - B \cdot \gamma \]
\[ H = \alpha \cdot B + \gamma \cdot E \]

The \( \gamma_{ij} \) have not traditionally been written explicitly in classical electromagnetic theory. However, they arise from the fact that, in the 4-dimensional formulation, the constitutive relations are described by a fourth rank tensor.
We will generalize the traditional definitions of the energy density, the symmetric stress tensor and the Poynting vector.

\begin{align}
E_n &= \frac{1}{2}(\alpha_{ij}B^iB^j + \epsilon_{ij}E^iE^j) - Q \\
T^{ij} &= -\frac{1}{2}(E^iD^j + E^jD^i + H^iB^j + H^jB^i) + \frac{1}{2}g^{ij}(\alpha_{mn}B^mB^n + \epsilon_{mn}E^mE^n) + g^{ij}Q \\
N^i &= \frac{1}{2}\epsilon^{ijk}(E_jH_k + c^2D_jB_k) \\
\end{align}

(2.4a) (2.4b) (2.4c)

The reason for these definitions is to make the 4-dimensional stress-energy tensor symmetric, thus ensuring that angular momentum is conserved and that gravitation can be included via a symmetric gravitational stress-energy tensor. The function \(Q\) will be chosen so that the particle solutions are force-free. The determination of \(Q\) has to be done for each particle separately. \(T^{ij}\) is defined with the opposite sign from what is usually used in 3-dimensions. It is useful because it lets \(T^{ij}\) be the spatial part of \(T^{\mu\nu}\), which is defined so that \(T^4_4 = -E_n\). Since the energy density does not include any contribution from the \(\gamma_{ij}\) terms, they can represent internal degrees of freedom. We will define the force density and the power loss density.

\begin{align}
F_i &= -T_{,ij} - c^{-2}\partial_iN_i \\
P_{\mu\nu} &= -N_{,\mu\nu} - \partial_iE_n \\
\end{align}

(2.5)

In this paper, we will show that time-independent solutions for which \(\mathbf{B} = 0\) and \(\gamma \neq 0\) can be used to represent particles with spin.

### 3. Electric Monopole Solutions With Spin

In spherical coordinates \((r, \theta, \varphi)\), let

\begin{align}
E &= f_0(r)e_r = -e_r\alpha'(r) \\
A &= 0 \\
\mathbf{a} &= c^2\mathbf{e} = c^2\epsilon_0f_0(r)(e_r\epsilon_r + e_\theta\epsilon_\theta + e_\varphi\epsilon_\varphi) \\
\gamma &= h(r)[(2e_r\epsilon_r - e_\theta\epsilon_\theta - e_\varphi\epsilon_\varphi)\cos(\theta) + (e_r\epsilon_r + e_\theta\epsilon_\theta)\sin(\theta)] \\
\end{align}

(3.1a) (3.1b) (3.1c)

Then

\begin{align}
D &= \epsilon_0f_0(r)f_0(r)\epsilon_r \\
\rho(r) &= \epsilon_0r^{-2}\{\partial_r[r^2f_0(r)f_0(r)]\} \\
H &= f_0(r)h(r)[2\cos(\theta)\epsilon_r + \sin(\theta)\epsilon_\theta] \\
j &= r^{-1}\{2f_0(r)h(r) + \partial_r[f_0(r)h(r)]\}\sin(\theta)\epsilon_\varphi \\
Q(r) &= \frac{1}{2}\epsilon_0f^2_0(r)f_0(r) + 2\epsilon_0\int dr r^{-1}f^2_0(r)f_0(r) \\
T &= \frac{1}{2}\epsilon_0f^2_0(r)f_0(r)[-e_r\epsilon_r + e_\theta\epsilon_\theta + e_\varphi\epsilon_\varphi] + Q(r)[e_r\epsilon_r + e_\theta\epsilon_\theta + e_\varphi\epsilon_\varphi] \\
E_n(r) &= \frac{1}{2}\epsilon_0f^2_0(r)f_0(r) - Q(r) = -2\epsilon_0\int dr r^{-1}f^2_0(r)f_0(r) \\
N &= \frac{1}{2}h(r)f^2_0(r)\sin(\theta)\epsilon_\varphi \\
\end{align}

