Mathematical Proof of Four-Color Theorem

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Abstract

The method and basic theory are far from traditional graph theory. Maybe they are the key factor of success. *K4 regions* (every region is adjacent to other 3 regions) are the max adjacent relationship, four-color theorem is true because more than 4 regions, there must be a non-adjacent region existing. Non-adjacent regions can be color by the same color and decrease color consumption.

Another important three-color theorem is that the border of regions can be colored by 3 colors. Every region has at least 2 optional colors. Every border region can be permuted without impact on other border regions.

1. Introduce

How many different colors are sufficient to color the regions on a map in such a way that no two adjacent regions have the same color? After examining a wide variety of different planar graphs, one discovers the apparent fact that every graph, regardless of size or complexity, can be colored with just four distinct colors.

The famous four color theorem, sometimes known as the four-color map theorem or Guthrie's problem. There had been numerous attempts to prove the supposition in mathematical history, but these so-called proofs turned out to be flawed. There had been accepted proofs that a map could be colored in using more colors than four, such as six or five, but proving that only four colors were required was not done successfully until 1976 by mathematicians Appel and Haken, although some mathematicians do not accept it since parts of the proof consisted of an analysis of discrete cases by a computer. But, at the present time, the proof remains viable. It is possible that an even simpler, more elegant, proof will someday be discovered, but many mathematicians think that a shorter, more elegant and simple proof is impossible.

In the mathematical field of graph theory, a complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete digraph is a directed graph in which every pair of distinct vertices is connected by a pair of unique edges (one in each direction). Kn denotes the complete graph on n vertices. K1 through K4 are all planar graphs. However, every planar drawing of a complete graph with five or more vertices must contain a crossing, and the non-planar complete graph K5 plays a key role in the characterizations of planar graphs.

2. Four color theorem

(2.1) For any subdivision of the spherical surface into non-overlapping regions, it is always possible to mark each of the regions with one of the numbers 1, 2, 3, 4, in such a way that no two adjacent regions receive the same number.

In fact, if the four-color theorem is true on spherical surface, it is also true on plane surface. Because the map is originate from sphere, and plane surface is part of spherical surface.

3. Strategy

K4 regions (every region is adjacent to other 3 regions) are the max adjacent relationship, four-color theorem is true because more than 4 regions, there must be a non-adjacent region existing. Non-adjacent regions can be color by the same color and decrease color consumption.

Another important theorem is that the border of regions can be colored by 3 colors. Every region has at least 2 optional colors. Every border region can be permuted without impact on other border regions.

4. Basic axiom

(4.1) Coloring the regions on a map has nothing to do with the region shape.

This is the only one axiom in proof. It’s obviously true. Color only depends on adjacent relationship.

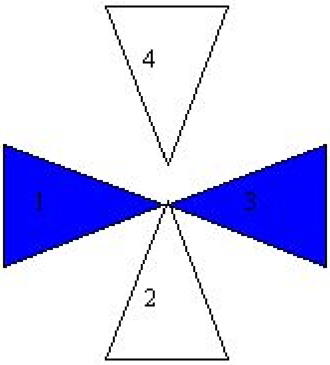
Theorem (4.2)

All color solutions for boundary adjacent regions can apply to point adjacent regions or non-adjacent regions.

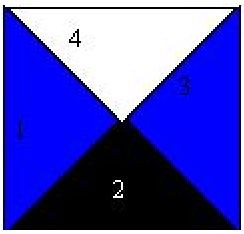
We define adjacent regions as those that share a common boundary of non-zero length. Regions, which meet at a single point or limited points, are not considered to be "adjacent".

Because point adjacent regions are not considered to be "adjacent", any color solution can apply to point adjacent regions, include the color solution of boundary adjacent regions. The free degree of non-adjacent region is limitless. So any color solution of boundary adjacent regions can apply to point adjacent regions and non-adjacent regions.

For example:



Scenario a: non-adjacent and point adjacent



Scenario b: boundary adjacent

All color solutions for Scenario b can apply to Scenario a.

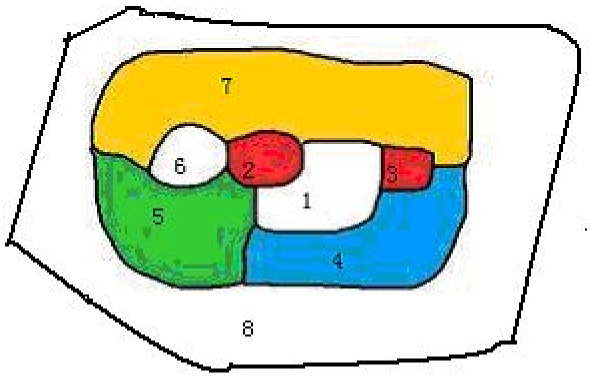
Theorem (4.3)

Any irregular regions map can transform into a circle regions map. The color solution for circle regions map can also apply to the irregular regions map

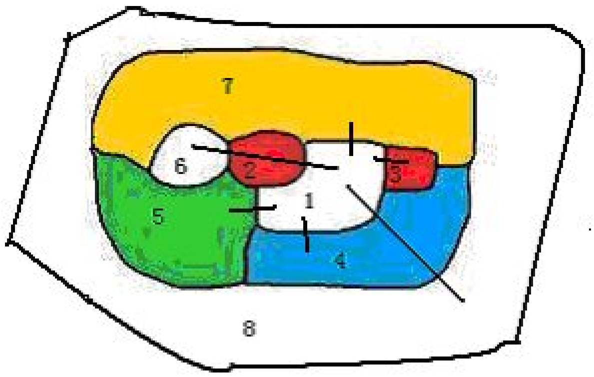
Because basic axiom (4.1) $\Rightarrow$ any irregular regions map can transform into circle-shaped, ring-shaped or fan-shaped.

If circle-shaped, ring-shaped or fan-shaped are point adjacent or non-adjacent, transform into boundary adjacent, finally, to transform into a circle map, ring-shaped and fan-shaped surround circle. Because of Theorem (4.2), the color solution of map transformed can apply to the color solution of map transforming before.

For example:



This an irregular map.



To ensure arbitrary map can be transform into circle map, first select a circle center, second draw a line from center to region, the least region number crossed over is the layer number of ring. From 1 to 6, the least region number is 2, from 1 to 8, the least region number is 2, and so both 6 and 8 are in layer 3 in circle map. Other regions are in layer 2.

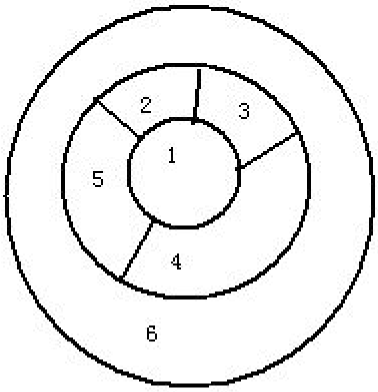
Layer	**Region**
Layer 1	1
Layer 2	2,3,4,5,7
Layer 3	6,8

To ensure to preserve the adjacent relationship in transforming,

Region	**Adjacent region**
1	2,3,4,5,7
2	1,5,6,7
3	1,4,7
4	1,3,7,8,5
5	1,2,4,6,7,8
6	2,5,7
7	1,2,3,4,5,6,8
8	4,5,7

In circle map, the necessary condition of adjacent relationship is between

2 adjacent layers, or between 2 adjacent regions (left, right) in the same layer. Such as below:



Region 1 (layer 1) is adjacent to 2,3(layer 2). Region 3 (layer 2) is adjacent to 2,4(layer 2).

If regions are not in the adjacent layer or more than 2 regions in the same layer, they are sure to be not adjacent. Such as, 6(layer 3) and 1(layer 1) are not adjacent; 2 and 4 are not adjacent in layer 2, because there are 3,4,5 in layer 2, they can’t all adjacent to 1.

With the 2 necessary condition of adjacent relationship, check the table one item by one item.

1	2,3,4,5,7
2	1,5,6,7
3	1,4,7
4	1,3,7,8,5
5	1,2,4,6,7,8
6	2,5,7
7	1,2,3,4,5,6,8

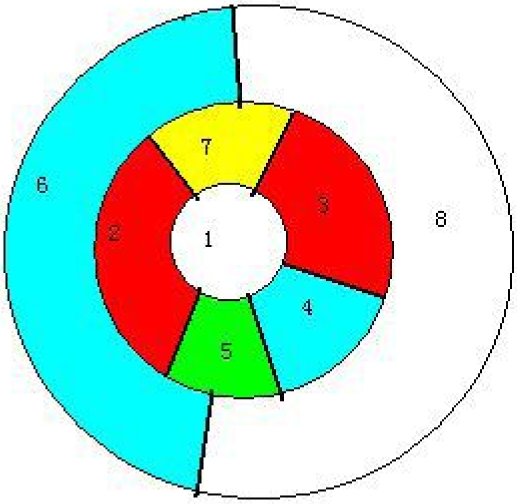
8	4,5,7

This is the method to check region 7 and others are similar.

First, remove regions in adjacent layers.

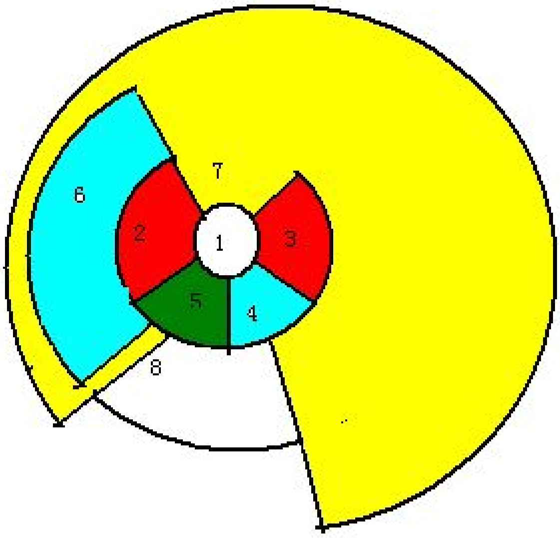
7	2,3,4,5

Because 2,3,4,5 are in the same layer and total 4 > 2, the region 7 can't preserver adjacency.

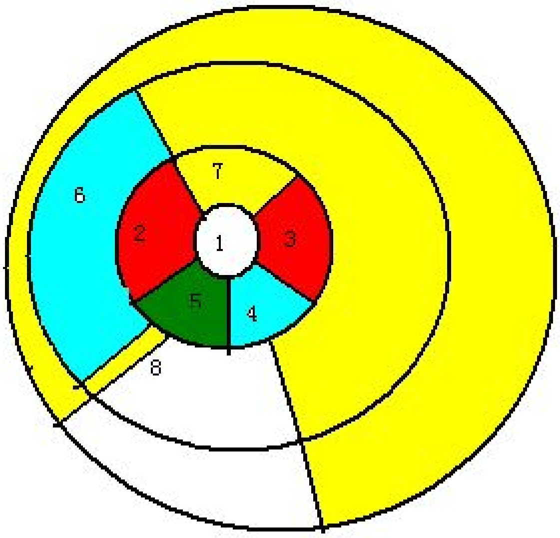


To preserver adjacency, it must to cross layers by region 4, 5 or 7.

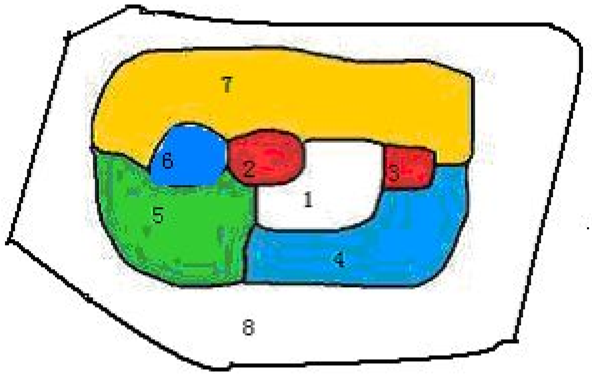
Go back to original map. Region 7 is across over region 6 to adjacent to 5, and is across over region 3 to adjacent to 4. The final map is below.



It equal to the standard circle map below.



Transform irregular map to circle map, 1 is circle center, 6 and 8 are in layer 2, 7 is across layer 2,3,4. Other regions are in layer 1. The boundary adjacent relation is never changed, but some point adjacent or non-adjacent relations are changed to boundary adjacent relation to match the circle map transforming.



The color solution for circle map transformed can apply to irregular map also. Region 6 has changed color, but there is no same color between boundary adjacent regions. It is a color solution qualified.

5. Terminology

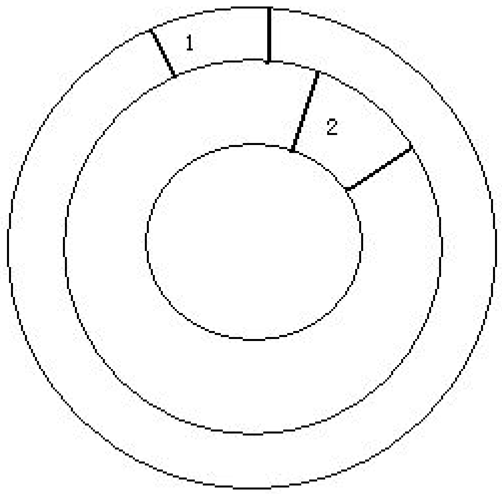
To describe conveniently, I have defined some terms in circle map.

Solution(G(n), color1,color2,…,colork) is a color solution qualified to color all of n regions of G(n) by color in *{color1,color2,…,colork}*.

G(n) is the n of adjacent regions in map.

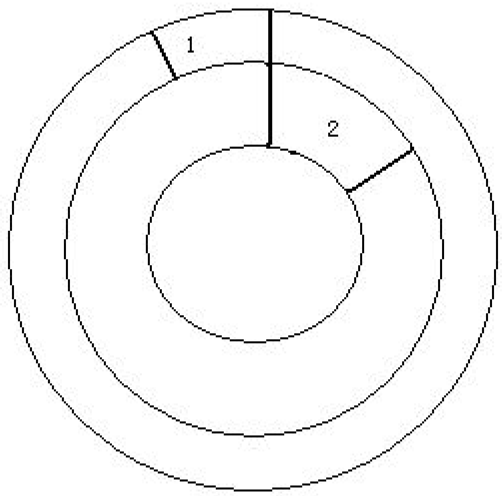
Non-adjacent regions as those no point met.

For example: 1 is non-adjacent to 2 in below circle map.