(3.2a) (3.2b) (3.2c) (3.2d) (3.2e) (3.2f) (3.2g) (3.2h)

For continuously differentiable functions, these solutions are force free and radiationless. At \(r = 0\), we must have \(f_0(0) = 0\). We must also have

\begin{align}
\lim_{r \to \infty}f_0(r) &= q(4\epsilon_0r^2)^{-1} \\
\lim_{r \to \infty}f_0(r) &= 1 \\
\lim_{r \to \infty}H &= \gamma(4\pi r^3)^{-1}[2\cos(\theta)e_r + \sin(\theta)e_\theta] \\
\end{align}

(3.3a) (3.3b) (3.3c)

The rest mass

\begin{align}
m_0 &= c^2\int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi E_n(r) \\
&= -8\pi\epsilon_0c^{-2}\int_0^\infty dr r^2 \int_r^\infty dr' (r')^{-1}f^2_0(r')f_0(r') \\
\end{align}

(3.4)

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Since
\[ e_r = \sin(\theta) \cos(\varphi) e_x + \sin(\varphi) \sin(\theta) e_y + \cos(\theta) e_z \]
\[ e_\theta = \cos(\theta) \cos(\varphi) e_x + \cos(\varphi) \sin(\theta) e_y - \sin(\theta) e_z \]
\[ e_\varphi = -\sin(\varphi) e_x + \cos(\varphi) e_y \] \hspace{1cm} (3.5)
the total angular momentum
\[ J_T = c^{-2} \int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi \mathbf{r} \times \mathbf{N} \]
\[ = \frac{4}{3} \pi c^{-2} \int_0^\infty dr r^3 h(r) f_x(r) e_z \] \hspace{1cm} (3.6)

The total current and the total angular moment of the current are defined by
\[ j_T = \int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi j \]
\[ = 0 \] \hspace{1cm} (3.7a)
\[ M_T = \int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi \mathbf{r} \times j \]
\[ = \frac{8}{3} \pi e_z \int_0^\infty dr \int_0^{2\pi} d\varphi \{ \partial_r [r f_r(r) h(r)] + 2 f_r(r) h(r) \} \] \hspace{1cm} (3.7b)

If there are external fields with potentials
\[ \phi_{\text{ext}} = -(E_{0x} x + E_{0y} y + E_{0z} z) \]
\[ \mathbf{A}_{\text{ext}} = \frac{1}{2} [(B_{0y} z - B_{0z} y) e_x + (B_{0x} x - B_{0z} z) e_y + (B_{0y} y - B_{0x} x) e_z] \]
then we have the constant fields
\[ E_0 = E_{0x} e_x + E_{0y} e_y + E_{0z} e_z \]
\[ B_0 = B_{0x} e_x + B_{0y} e_y + B_{0z} e_z \] \hspace{1cm} (3.9)

If we assume that the external fields do not, to a first approximation, modify \( f_r(r) \), \( h(r) \) and \( Q(r) \), then the total force and the total torque are
\[ F_T = \int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi \mathbf{F} \]
\[ = 4\pi \varepsilon_0 r^2 f_r(r) f_x(r) |_{r=0}^{r=\infty} E_0 \]
\[ = q E_0 \] \hspace{1cm} (3.10a)
\[ W_T = \int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi \mathbf{r} \times \mathbf{F} \]
\[ = 2\pi r^3 h(r) f_r(r) |_{r=0}^{r=\infty} e_z \times B_0 \]
\[ = \frac{1}{2} \gamma e_z \times B_0 \] \hspace{1cm} (3.10b)

The factor of \( \frac{1}{2} \) in \( W_T \) distinguishes this result from the normal magnetic dipole, \( W_T = \mu_m e_z \times B \). Thus the numerical values for \( \gamma \) are related to the numerical values reported for \( \mu_m \) by
\[ |\gamma| = 2 |\mu_m| \] \hspace{1cm} (3.11)

We can define an effective rest mass energy for the particle by subtracting the unperturbed energy density of the external field. In this case, referring back to the general definition in (2.4a),
\[ m_{\text{eff}} c^2 = \int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi [E_n - \frac{1}{2} \varepsilon_0 (B_0^2 c^2 + E_0^2)] \]
\[ = m_0 c^2 + 2\pi \varepsilon_0 (B_0^2 c^2 + E_0^2) \int_0^\infty dr r^2 [f_r(r) - 1] \] \hspace{1cm} (3.12)
Even though the particle is at rest, we can define an effective total field momentum and an effective total angular momentum by subtracting the unperturbed Poynting vector at infinity. In this case,

\[ N_0 = c^2 \epsilon_0 [q(4\pi \epsilon_0 r^2)^{-1}e_r + E_0] \times B_0 \]  
\[ P_{\text{eff}} = c^{-2} \int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi \ (N - N_0) \]

\[ = 4\pi \epsilon_0 E_0 \times B_0 \int_0^\infty \, dr \, r^2 [f_e(r) - 1] \]  
\[ J_{\text{eff}} = c^{-2} \int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi \ r \times (N - N_0) \]

\[ = \frac{4}{3} \pi c^{-2} \int_0^\infty dr \, r^3 h(r) f^2_e(r) e_z + \frac{2}{3} B_0 \int_0^\infty dr \, r [q - 4\pi \epsilon_0 r^2 f_e(r) f_e(r)] \]  

Thus we can define the angular momentum about another point \( P \) as

\[ J_P = J_{\text{eff}} + r_P \times P_{\text{eff}} \]  

(3.14)

where \( r_P \) is the radius vector from the point \( P \) to the center of the particle. In standard quantum theory, this is valid for orbital angular momenta, but not for spin. The total current and the total angular moment of the current are

\[ j_T = 0 \]  
\[ M_T = \frac{8}{3} \pi \epsilon_z \int_0^\infty dr \, r^2 \{ \partial_r [r f_e(r) h(r)] + 2 f_e(r) h(r) \} - \frac{8}{3} \pi \epsilon_0 c^2 B_0 \int_0^\infty dr \, r^3 f'_e(r) \]  

(3.15a)  

(3.15b)

In classical theory, every static magnetic field is associated with a current of moving charges. In quantum theory, static magnetic fields are associated either with a current of moving charges or with a fixed array of particles that have spin. In this theory, \( B = 0 \) for spin fields, but not for moving charges. This would seem to be one reason why it has been difficult to explore the mathematical transition between quantum theory and classical theory. For a particle moving with a uniform velocity in an inertial frame, a transformation of the fields to the rest frame of the particle will show that the Lorentz force law will be modified if the source of the external magnetic field is due to spins rather than to moving charges.

4. Particular Solutions

For any given total charge \( q \), it is obviously possible to construct particles with many different rest masses. As an example of solutions for which \( f_e(r) \) approaches \( q(4\pi \epsilon_0 r^2)^{-1} \) exponentially, we can look at

\[ f_e(r) = q(4\pi \epsilon_0)^{-1} \{ 1 - \exp[-(r/r_0)^3] \} r^{-2} \]  
\[ f_e(r) = 1 + \lambda \exp[-(r/r_0)^3] \]  
\[ h(r) = \gamma \epsilon_0 q^{-1} \{ 1 - \exp[-(r/r_0)^3] \} r^{-1} \]  

Then

\[ m_0 c^2 = q^2 (6\pi \epsilon_0 r_0)^{-1} [2 - 2^{1/3} + (-1 + 2^{1/3} - 3^{1/3}) \lambda] \Gamma(2/3) \]  
\[ J_T = \gamma q [4 \pi^2 \epsilon_0 r_0^{-1} (1 - 2^{1/3} + 3^{-2/3}) \Gamma(2/3) e_z] \]  
\[ m_{\text{eff}} c^2 = m_0 c^2 + \frac{8}{3} \pi \epsilon_0 \lambda r_0^3 (B_0^2 c^2 + E_0^2) \]  
\[ P_{\text{eff}} = \frac{4}{3} \pi \epsilon_0 \lambda r_0^3 E_0 \times B_0 \]  
\[ J_{\text{eff}} = J_T + \frac{4}{3} [-2 + (-2 + 2^{1/3}) \lambda] q r_0^2 \Gamma(2/3) B_0 \]  
\[ M_{\text{eff}} = \frac{2}{3} \gamma e_z + \frac{8}{3} \pi \epsilon_0 c^2 \lambda r_0^3 B_0 \]  