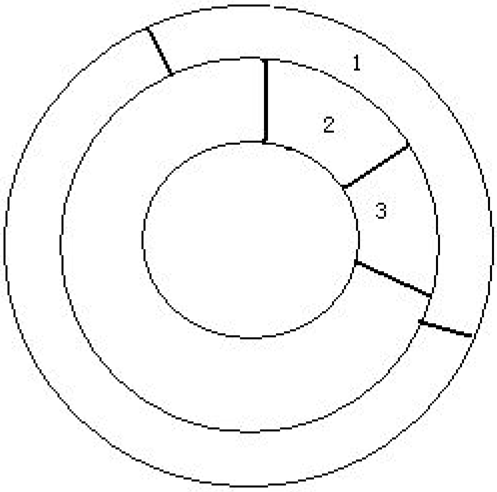
Point adjacent regions as those that meet at a single point or limited points. *Point adjacent* regions are also non-adjacent.

For example: 1 is point adjacent to 2 in below circle map.



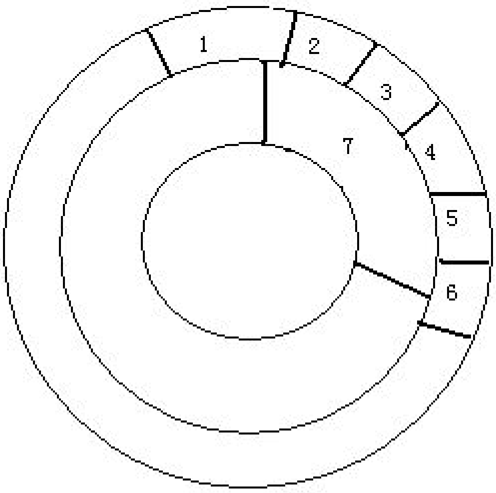
Boundary adjacent regions as those that share a common boundary of non-zero length.

For example: 1, 2, 3 are all boundary adjacent to other 2 regions in below circle map.



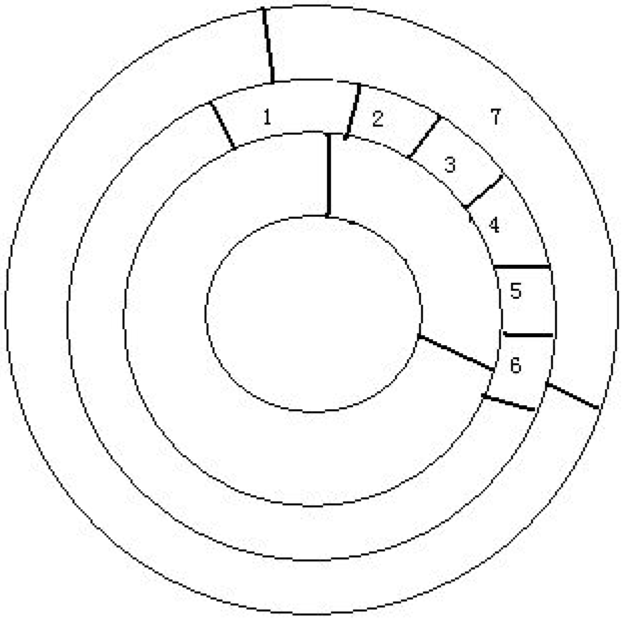
Covered is that the least regions in upper ring have covered one nation in lower ring. Especially, the least N regions in upper ring covering 1 nation in lower ring calls *N Covered*, all the regions in upper ring covering 1 nation in lower ring calls *full Covered.*

For example: 1, 2, 3, 4, 5, 6 are covering 7 in below circle map, that is 6 covering.



Supported is that the least regions in lower ring have covered one nation in upper ring. Especially, the least N regions in lower ring covering 1 nation in upper ring calls *N Supported*, all the regions in lower ring covering 1 nation in upper ring calls *full Supported.*

For example: 1, 2, 3, 4, 5, 6 are supporting 7 in below circle map, that is 6 supporting.



Color is to color region by one or more than one colors. It is recoded as *Color(region) = { color}.* If a region can be colored by more than 1 colors, it can be recoded as *Color(region) = {color1/color2…/colork}.* Such as *Color(3) = {yellow/green/gray}.* Region 3 is colored by *{yellow}* now, but region 3 has the freedom to color by *{ gree}* or *{gray}.*

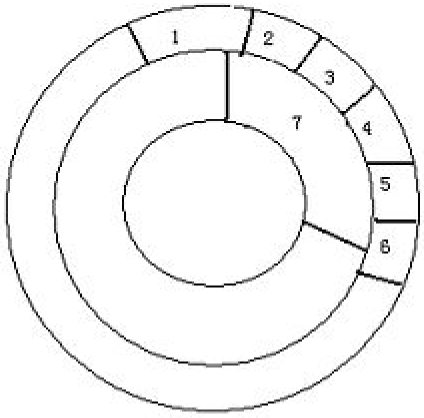
Main color is *{color1}* in *Color(region) = {color1/color2…/colork}*, which color the region in fact.

Backup color is *{color2…/colork}* in *Color(region) = {color1/color2…/colork}*, which doesn't color the region in fact, but which has the freedom to color by *{color2…/colork}.*

Optional colors are all of *main color* and *backup color.*

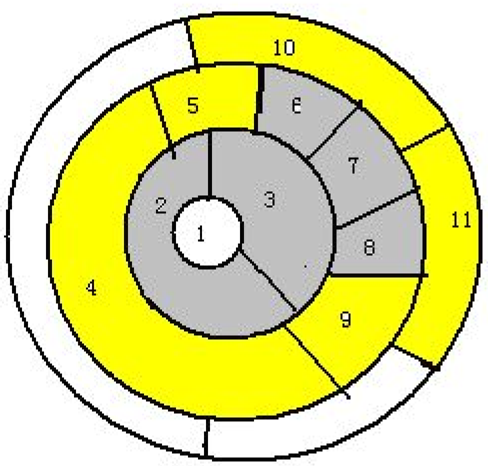
Region number is the total region number of a ring.

For example: region number of ring 3 is 7 in below circle map. Record as *Region(3) = 7*.



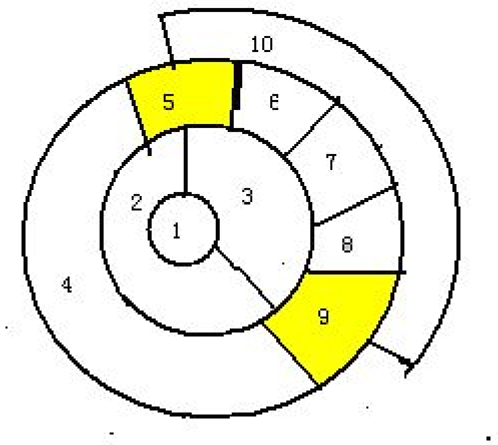
Border regions are all the boundary regions between colored and uncolored. It's the frontier of regions colored.

Inner regions are the entire boundary regions closed by *Border regions*. It's the home of regions colored. For example:



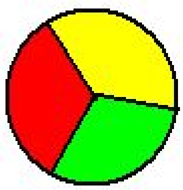
Regions 1 to 11 are colored, the *Border regions* are marked as yellow color, which are close border to seal all regions colored. *Inner regions* are marked as gray color.

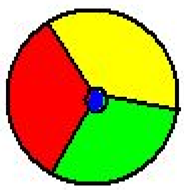
Adjacent border regions are a region being adjacent to *border regions,* there are 1 or 2 regions are adjacent to the region, but not *full covered.* For example:



Region 5 and 9 are adjacent to region 10, but not covered by region 10. Region 5 and 9 are *adjacent border regions.*

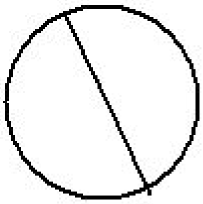
Empty region is a point, which is not a real region, only a proving tool. For example, 3 regions is equivalent to 3 regions and a *Empty region.*



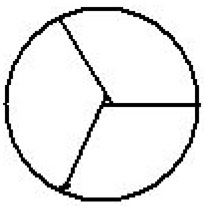


Kn regions are k regions are all adjacent. Anyone of k region is adjacent to other k-1 regions.