(4.2a)  

(4.2b)  

(4.2c)  

(4.2d)  

(4.2e)  

(4.2f)
If we set the z-component of \( \mathbf{J}_T \) to \( \frac{1}{2} \hbar \) and use (3.11), (4.2a) and (4.2b) with \( \lambda = -1 \), then we obtain the standard magneton result, \( 2\mu_m = \gamma = q\hbar \mu_0^{-1} \). Adding additional terms to the example above makes it possible to satisfy the conditions \( m_{\text{eff}} = m_0 \), \( \mathbf{P}_{\text{eff}} = 0 \) and \( \mathbf{J}_{\text{eff}} = \mathbf{J}_T \).

\[
\begin{align*}
  f_e(r) &= q(4\pi \epsilon_0)^{-1} [1 - (1 + \lambda_e/r_0)^3] r^{-2} \\
  f_e(r) &= 1 + [\lambda_0 + \lambda_1(r/r_0)^3 + \lambda_2(r/r_0)^6] \exp[-(r/r_0)^3] \\
  h(r) &= \gamma \epsilon_0 q^{-1} [1 - (1 + \lambda_h(r/r_0)^3] r^{-1} \\
  \lambda_1 &= \frac{-3[72 + (32 + 3 \cdot 2^{1/3}) \lambda_0] - 16(9 + 2^{1/3} \lambda_0) \lambda_0}{3((-8 + 7 \cdot 2^{1/3}) + 10 \cdot 2^{1/3} \lambda_0)} \\
  \lambda_2 &= -\frac{1}{2} (\lambda_0 + \lambda_1) \\
  m_{\text{eff}}^2 &= \left( \frac{2}{\gamma} \right)^2 \left( \frac{17986}{\gamma} \right) \left( \frac{\epsilon_0 \hbar}{r_0} \right)^{-1} \\
  J_T &= \gamma q \left( \frac{17986}{\gamma} \right) \left( \frac{\epsilon_0 \hbar}{r_0} \right)^{-1} \\
  M_T &= \frac{2}{\gamma} \epsilon_0 e_z \\

\end{align*}
\]

An example of a neutral particle for which \( \mathbf{F}_T = 0 \) and \( \mathbf{W}_T = 0 \) and for which \( m_{\text{eff}} = m_0 \), \( \mathbf{P}_{\text{eff}} = 0 \) and \( \mathbf{J}_{\text{eff}} = \mathbf{J}_T \) is

\[
\begin{align*}
  f_e(r) &= \beta r \exp(-r/r_0) \\
  f_e(r) &= 1 + [\lambda_0 + \lambda_1(r/r_0)^3 + \lambda_2(r/r_0)^6] \exp(-r/r_0) \\
  h(r) &= \eta r \exp(-r/r_0) \\
  \lambda_1 &= -\frac{1}{2} (256 + 3 \lambda_0) \\
  \lambda_2 &= \frac{1}{2} (192 + \lambda_0) \\
  m_{\text{eff}}^2 &= 2 \beta^2 \pi \epsilon_0 \hbar \left( -10101 + 64 \lambda_0 \right) (2187)^{-1} \\
  J_T &= 320 \pi \beta^2 \eta \gamma \left( \frac{17986}{\gamma} \right)^{-1} e_z \\
  M_T &= 0
\end{align*}
\]

There are, as yet, no quantization conditions to specify the allowable solutions.