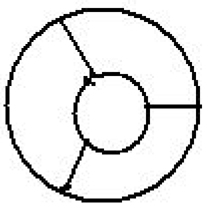
For example: *k2 regions* are in below circle map.



k3 regions are in below circle map. Anyone region is adjacent to other two regions.



K4 regions are in below circle map. Anyone region is adjacent to other three regions.



Graph theory has proven *K4 regions* are the max adjacent relationship in planar graph.

6. Preliminary theorem

(6.1) *K4 regions* have only 3 scenarios in circle map.

Because in a ring, one region can only adjacent to 2 regions at most => there are at most 3 regions in a ring, but *K4 regions* have 4 regions > 3

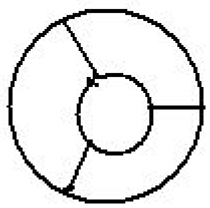
=> *K4 regions* are at least in 2 rings.

If total of rings >=3, there must be one ring insulating another ring. =>There must be 2 regions being non-adjacent. => Total of rings <=2,

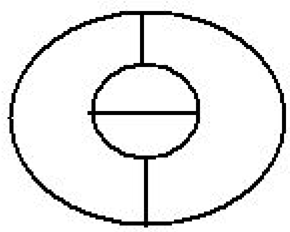
Because total of rings >=2 and total of rings < 2 => total of rings =2.

Total of rings =2 and *K4 regions* have 4 regions => *K4 regions* have only 3 scenarios in circle map. I.e.

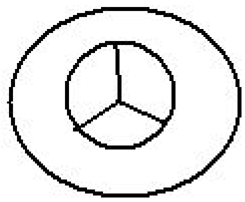
(6.1.1) *region(1) = 1, region (2) = 3.*



(6.1.2) *region(1) = 2, region (2) = 2.*



(6.1.3) *region(1) = 3, region (2) = 1.*

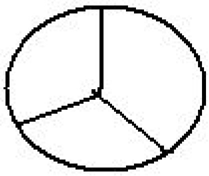


Is there any region across rings? No. Because there is only 2 adjacent rings, regions can keep adjacent relationship in adjacent rings, don't' need to cross rings.

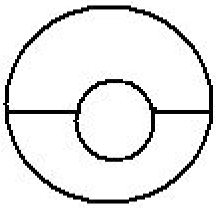
(6.2) *K3 regions* have only 3 scenarios in circle map.

Similarly, we can get 3 scenarios.

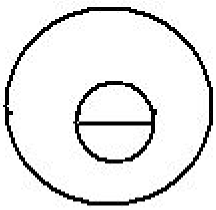
(6.2.1) *region (1) = 3.*



(6.2.2) *region(1) = 1, region (2) = 2.*



(6.2.3) *region(1) = 2, region (2) = 1.*



To prove conveniently, we can unify (6.1) and (6.2). For *K3 regions*, we regard there is a *empty region* in *inner regions*. Then *K3 regions* become *K4 regions*

(6.2.1) *region (1) = 1 empty region , region (2) = 3.*

(6.2.2) *region (1) = 1+ 1 empty region, region (2) = 2.*

(6.2.3) *region (1) = 2+ 1 empty region, region (2) = 1.*

(6.3) Three-color theorem

G(n) is the n of adjacent regions in map.

B(k) is the k of adjacent border regions of G(n).

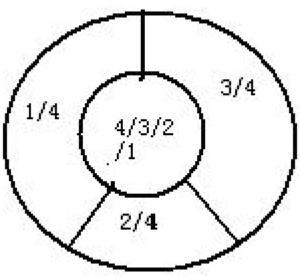
P(n): to a g(n) map with b(k) border regions, there is a Solution(G(n), 1,2,3,4)

1. Three colors {1,2,3} are sufficient to border regions, i.e. there is a *Solution(b(k), 1,2,3)*, which is satisfied *Solution(G(n), 1,2,3,4)* also;

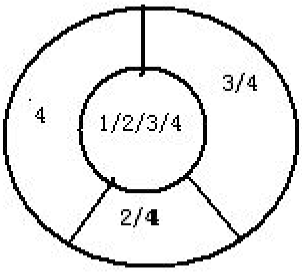
2. Every region has at least 2 *optional colors*, i.e. if region *color(r(i))={1/2}*, there are 2 solutions, *Solution1(G(n), 1,2,3,4)* and *Solution2(G(n), 1,2,3,4)* . Solution1 satisfies *color(r(i))={1}* and Solution2 satisfies *color (r(i))={2}*

3. Every *border region* can change into another *optional color*, which is no impact on other *border regions*. I.e if border region *Color (b(i))={1/2},* there are 2 solutions, *Solution1(b(k), 1,2,3), Solution2(b(k), 1,2,3)*. The only difference of them is *Color (b(i))={1}* and *Color (b(i))={2}*.

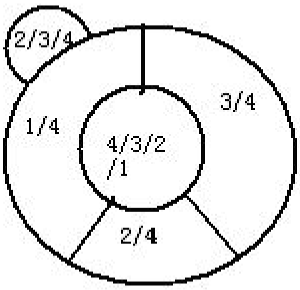
Such as a K4 regions, the possible color is below



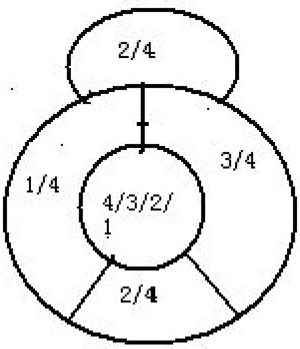
Change one border region color, other border regions are intact. E.g.



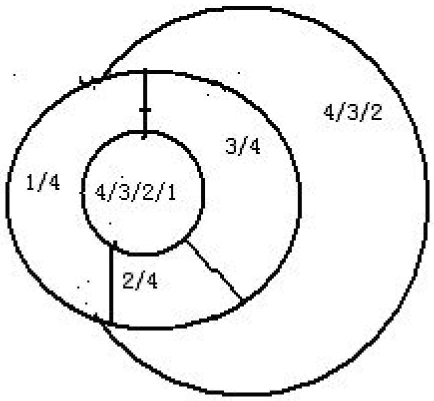
When add a region, which is adjacent to 1 border region and has 3 optional colors. When the region changes to other optional colors, which is no impact on other border regions and inner regions.



When add a region, which is adjacent to 2 border regions and has 2 optional colors. When the region changes to other optional colors, which is no impact on other border regions and inner regions.

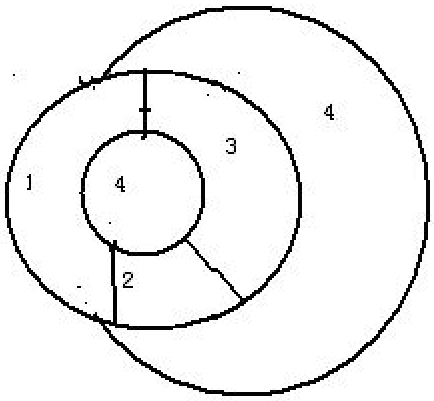


When add a region, which is adjacent to 3 border regions and has 3 optional colors.