5. Maxwell’s Equations in 4-Dimensions

The electromagnetic fields and the current density are defined by

\[
\begin{align*}
  f_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu,\mu} - A_{\mu,\nu} \\
  p^{\mu\nu} &= \frac{1}{2} \chi^{\mu\rho\sigma} f_{\rho\sigma} \\
  j^\mu &= p^{\mu\nu} \epsilon_{\nu} \\
\end{align*}
\]

where \( f_{\mu\nu} \) and \( p^{\mu\nu} \) are antisymmetric and the constitutive tensor, \( \chi^{\mu\rho\sigma} \), has the symmetries

\[
\begin{align*}
  \chi^{\mu\rho\sigma} &= -\chi^{\nu\mu\rho} \\
  \chi^{\mu\nu\rho} &= -\chi^{\nu\rho\mu} \\
  \chi^{\mu\rho\sigma} &= \chi^{\rho\sigma\mu} \\
\end{align*}
\]

In terms of the 3-dimensional potentials, \( A_\mu = c (A, -\phi) \). We will define the stress-energy tensor and the force density.

\[
\begin{align*}
  T^{\mu\nu} &= \frac{1}{2} (f^{\mu}_{\tau} p^{\nu}_{\tau} + f^{\nu}_{\tau} p^{\mu}_{\tau}) - g^{\mu\nu} (\frac{1}{4} f_{\kappa\tau} p^{\kappa\tau} - Q) \\
  f_\mu &= -T^{\mu}_{\nu} \epsilon^\nu \\
\end{align*}
\]
6. Einstein-Maxwell Equations

Eisenhart [1] shows that the most general symmetric connection can be written in the form

\[
\hat{\Gamma}^\mu_{\alpha\beta} = a^\mu_{\alpha\beta} + \Gamma^\mu_{\alpha\beta} \quad a^\mu_{\alpha\beta} = a^\mu_{\beta\alpha} \quad \Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}
\]

(6.1)

where \(a^\mu_{\alpha\beta}\) is a tensor and \(\Gamma^\mu_{\alpha\beta}\) is the metric connection. The curvature tensor for \(\hat{\Gamma}^\mu_{\alpha\beta}\) can be written as [1, eq. 5.15],

\[
B^\mu_{\nu\rho\sigma} = R^\mu_{\nu\rho\sigma} + a^\mu_{\nu\rho,\sigma} - a^\mu_{\nu\sigma,\rho} + a^\alpha_{\nu\sigma} a^\mu_{\alpha\rho} - a^\alpha_{\nu\rho} a^\mu_{\alpha\sigma}
\]

(6.2)

where \(R^\mu_{\nu\rho\sigma}\) is the Riemann curvature tensor for the metric \(g_{\mu\nu}\). A semicolon denotes covariant differentiation with respect to the metric connection, \(\Gamma^\mu_{\alpha\beta}\); a colon will denote covariant differentiation with respect to the general symmetric connection, \(\hat{\Gamma}^\mu_{\alpha\beta}\); and a comma will denote partial differentiation with respect to the coordinates. (This notation is somewhat different from that used by Eisenhart. He uses the Christoffel symbols for the metric connection and \(\Gamma^\mu_{\alpha\beta}\) for the general symmetric connection. More importantly, he usually uses a comma to denote covariant differentiation with respect to the general symmetric connection.) It is important to note that \(g_{\mu\nu,\rho} = -g_{\mu\nu} a^\alpha_{\rho\nu} - g_{\mu\nu} a^\alpha_{\nu\rho} \neq 0\). For that reason, the equations are expressed in terms of covariant differentiation with respect to the metric connection, \(\Gamma^\mu_{\alpha\beta}\). Define

\[
B^\mu_{\nu\rho\sigma} = B^\mu_{\nu\rho\sigma} = R^\mu_{\nu\rho\sigma} + a^\mu_{\nu\rho,\sigma} - a^\mu_{\nu\sigma,\rho} + a^\alpha_{\nu\sigma} a^\mu_{\alpha\rho} - a^\alpha_{\nu\rho} a^\mu_{\alpha\sigma}
\]

(6.3)

Define symmetric and antisymmetric parts in the following way:

\[
S^\nu_{\sigma} = \frac{1}{2}(B^\nu_{\sigma\nu} + B^\nu_{\sigma\nu}) - R^\nu_{\sigma\nu} = a^\mu_{\nu\sigma,\mu} - \frac{1}{2}(a^\mu_{\nu\sigma,\mu} + a^\mu_{\sigma\nu,\mu}) + a^\alpha_{\nu\sigma} a^\mu_{\alpha\mu} - a^\alpha_{\nu\rho} a^\mu_{\alpha\rho}
\]

\[
A^\nu_{\sigma} = \frac{1}{2}(B^\nu_{\sigma\nu} - B^\nu_{\sigma\nu}) = -\frac{1}{2}(a^\mu_{\nu\sigma,\mu} - a^\mu_{\nu\mu,\sigma})
\]