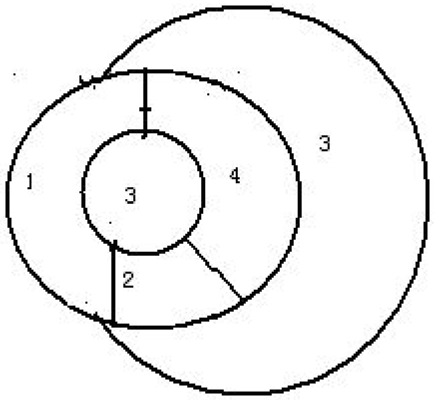


Only explain this scenario, others are similar.

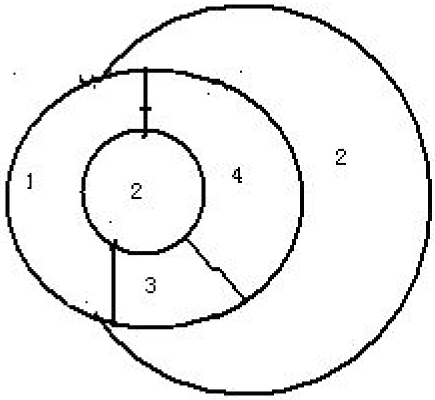
1. To be colored by {4}.



2. To be colored by {3}.

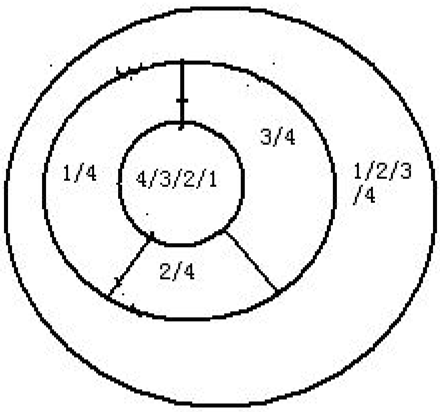


3. To be colored by {2}.



The region colored by {1} is intact and {2/3/4} can color the region added. Change the region color into another *optional color*, other *border regions* are intact.

When add a region, which is *full covering* all *border regions* and has 4 *optional colors*, because there is no *border regions* except for the region added now.

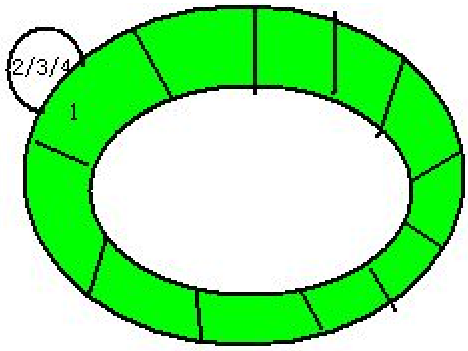


It's easy to verify every region in *border regions* has at least 2 *optional colors* when total of regions is not above 5.

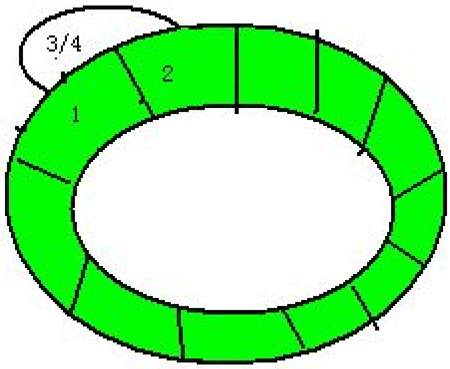
(6.3.1) P(k): when total of regions equals to k(k>5), (6.3) Three-color theorem is true.

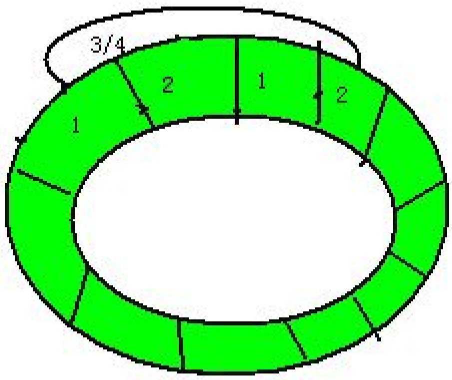
Then add the $(k+1)^{th}$ region, total of regions become k+1. Then we can deduce P(k+1) from P(k).

If the $(k+1)^{th}$ region is adjacent to 1 region of *border regions,* it has 3 *optional colors*. Change the region color to other *optional color*, which is no impact on other *border regions* and *inner regions*.

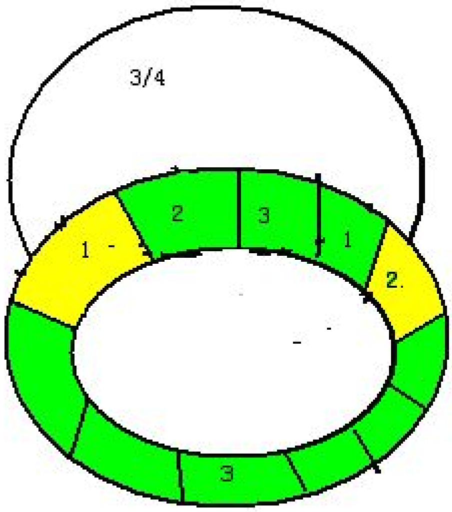


If the $(k+1)^{th}$ region is adjacent to 2 colors of *border regions*, it has 2 *optional colors*. Change the region color to other *optional color*, which is no impact on other *border regions* and *inner regions*.





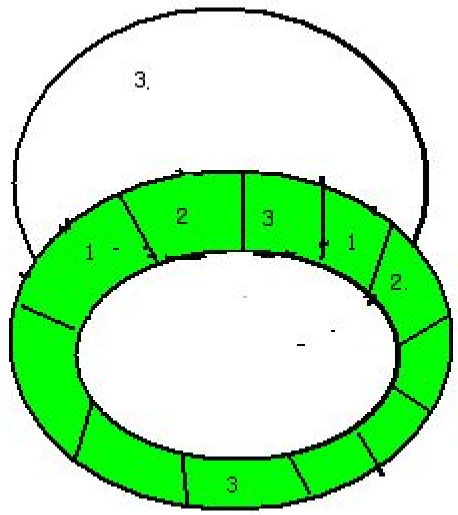
If the $(k+1)^{th}$ region is adjacent to more than 2 colors of *border regions*, it has at least 2 *optional colors*. The proof is below.



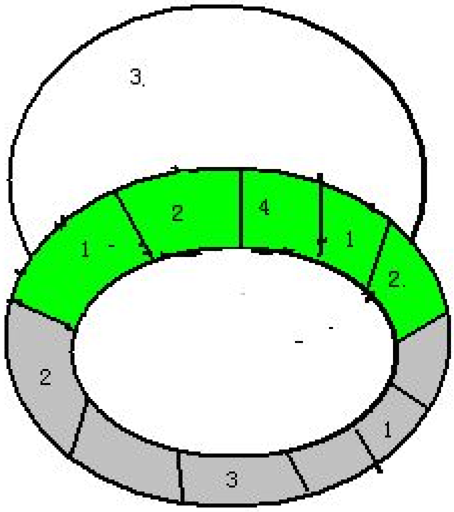
$(k+1)^{th}$ region's color depends on the *adjacent border regions*. E.g regions in yellow color. {1,2,3,4} – {1,2} = {3/4}

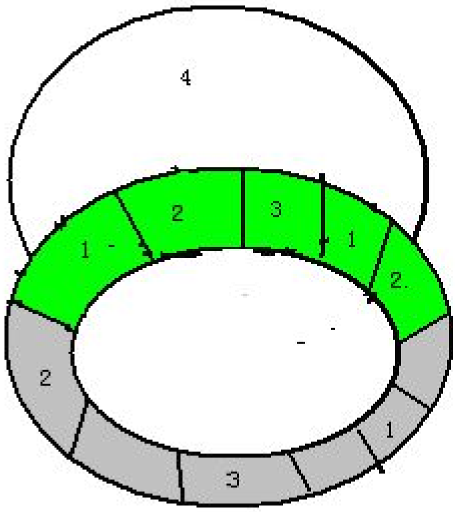
Because $(k+1)^{th}$ region's color is adjacent to 3 colors, {4} is OK.

How to color {3}? First to color $(k+1)^{th}$ region as {3}.



Because (6.3.1), *border regions* of G(k) have at least 2 *optional colors* and *border regions* of G(k) being *full covered* by $(k+1)^{th}$ region and colored by {3} can change to other *optional color* and this color must in {1,2,4}. Other *border regions* of G(k) are intact. Then $(k+1)^{th}$ region can be colored by {3/4} and colors of *border regions* of G(k+1) is still in {1,2,3}. Change the region color to other *optional color*, which is impact on *inner regions*, and other *border regions* are intact.





So when total of regions equals to k+1, (6.3) Three-color theorem is true.

When n <4, 3 colors are sufficient.

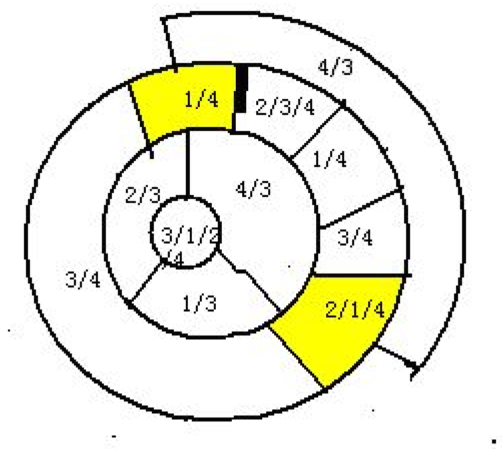
When n = 4, 3 colors are sufficient for *border regions*.

When n = 5, we can verify 3 colors are sufficient for *border regions*.

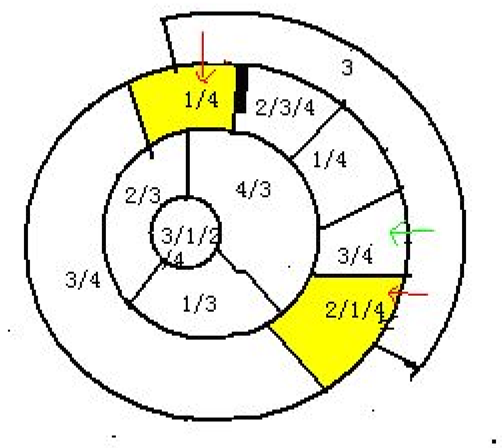
P(k): if n = k, 3 colors are sufficient for *border regions,* we can deduce P(k+1): n = k+1, 3 colors are sufficient for *border regions.*

So to all of n, in G(n) map, (6.3) Three-color theorem is true.

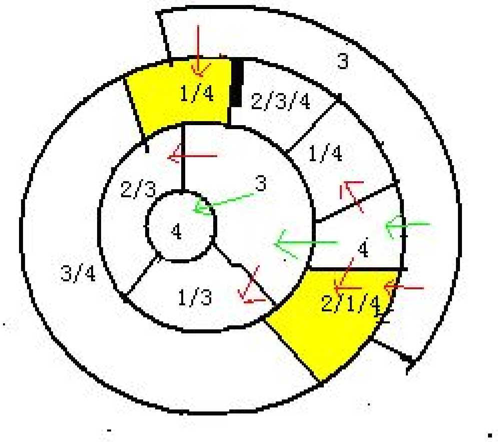
Some people wonder how the *inner regions'* color permutes? We can find that the *optional colors* depend on the *adjacent border regions.* For example:



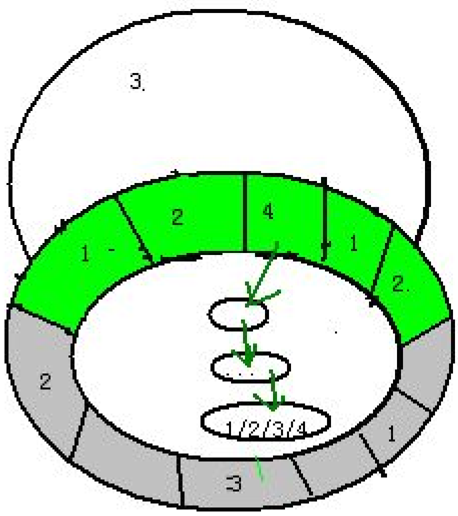
The region colored by {4/3} = {1,2,3,4} – {1,2}. When color change from {4} to {3}, {3} need to change, it is no impact to its *adjacent border regions*, but only to *inner regions*. For example:



Red arrow denotes no impact; green arrow denotes impact. It's the same as next region permuted, which is no impact on *adjacent border* *regions*, but only *inner regions*.

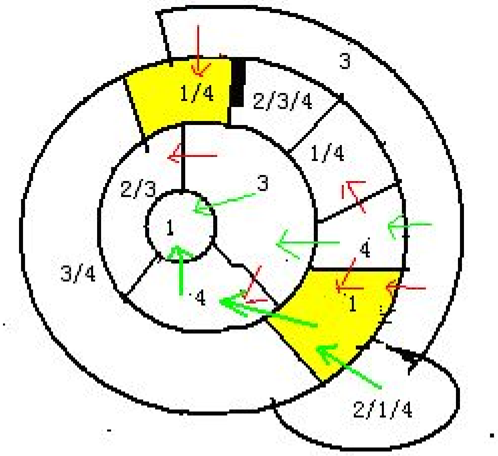


Base on the algorithm, we can guarantee the core center is a *K4 regions*, which can guarantee at least 1 region with 4 *optional colors*, and any impact will stop here. E.g.



All impacts are towards *inner regions*, can stop at the region with more *optional colors*. The worst scenario is impact stop at the region with 4 *optional colors*, which can stop any type of impact. The better scenario is impact stop at the region, which is no impact on *inner regions*.

More impacts are similar to 1 impact, because more impacts can be handled one by one. The region with 4 *optional colors* still exists. E.g.



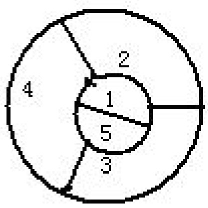
(6.4) *K4 regions* are the max adjacent relationship in circle map.

Traditional graph theory has proven this theorem. Here is a new proof.

(6.4.1) Suppose *K5 regions* are the max adjacent relationship in circle map.

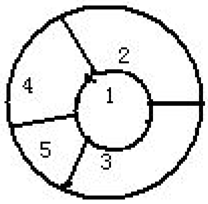
Base on the *K4 regions* scenario of (6.1.1), let me to add a new country in circle map, which is *K5 regions*.

(6.4.1.1) The new country is in ring 1, like below circle map.