(6.4a, 6.4b)

If we consider

\[
a^\mu_{\nu\mu} = 0 \quad A^\nu_{\sigma\nu} = 0 \quad S^\mu_{\nu\sigma} = a^\alpha_{\mu\nu\sigma} - a^\beta_{\nu\mu\alpha}
\]

(6.5)

then we can write a generalized form of the Einstein-Maxwell equations

\[
G^\mu_{\nu} + S^\mu_{\nu} = 8\pi Gc^{-4}T^\mu_{\nu}
\]

(6.6)

where

\[
G^\mu_{\nu} = R^\mu_{\nu\rho\sigma} - \frac{1}{2}g_{\mu\nu}R
\]

(6.7)

and \(G\) is Newton’s gravitational constant. For a particle at rest, we will see that the spin is described by the non-Riemannian part of the symmetric connection. In the rest frame of the particle, let the metric be given by

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f^{-1}_g(r)dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 - c^2f_g(r)dt^2
\]

(6.8)

and let the only non-zero components of \(a^\mu_{\nu\sigma}\) be

\[
a^3_{44} = \zeta_1(r, \theta) \quad a^4_{33} = \zeta_2(r, \theta)
\]

(6.9)

The only non-zero components of \(S^\mu_{\nu}\) are

\[
S^3_{44} = S^4_{33} = -a^3_{44}a^4_{33} = -\zeta_1(r, \theta)\zeta_2(r, \theta)
\]

(6.10)

If

\[
A_\mu = (0,0,0, -c\phi(r)) \quad \phi(r) = -\int dr f_\phi(r)
\]

(6.11)

and if the metric and non-metric components of the constitutive tensor are specified by

\[
\chi_{\mu\nu\rho\sigma} = \epsilon_0 f_\epsilon(r)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})
\]

\[
\chi_{3241} = -2r^2h(r)\sin(\theta)\cos(\theta)
\]

\[
\chi_{3242} = -r^2f_g^2(r)\chi_{3141} = -r^3h(r)f_g(r)\sin^2(\theta)
\]

\[
\chi_{2143} = -\chi_{3142} = r^3h(r)\sin(\theta)\cos(\theta)
\]

(6.12)
then the non-zero components of $T_{\mu\nu}$ and $G_{\mu\nu}$ are
\begin{align}
T_{44} &= -c^2 f_4'(r)T_{11} = -2c^2 \epsilon_0 f_6(r) \int dr \, r^{-1} f_6'(r)f_6(r) \\
T_{33} &= \sin^2(\theta)T_{22} = \epsilon_0 r^{-2}[f_2'(r)f_6(r) + 2 \int dr \, r^{-1} f_2'(r)f_6(r)] \sin^2(\theta) \\
T_{34} &= -\frac{1}{2} r h(r)f_2'(r) \sin^2(\theta) \\
G_{44} &= -c^2 f_4'(r)G_{11} = -c^2 r^{-2} f_6(r)[-1 + f_6(r) + r f_6'(r)] \\
G_{33} &= \sin^2(\theta)G_{22} = [rf_6'(r) + \frac{1}{2} r^2 f_6''(r)] \sin^2(\theta)
\end{align}
Equations (6.6) reduce to
\begin{align}
-1 + f_6(r) + rf_6'(r) &= 16\pi G c^{-4} \epsilon_0 r^2 \int dr \, r^{-1} f_6'(r)f_6(r) \\
rf_6'(r) + \frac{1}{2} r^2 f_6''(r) &= 8\pi G c^{-4} \epsilon_0 r^2[f_2'(r)f_6(r) + 2 \int dr \, r^{-1} f_2'(r)f_6(r)] \\
\zeta_1(r, \theta)\zeta_2(r, \theta) &= 4\pi G c^{-4} rh(r)f_2'(r) \sin^2(\theta)
\end{align}
Integrating (6.14b) and substituting into (6.14a) gives
\begin{align}
f_6(r) = 1 + 16\pi G \epsilon_0 c^{-4} r^{-1} \int_0^r dr' (r')^2 \int_0^{r'} dr'' (r'')^{-1} f_6'(r'')f_6(r'')
\end{align}
Comparison with the Schwarzschild metric, for which $f_6(r) = 1 - 2Gm_0c^{-2}r^{-1}$, shows that
\begin{align}
m_0 = -8\pi \epsilon_0 c^{-2} \int_0^\infty dr \, r^2 \int_0^r dr' (r')^{-1} f_6'(r')f_6(r')
\end{align}
which agrees with (3.4). The entire rest mass is electromagnetic.