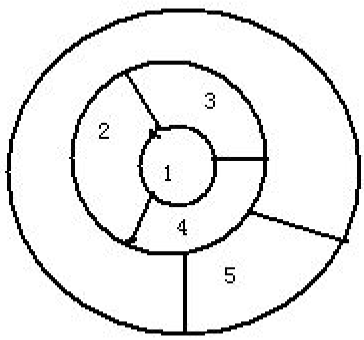
Because they are *K5 regions*, country 1 is full covered by country 2,3,4 and country 5 is also full covered by country 2,3,4. 3 countries cover both country 1 and 5. And countries 2,3,4 are in a ring, so the overlap countries are at most 2 countries. Then the total countries in ring 2 , denoted as *Country(2) >= 3+3-2 = 4>3.* It's contradictory.

(6.4.1.2) The new country is in ring 2, like below circle map.



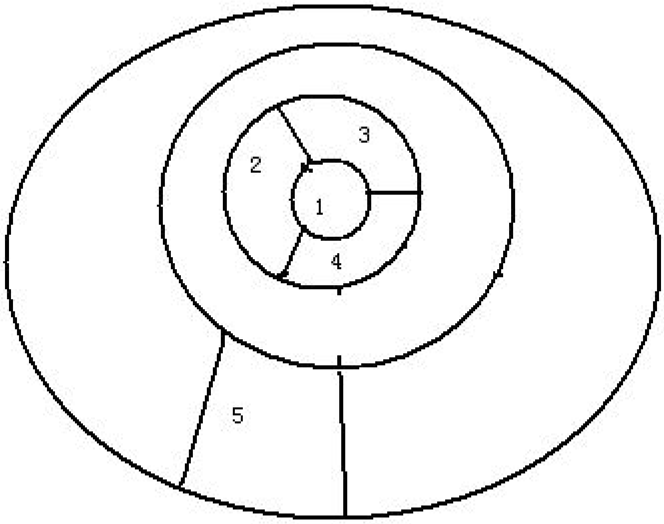
Because countries 2,3,4,5 are in the same ring 2, country 5 can be only *Boundary adjacent* with 2 countries in the same ring. But there are 3 other counties in the ring 2, so one country must be *Non-adjacent* with country 5.

(6.4.1.3) The new country is in a new ring 3, like below circle map.



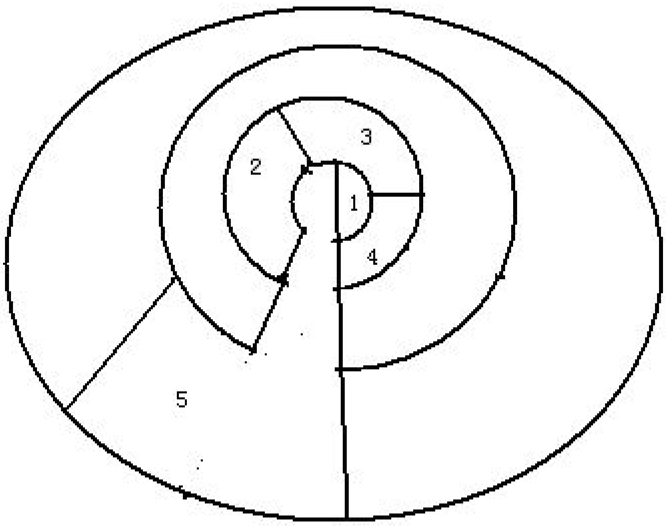
Country 5 is a new country in new ring. Obviously, country 5 is always *Non-adjacent* with country 1, because they are in ring 1 and ring 3, ring 2 has insulated them.

(6.4.1.4) The new country is in a new ring k, (k>3), like below circle map.

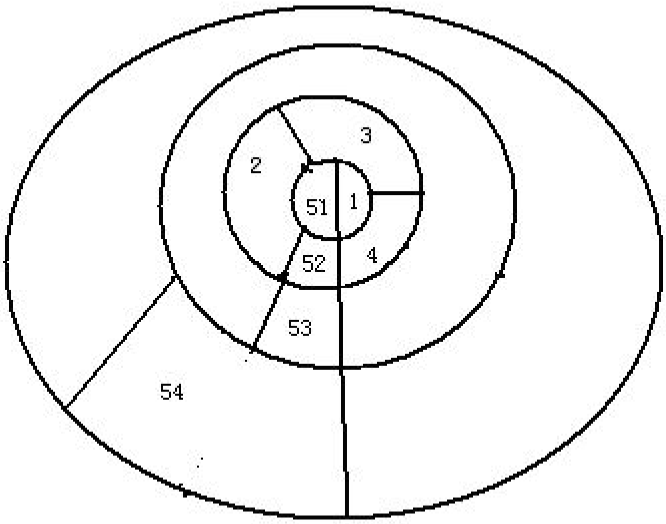


Country 5 is a new country in new ring. Obviously, country 5 is always *Non-adjacent* with country 1, because they are in ring 1 and ring k, ring 2 has insulated them.

(6.4.1.5) The new country is across ring 2, like below circle map.

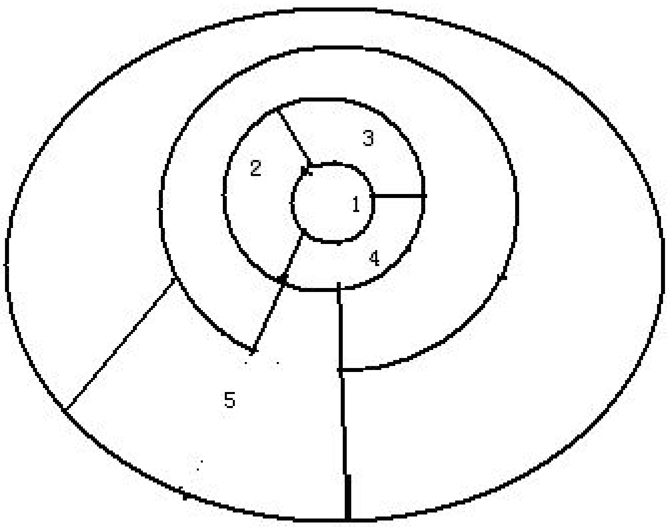


We can divide country 5 into multiple countries in each ring like below.



Similar to scenario (6.4.1.2), because countries 2,3,4,52 are in the same ring 2, country 52 can be only *Boundary adjacent* with 2 countries in the same ring. Select one of them (such as country 4) to find another *non-adjacent country*. Country 4 can be only *Boundary adjacent* with 2 countries in the same ring. But there are 3 other counties in the ring 2, so one country must be *Non-adjacent* with country 4.

(6.4.1.6) The new country is not across ring 2, like below circle map.



Similar to scenario (6.4.1.3), country 5 is always *Non-adjacent* with country 1, because ring 2 has insulated them.

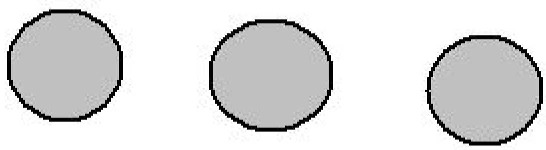
The same method can prove scenarios of (6.1.2) and (6.1.3).

All scenarios are contradictory, so supposition (6.4.1) is false and *K4 regions* are the max adjacent relationship in circle map.

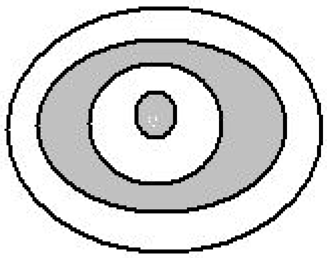
7. Four color theorem

There are N regions. Firstly, search max adjacent relationship.

If no *K2 regions,* all regions are *non-adjacent*, one color is sufficient.