7. Electromagnetic Waves

In a cylindrically symmetric space with the metric given by
\begin{align}
ds^2 = dr^2 + r^2 d\varphi^2 + dz^2 - c^2 dt^2
\end{align}
let us look at electromagnetic waves of the form
\begin{align}
A_\mu &= cf(z-ct) f_{em}(r)(1, 0, 0, 0) \\
\chi_{\mu\nu\rho\sigma} &= \epsilon_0 (g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}) \\
Q &= 0 \\
c^2 a_{33} &= -ca_{43} = a_{14} = [f'(z-ct)]^2 f_6(r)
\end{align}
Then $G_{\mu\nu} = 0$ and the non-zero components of $T_{\mu\nu}$ and $S_{\mu\nu}$ are
\begin{align}
T_{44} &= -ct_{34} = c^2 T_{33} = c^4 \epsilon_0 [f'(z-ct)]^2 f_{em}^2(r) \\
S_{44} &= -cS_{34} = c^2 S_{33} = [f'(z-ct)]^2 [f_6'(r) + r^{-1} f_6(r)]
\end{align}
Thus $T^{\mu\nu}_{;\nu} = 0$ and $S^{\mu\nu}_{;\nu} = 0$ and equations (6.6) reduce to
\begin{align}
f_6(r) = 8\pi G \epsilon_0 r^{-1} \int_0^r dr' r' f_{em}^2(r')
\end{align}
For this type of wave, \( j_\mu \neq 0 \), but \( j_\mu j^\mu = 0 \). However,

\[
\tilde{\tilde{\mathcal{J}}}_\mu = \int_0^\infty dr \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \, j^\mu \\
= 2\pi\epsilon_0 \left[ rf_{em}(r)\right]_r^\infty f(z-ct)|z=-\infty (0, 0, c, 1) \\
= 0
\]

(7.5)

As an example, consider

\[
f_{em}(r) = r \exp(-\beta^2 r^2) \quad (7.6a) \\
f_a(r) = \pi G\epsilon_0 (r\beta^4)^{-1}[1 - (1 + 2\beta^2 r^2) \exp(-2\beta^2 r^2)] \\
\]

(7.6b)

In free space, electromagnetic waves are usually assumed to have zero current, \( j_\mu = 0 \). However, if we admit the possibility of non-zero field currents such that \( j_\mu j^\mu = 0 \), then we have a class of force-free wave solutions that have a spatial variation in the plane perpendicular to the direction of propagation. These null-vector field currents are intrinsic to the structure of the wave; they are not an external source. Furthermore, this class of solutions introduces terms only in the non-Riemannian part of the curvature tensor.

8. Conclusions

We have made use of the properties of the constitutive tensor together with a generalized form of the stress-energy tensor to introduce a classical form of spin for charged particles. The solutions are force-free and radiationless. We have constructed particular solutions that have some of the properties required for the elementary particles. We have also shown that the spin terms can be incorporated into the Einstein-Maxwell equations by adding curvature terms arising from a general symmetric connection. These additional curvature terms can also be used to construct a new class of electromagnetic waves.

Acknowledgments

Many of the calculations were done using Mathematica® 8.01 [2] with the MathTensor™ 2.2.1 [3] Application Package. Stephen C. Young double-checked the derivations and suggested several clarifications.

References


2. Wolfram Research, Mathematica® 8.01 (http://www.wolfram.com/).

3. L. Parker and S.M. Christensen, MathTensor™ 2.2.1 (http://smc.vnet.net/MathTensor.html).