Max adjacent relationship is *K2 regions,* all regions are at most adjacent to one region, and two colors are sufficient.

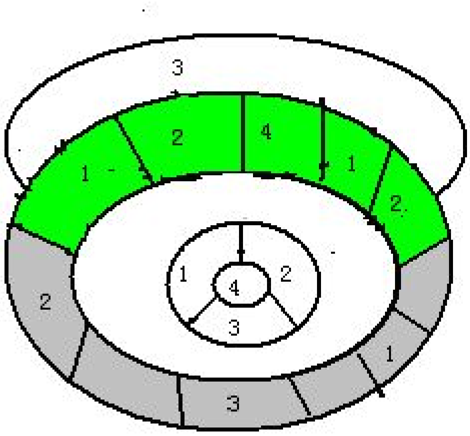


Max adjacent relationship is *K3 regions,* we can add a *empty region* in ring 1, it becomes *K4 regions. border regions* are formed.

Max adjacent relationship is *K4 regions,* because *K4 regions* have two rings, *border regions* are formed now.

Base on *K4 regions,* we can select a region which is adjacent to *border regions*, and don't stop coloring these regions one by one until all of the N regions are colored.

Because of (6.3) Three-color theorem, we can use 3 colors in {1,2,3} to color *border regions* and 4 colors in {1,2,3,4} to color *inner regions.*



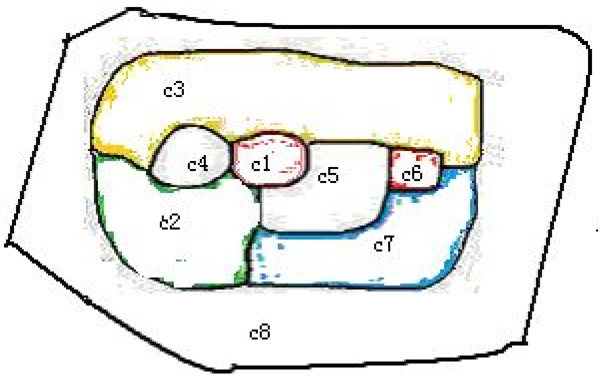
Above proof indicates *border regions'* colors are in {1,2,3}, and *inner regions'* colors are in {1,2,3,4}.

Finally, four-color theorem is proven now!

8. Verification and Demo

One example to verify and explain:

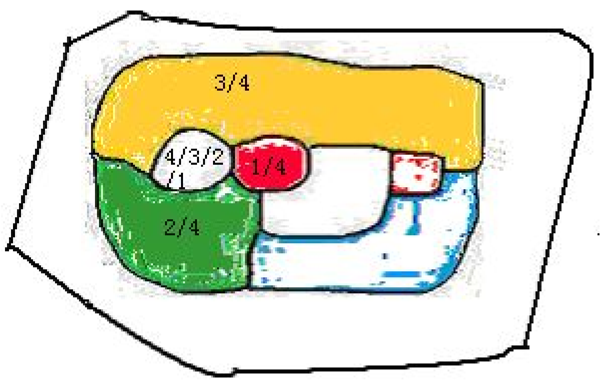
To descript clearly, all regions are marked the number by the color order in advance. The order number (region number) map is below,



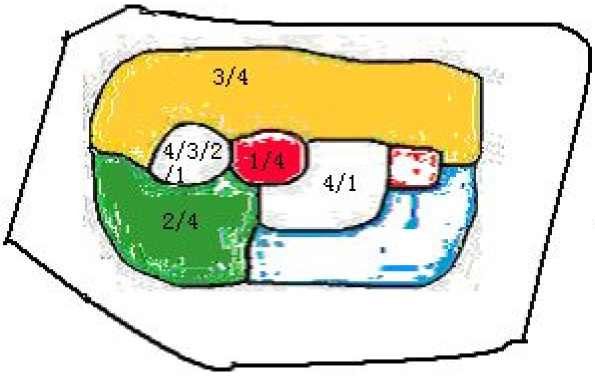
The algorithm bases on (7.4), is simpler than the example in section 7.

1. Search and color max adjacent relationship of complete graph.
2. Find *non-adjacent regions* and colored with more possibilities.

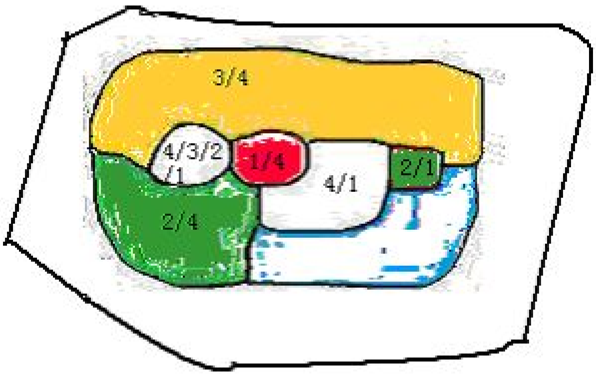
Firstly, search *K4 regions* and colored by {1,2,3,4}



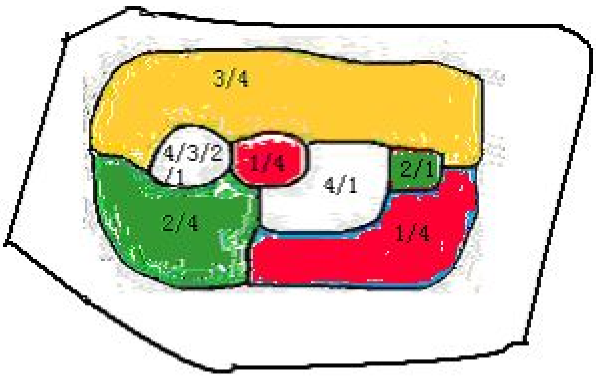
Next, we can select an adjacent region c5, which is adjacent to 3 regions in border. 2 *adjacent border regions* are {2,3}. So it is colored by color of inner region {4/1}.



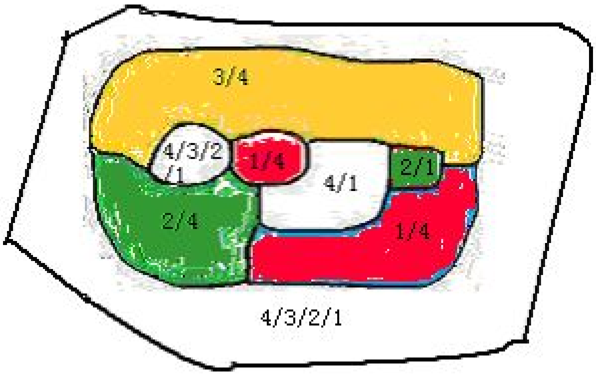
Next, we can select an adjacent region c6, which is adjacent to 2 regions in border. 2 *adjacent border regions* are {3,4}, so color is {2/1}.



Next, we can select an adjacent region c7, which is adjacent to 3 regions in border regions. *adjacent border regions* are {2,3}, so color is {1/4}.



Next, we can select an adjacent region c8, which is adjacent to all border regions. It is *full covered* all *border regions*. So it can be colored by {4/1/2/3}.



Every region has at least 2 *optional colors*. All regions use 4 colors.

9. Conclusion

Four-color theorem is an interesting phenomenon, but there is a rule hidden the phenomenon. The max adjacent relationship on a surface decides how many colors are sufficient. More than max adjacent regions, there is a *non-adjacent region*. The *non-adjacent region* can decrease color consumption. Every region has at least 2 *optional colors*, which can be permuted with only impacts on *inner regions*.

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